The Scottish Book

Mathematics from the Scottish Café

Edited by R. Daniel Mauldin

(Excerpt of Problems 73-74)

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MAZUR

LET E BE A SPACE OF type (F) with the following property: If $Z \subseteq E$ is a compact set, then the smallest closed convex set containing Z is also compact. Is E then a space of type (F_0)? [See Problem 26 for a definition of (F_0).]

Commentary

Mazur's theorem states that the closed convex hull of a compact subset of a Banach space is compact [1]. Problem 72 then asks for a partial converse to this result. It was answered by Mazur and Orlicz [2] who showed that an F-space X is locally convex if and only if whenever $x_n \to 0$ in X, $t_n \geq 0$ and $\Sigma t_n \leq 1$ then the series $\Sigma t_n x_n$ is bounded (i.e., has bounded partial sums). Thus if the convex hull of every compact set is bounded, then X is locally convex. In particular, spaces of type (F_0) are locally convex and the answer to Problem 72 is affirmative.

References

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MAZUR, ORLICZ

LET c_n BE THE SMALLEST number with the property that if $F(x_1, \ldots, x_n)$ is an arbitrary symmetric n-linear operator [in a space of type (B) and with values in such a space], then

$$\sup_{\|x_i\| \leq 1,\, i = 1,2,\ldots,n} \|F(x_1, \ldots, x_n)\| \leq c_n \sup_{\|x\| \leq 1} \|F(x, \ldots, x)\|.$$

It is known (Mr. Banach) that c_n exists. One can show that the number c_n satisfies the inequalities

$$\frac{n^n}{n!} \leq c_n \leq \frac{1}{n!} \sum_{k=1}^{n} \binom{n}{k} \cdot k^n.$$

Is $c_n = n^n/n!$?

Commentary

The answer to this problem is yes for any real normed linear spaces and it is now a standard fact in the field of infinite dimensional holomorphy. R.S. Martin proved that $c_n \leq n^n/n!$ in his 1932 thesis [11] with the aid of an n-dimensional polarization formula. His argument was published a few years later in [15] by A.E. Taylor. Although this polarization formula was known to Mazur and Orlicz [12, p. 52], it appears that they used the case $x = 0$ rather than the case $x = -\Sigma_1^n h_k/2$. which gives the best estimate. Extremal examples in ℓ^1 and $L[0,1]$ showing that $c_n = n^n/n!$ are given in [5], [10], and [16]. For expositions, see [4, p. 48] and [13, p. 7]. Note that by the Hahn-Banach theorem, there is no loss of generality in this problem if all multilinear mappings are taken to be complex valued.

S. Banach [1] showed in 1938 that $c_n = 1$ when only real Hilbert spaces are considered. (He also assumed separability, though this assumption is not needed.) His result can be deduced quite easily from [3, Satz 9] or [9, Th. IV]. For modern expositions, see [2, p. 62], [6] or [8]. It is shown in [6] that Banach's result and an improvement by Szego of Bernstein's inequality for trigonometric polynomials are easily deduced from each other. For complex L^p-spaces, $1 \leq p < \infty$, it is conjectured in [6] that

$$c_n \leq \left(\frac{n^n}{n!}\right)^{\frac{|p-2|}{p}},$$

and this is proved when n is a power of 2. It also follows from [6] that

$$c_n \leq \frac{n^{\frac{n}{2}}(n+1)^{\frac{(n+1)}{2}}}{2^n n!}$$

holds for J^*-algebras [7]. (In particular, the space $C(S)$ of all continuous complex-valued functions on a compact Hausdorff space S and more generally, any B^*-algebra is a J^*-algebra.) Since $c_n = 1$ for the space $C(S)$ with S a two point set [6, p. 154], it is natural to ask whether this holds for any compact Hausdorff space S. If so, then it is easy to deduce that the Bernstein inequality holds for polynomials on $C(S)$. (See [6, p. 149].)

A natural generalization of Problem 73 is the following: Let $k_1, \ldots, k_n$ be nonnegative integers whose sum is n and let $c(k_1, \ldots, k_n)$ be the smallest number with the property that if F is any symmetric n-linear mapping of one real normed linear space into another, then

$$\sup_{|x_i| \leq 1, i = 1,2,\ldots,n} \|F(x_1^{k_1} \ldots x_n^{k_n})\| \leq c(k_1, \ldots, k_n) \sup_{\|x\| \leq 1} \|F(x, \ldots, x)\|,$$

where the exponents denote the number of coordinates in which the base variable appears. It is shown in [6] that if only complex normed linear spaces and complex scalars are considered, then

$$c(k_1, \ldots, k_n) = \frac{k_1! \ldots k_n!}{k_1^{k_1} \ldots k_n^{k_n}} \frac{n^n}{n!} \tag{1}$$

(where $o^0 = 1$) but there are many cases where (1) does not hold when real normed linear spaces are considered.

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MAZUR, ORLICZ

GIVEN IS A POLYNOMIAL

$$W(t_1, \ldots, t_n) = \sum_{k_1 + \ldots + k_n = n} a_{k_1, \ldots, k_n} t_1^{k_1} \ldots t_n^{k_n}$$

in real variables $t_1, \ldots, t_n$, homogeneous and of order n; let us assume that $|W(t_1, \ldots, t_n)| \leq 1$ for all $t_1, \ldots, t_n$ such that $|t_1| + \ldots + |t_n| \leq 1$. Do we then have

$$|a_{k_1, \ldots, k_n}| \leq \frac{n^n}{k_1! \ldots k_n!}?$$

Commentary

The answer to this problem is yes and the solution is an easy consequence of the solution to Problem 73. Indeed, let F be the symmetric n-linear map on ℓ_n^1 such that $W(x) = F(x, \ldots, x)$ for all $x \in \ell_n^1$. Then applying the multinomial theorem [12, p. 52] with $x = t_1e_1 + \ldots + t_ne_n$ and the uniqueness of the representation for $W(t_1, \ldots, t_n)$, we obtain

$$a_{k_1 \ldots k_n} = \frac{n!}{k_1! \ldots k_n!} F(e_1^{k_1} \ldots e_n^{k_n}), \tag{2}$$

where $e_1, \ldots, e_n$ is the standard basis for ℓ_n^1. The desired estimate follows. Note that the problem of determining the best estimate $\alpha(k_1, \ldots, k_n)$ in Problem 74 is equivalent to the problem of determining $c(k_1, \ldots, k_n)$; for, if F is a symmetric n-linear map satisfying $\|F(x, \ldots, x)\| \le 1$ for all $\|x\| \le 1$ and if $\|x_1\| \le 1, \ldots, \|x_n\| \le 1$, then the polynomial $W(t_1, \ldots, t_n) = F(x, \ldots, x)$, where $x = t_1x_1 + \ldots + t_nx_n$, satisfies the hypotheses of Problem 74 and

$$a_{k_1 \ldots k_n} = \frac{n!}{k_1! \ldots k_n!} F(x_1^{k_1} \ldots x_n^{k_n}).$$

Thus

$$\alpha(k_1, \ldots, k_n) = \frac{n!}{k_1! \ldots k_n!} c(k_1, \ldots, k_n). \tag{3}$$

The general problem of obtaining estimates on the coefficients of polynomials in m variables which satisfy a given growth condition on R^m can be solved with the aid of the following generalized polarization formula: Let $W(t_1, \ldots, t_m)$ be any homogeneous polynomial of degree n and let $a_{k_1 \ldots k_m}$ be the coefficient of $t_1^{k_1} \ldots t_m^{k_m}$ in its expansion. For each $i = 1, \ldots, m$, choose distinct real numbers $x_{i0}, \ldots, x_{ik_i}$ and put

$$\Gamma_{ij} = \prod_{\ell \ne j} (x_{ij} - x_{i\ell}),\ 0 \le j \le k_i,$$

with $\Gamma_{ij} = 1$ if $k_i = 0$. Then

$$a_{k_1 \ldots k_m} = \sum \frac{W(x_{1j_1}, \ldots, x_{mj_m})}{\Gamma_{1j_1} \ldots \Gamma_{mj_m}} \tag{4}$$

where the sum is taken over all $0 \le j_1 \le k_1, \ldots, 0 \le j_m \le k_m$ Note that one can convert any polynomial p of degree $\le n$ in $m - 1$ variables to a homogeneous polynomial W of degree n in m variables by defining

$$W(t_1, \ldots, t_m) = t_m^n\, p\left(\frac{t_1}{t_m}, \ldots, \frac{t_{m-1}}{t_m}\right).$$

One can obtain estimates on the left-hand side of (4) by estimating the right-hand side of (4) and minimizing. (A reasonable first choice is $x_{ij} = k_i/2 - j$.) To prove (4), observe that if $p(t)$ is a polynomial of degree $\leq k_i$, then the coefficient of t^{k_i} in the Lagrange interpolation formula for p is $\sum_{j=0}^{k_i} p(x_{ij})/\Gamma_{ij}$ and apply this to each variable of W.

For example, we show that the improved estimate

$$|a_{k_1 \dots k_n}| \leq \frac{n^n}{k_1! \dots k_n!} r^\ell$$

holds in Problem 74, where

$$r = \frac{1 + e^{-2}}{2} \quad \ell = \sum_{i=1}^{n} \left[\frac{k_i}{2}\right] .$$

Indeed, choose $x_{i0} = 2$, $x_{i1} = 0$, $x_{i2} = -2$ for $i = 1, \dots, \ell$, $x_{i0} = 1$, $x_{i1} = -1$ for $i = \ell + 1, \dots, n - \ell$, and $x_{i0} = 0$ for $i = n - \ell + 1, \dots, n$. Then by (4),

$$|a_{2 \dots 21 \dots 10 \dots 0}| \leq \frac{1}{4^\ell} \sum_{j=0}^{\ell} \binom{\ell}{j} (n - 2j)^n \leq 2^{-\ell} n^n r^\ell, \tag{6}$$

where the last inequality follows from $(1 - 2j/n)^n \leq e^{-2j}$. Clearly

$$c(k_1, \dots, k_n) \leq c(2, \dots, 2,1, \dots, 1,0, \dots, 0)$$

and this together with (3) and (6) implies (5).

A related problem of interest is to find a Banach space analogue of Markov's theorem; that is, to find the smallest number $M_{n,k}$ with the property that if P is any polynomial of degree $\leq n$ mapping one real normed linear space into another, then

$$\sup_{\|x\| \leq 1} \|\hat{D}^k P(x)\| \leq M_{n,k} \sup_{\|x\| \leq 1} \|P(x)\|,$$

where $\hat{D}^k p(x)y = d^k/dt^k\, p(x + ty)|_{t=0}$. It is not difficult to show that

$$T_n^{(k)}(1) \leq M_{n,k} \leq 2^{2k-1} T_n^{(k)}(1), \tag{7}$$

where T_n is the Chebyshev polynomial of degree n. (See [14, p. 119].) Indeed, let P be any real-valued polynomial of degree $\leq n$ on a real normed linear space and suppose $|P(x)| \leq 1$ for all $\|x\| \leq 1$. Let $\|x\| \leq 1$, $\|y\| \leq 1$ and $-1 \leq s \leq 1$. Define $q(t) = P(\phi(t))$, where $\phi(t) = [x - sy + t(x + sy)]/2$, and note that $\|\phi(t)\| \leq (1 + t)/2 + (1 - t)/2 = 1$ when $-1 \leq t \leq 1$. Then q is a polynomial of degree $\leq n$ satisfying $|q(t)| \leq 1$ for $-1 \leq t \leq 1$, so $|q^{(k)}(1)| \leq |T^{(k)}(1)|$ by [14, 1.5.11] and clearly $q^{(k)}(1) = 2^{-k}\hat{D}^k P(x)(x + sy)$. Hence the map $s \to \hat{D}^k P(x + sy)$ is a polynomial of degree $\leq k$ with bound $2^k T_n^{(k)}(1)$ on $[-1,1]$ and $\hat{D}^k P(x)y$ is the coefficient of s^k in this polynomial. Therefore,

$$|\hat{D}^k P(x)y| \leq 2^{k-1}[2^k T_n^{(k)}(1)]$$

by [14, p. 57]. Thus (7) follows by the Hahn-Banach theorem.

Note that the value of $M_{n,k}$ is unchanged when only real-valued polynomials on ℓ_2^1 are considered. It is shown in [6] and [9] that $M_{n,1} = n^2$ when only real Hiblert spaces are considered and it would be interesting to know whether $M_{n,k} = T_n^{(k)}(1)$ for all $1 \leq k \leq n$ in this case. See also [17].

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IN THE EUCLIDEAN n-dimensional space E, or, more generally, in a space of type (B) there is given a polynomial $W(x)$. α is a number $\neq 0$. If a polynomial $W(x)$ is bounded in an ϵ-neighborhood of a certain set $R \subset E$ is it then bounded in a δ-neighborhood of the set αR (which is the set composed of elements αx for $x \in R$)? (See Problem 55.)

Addendum. From the solution of Problem 55, it follows that the theorem is true in the case of a Euclidean space.

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