

Removable singularities in C^* -algebras of real rank zero

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Abstract

Let $\mathfrak{A}$ be a C^* -algebra with identity and real rank zero. Suppose a complex-valued function is holomorphic and bounded on the intersection of the open unit ball of $\mathfrak{A}$ and the identity component of the set of invertible elements of $\mathfrak{A}$. We give a short transparent proof that the function has a holomorphic extension to the entire open unit ball of $\mathfrak{A}$. The author previously deduced this from a more general fact about Banach algebras.

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1. Preliminary definitions and theorems.

Recall [1] that a C^* -algebra is a closed complex subalgebra $\mathfrak{A}$ of the Banach algebra $\mathcal{B}(H)$ of all bounded linear operators on a Hilbert space with the operator norm such that $\mathfrak{A}$ contains the adjoints of each of its elements. All our C^* -algebras contain the identity operator I .

To give a basic example, let S be a compact Hausdorff space and let $C(S)$ be the algebra of all continuous complex-valued functions on S with the sup norm. Then there exist a Hilbert space H , a C^* -algebra $\mathfrak{A}$ in $\mathcal{B}(H)$ and an isomorphism $\rho : C(S) \rightarrow \mathfrak{A}$ that preserves norms and adjoints. To see this, let H be the Hilbert space having the same dimension as the cardinality of S and let $\{e_s : s \in S\}$ be an orthonormal basis for H . Then we may take $\rho(f)$ to be the multiplication operator defined by $\rho(f)(e_s) = f(s)e_s$ for all $s \in S$ and $f \in C(S)$.

More generally, one can define a Banach algebra that is an abstraction of a C^* -algebra and show that an isomorphism like the above exists. Specifically, a B^* -algebra is a complex Banach algebra A with an involution $*$ such that $\|x^*x\| = \|x\|^2$ for all $x \in A$. Then a norm and adjoint preserving isomorphism ρ of A onto a C^* -algebra exists by the Gelfand-Naimark theorem [1, p. 209].

^{*}Dedicated to Richard Aron with gratitude

We now turn to some basic facts about complex-valued holomorphic functions defined on a domain D in a complex Banach space X . We say that a function $f : D \rightarrow \mathbb{C}$ is holomorphic if for each $x \in D$ there exists a continuous complex-linear functional $\ell \in X^*$ such that

$$\lim_{y \rightarrow 0} \frac{f(x+y) - f(x) - \ell(y)}{\|y\|} = 0.$$

Clearly, if f is holomorphic in D then the function $\phi(\lambda) = f(x + \lambda y)$ is holomorphic (in the usual sense) in a neighborhood of the origin for each $x \in D$ and $y \in X$. It is well known [7, Theorem 3.17.1] that this property also implies holomorphy when f is locally bounded in D . One can extend many classical results about holomorphic functions by applying the above property. For example, this is true for the following elementary form of the identity theorem [7, Theorem 3.16.4].

Proposition 1. *Let D be a domain in a complex Banach space X and let $f : D \rightarrow \mathbb{C}$ be holomorphic in D . If f vanishes on a ball in D then f vanishes everywhere in D .*

By definition, a ball is a set of the form

$$B_r(x_0) = \{x \in X : \|x - x_0\| < r\},$$

where $x_0 \in X$ and $r > 0$.

We will need the following elementary version of Taylor's theorem, which can be proved as in [7, Theorem 3.17.1], and a simple converse, which can be obtained from the Weierstrass M-test and [7, Theorem 3.18.1].

Proposition 2. *Let X be a complex Banach space and let $x_0 \in X$ and $r > 0$. If $f : B_r(x_0) \rightarrow \mathbb{C}$ is a bounded holomorphic function, then for each n there is a continuous complex-homogeneous polynomial $P_n : X \rightarrow \mathbb{C}$ of degree n such that*

$$f(x) = \sum_{n=0}^{\infty} P_n(x - x_0) \quad \text{for } x \in B_r(x_0). \quad (1)$$

Conversely, if for each n there is a continuous complex-homogeneous polynomial $P_n : X \rightarrow \mathbb{C}$ of degree n and if

$$\|P_n\| \leq \frac{M}{r^n}, \quad n = 0, 1, \dots \quad (2)$$

for some positive constants r and M , then the function f given by (1) is holomorphic in $B_r(x_0)$.

For example, if (1) holds then

$$P_n(y) = \frac{1}{n!} \left. \frac{d^n}{dt^n} f(x_0 + ty) \right|_{t=0}, \quad n = 0, 1, \dots \quad (3)$$

for all $y \in X$. If f is holomorphic on $B_r(x_0)$ and M is a bound for f , then (2) is a consequence of the classical Cauchy estimates. As usual,

$$\|P_n\| = \sup\{|P_n(x)| : \|x\| \leq 1, x \in X\}.$$

2. Real rank zero.

Definition 1. (See [2].) Let $\mathfrak{A}$ be a C^* -algebra and let $\mathcal{S}$ be the set of self-adjoint elements of $\mathfrak{A}$. Then $\mathfrak{A}$ has real rank zero if the elements of $\mathcal{S}$ with finite spectra are dense in $\mathcal{S}$.

As shown by Brown and Pedersen [2], many interesting C^* -algebras have real rank zero. For example, the C^* -algebra $\mathcal{B}(H)$ of all bounded linear operators on a Hilbert space H has real rank zero. More generally, any von Neumann algebra has real rank zero. The space $C(S)$ of all continuous functions on a compact Hausdorff space S has real rank zero if and only if S is totally disconnected. (It is a von Neumann algebra only if S is extremely disconnected.) Also, any AF-algebra has real rank zero. If $\mathcal{BC}(H)$ is the C^* -algebra of all compact operators on H , then $\mathbb{C}I + \mathcal{BC}(H)$ has real rank zero as does the Calkin algebra $\mathcal{B}(H)/\mathcal{BC}(H)$. Note that the set of invertible elements of the Calkin algebra has a different component for each value of the Fredholm index and thus is not connected. See [3] for further details and references.

Let $\mathfrak{A}$ be a C^* -algebra with identity, let

$$\mathfrak{A}_0 = \{A \in \mathfrak{A} : \|A\| < 1\}$$

be the open unit ball of $\mathfrak{A}$ and let $\mathfrak{A}_{\text{inv}}^e$ be the identity component of the set of invertible elements of $\mathfrak{A}$. Our main result is the following:

Theorem 1. Suppose $\mathfrak{A}$ has real rank zero and let f be a complex-valued function that is holomorphic and bounded on the intersection of the domains $\mathfrak{A}_0$ and $\mathfrak{A}_{\text{inv}}^e$. Then f has a holomorphic extension to $\mathfrak{A}_0$.

The author does not know even in the commutative case whether the removable singularity property of Theorem 1 characterizes C^* -algebras of real rank zero. However, it is shown in [4] that $C(S)$ does not have this property when S contains the homeomorphic image of an interval.

The proof given below of the previous theorem depends on two important facts about the identity component $\mathcal{U}$ of the set of unitary operators in $\mathfrak{A}$. The first is a maximum principle that is a special case of [6, Theorem 8] and [5, Theorem 9] and the second is a density theorem due to Huaxin Lin [8].

Proposition 3. Let $f : \mathfrak{A}_0 \rightarrow \mathbb{C}$ be a holomorphic function having a continuous extension to the closed unit ball $\mathfrak{A}_1$ of $\mathfrak{A}$. If $|f(U)| \leq 1$ for all $U \in \mathcal{U}$ then $|f(A)| \leq 1$ for all $A \in \mathfrak{A}_1$.

Proposition 4. *If $\mathfrak{A}$ has real rank zero then the set of unitaries in $\mathcal{U}$ with finite spectrum is dense in $\mathcal{U}$.*

Proof of Theorem 1. Given any ϵ with $0 < \epsilon < 1/2$, let $r = 1 - \epsilon$. The set $D = B_r(\epsilon I) \cap \mathfrak{A}_{\text{inv}}^e$ is open since $\mathfrak{A}_{\text{inv}}^e$ is open and one can deduce that D is connected from the fact that $B_r(\epsilon I)$ contains a neighborhood of 0. By Proposition 1, it suffices to show that there exists a function f_ϵ that is holomorphic in the ball $B_r(\epsilon I)$ and satisfies $f_\epsilon(A) = f(A)$ for all $A \in D$. Since the function f is holomorphic in a ball with center at $x_0 = \epsilon I$, it follows from Proposition 2 that

$$f(A) = \sum_{n=0}^{\infty} P_n(A - \epsilon I) \quad (4)$$

for all A in this ball. Thus by the converse part of Proposition 2, it suffices to show that

$$\|P_n\| \leq \frac{M}{r^n}, \quad n = 0, 1, \dots, \quad (5)$$

where M satisfies $|f| \leq M$ on $\mathfrak{A}_0 \cap \mathfrak{A}_{\text{inv}}^e$, since then the function

$$f_\epsilon(A) = \sum_{n=0}^{\infty} P_n(A - \epsilon I)$$

is holomorphic on $B_r(\epsilon I)$ and agrees with f on D by Proposition 1.

Let $B \in \mathfrak{A}$ with $\|B\| \leq 1$ and suppose the spectrum $\sigma(B)$ is finite. Define $\phi(\lambda) = f(\epsilon I + \lambda B)$. If $|\lambda| < r$ then $\epsilon I + \lambda B \in \mathfrak{A}_0$, $\epsilon I + \lambda B \in \mathfrak{A}_{\text{inv}}^e$ and $|\phi(\lambda)| \leq M$ for all but finitely many λ . By the classical Riemann removable singularity theorem, the function ϕ has a holomorphic extension to the disc $|\lambda| < r$ with $|\phi| \leq M$. Hence $|\phi^{(n)}(0)| \leq n!M/r^n$ by the Cauchy estimates so

$$|P_n(B)| \leq \frac{M}{r^n} \quad (6)$$

by (3).

By Proposition 4, inequality (6) holds whenever B is in the identity component of the set of unitary elements of $\mathfrak{A}$ and hence for all $B \in \mathfrak{A}$ with $\|B\| \leq 1$ by Proposition 3. This establishes (5) and completes the proof.

The proof of Theorem 1 given in [4] does not require Proposition 4 but the argument is less straightforward. See [4] for further results, examples and references.

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