EVOLUTION EQUATIONS AND GEOMETRIC FUNCTION THEORY IN J^*-ALGEBRAS[1]

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Dedicated to Professor Ky Fan on the occasion of his 85th birthday

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Abstract

In a series of papers Ky Fan developed a geometric theory of holomorphic functions of proper contractions on Hilbert spaces in the sense of the functional calculus. His results are a powerful tool in the study of the discrete-time semigroups of l-analytic functions defined by iterating such a function on the open unit ball of the space of bounded linear operators on a Hilbert space. In this paper we examine the asymptotic behavior of continuous semigroups of l-analytic functions. We establish infinitesimal versions of Ky Fan's results as well as of the classical Julia–Carathéodory and Wolff Theorems by developing the generation theory of continuous one-parameter semigroups of l-analytic functions. We then introduce a general approach to the study of geometric properties of univalent functions in Banach spaces by using the linear one-parameter semigroups defined on the space of holomorphic mappings. Applying our results on the asymptotic behavior of semigroups of l-analytic functions with no stationary point, we describe l-analytic functions which are star-like with respect to a boundary point. All our considerations are carried out in the framework of J^*-algebras with identity, which include, for example, C^*-algebras and certain Cartan factors.

0 Introduction

In a series of papers Ky Fan [14]–[17] developed a geometric theory of holomorphic functions of proper contractions on Hilbert spaces in the sense of the functional calculus. A special class of holomorphic mappings (which we will call l-analytic functions), defined by the Riesz–Dunford integral on the space $L(\mathcal{H})$ of bounded linear operators on a Hilbert space $\mathcal{H}$, is of great interest in the functional calculus of operator theory (see, for example, [10], [14]–[17], [3]). Generalizing the von Neumann Theorem [10], Fan [14]–[17] extended to l-analytic functions the classical Schwarz Lemma, Julia's Lemma and Wolff's Theorem (as a boundary version of the Schwarz Lemma). Ando and Fan [3] also proved several general operator inequalities in the spirit of Pick and Julia which yield the above-mentioned results. These results are a powerful tool in the functional calculus as well as in the study the discrete-time semigroups of l-analytic functions defined by iterating such a function on the open unit ball of $L(\mathcal{H})$.

We begin with a brief description of these results.

For a complex function f which is analytic on the open unit disk $\Delta = \{z \in \mathbb{C} : |z| < 1\}$, $f(A)$ will denote the operator on $\mathcal{H}$ defined by the usual Riesz–Dunford integral ([10, p. 568]):

$$f(A) = \frac{1}{2\pi i} \int_{\Gamma} f(z)(zI - A)^{-1} dz, \tag{0.0.1}$$

where Γ is a positively oriented simple closed rectifiable contour lying in Δ and containing the spectrum of A in its interior. (This definition is equivalent to the definition using the power series expansion of f about the origin.)

We now quote Ky Fan's version of **Wolff's Theorem** for operators.

Theorem A [16] *Let the complex function f be analytic on the open unit disk Δ such that $f(\Delta) \subset \Delta$ and $f(z) \neq z$ for every $z \in \Delta$. Let $f^{[n]}$ denote the n-th iterate of f. Then there exist a complex number w with $|w| = 1$ and non-negative numbers $c(w, A)$ and $r(w, A)$ such that*

$$\left\| f^{[n]}(A) - c(w, A)wI \right\| \leq r(w, A) \tag{0.0.2}$$

holds for $n = 1, 2, \ldots$ and any operator A on $\mathcal{H}$ with $\|A\| < 1$. Furthermore, the relations

$$\|A - c(w, A)wI\| = r(w, A) \tag{0.0.3}$$

and

$$c(w, A) + r(w, A) = 1 \tag{0.0.4}$$

hold for any complex number w with $|w| = 1$ and any operator A on $\mathcal{H}$ with $\|A\| < 1$.

In the case when the Hilbert space $\mathcal{H}$ is one-dimensional (i.e., when $\mathcal{H}$ is the complex plane), the above theorem reduces to Wolff's theorem, which has the following geometric interpretation:

If D_z denotes the closed disk with center $c(w, z)w$ and radius $r(w, z)$, then D_z is the closed disk containing z on its boundary and internally tangent to the unit circle at w. Inequality (0.0.2) asserts that $f^{[n]}(z) \in D_z$.

Another elegant result concerning iteration theory is **Julia's Lemma for Operators**.

Theorem B ([14]–[15]) *Let f be a complex function analytic on the open unit disk $\Delta = \{z \in \mathbb{C} : |z| < 1\}$ with $|f(z)| < 1$ for $z \in \Delta$. Let $\{z_n\} \subset \Delta$ be such that*

$$\lim_{n\to\infty} z_n = 1, \qquad \lim_{n\to\infty} f(z_n) = 1 \tag{0.0.5}$$

and

$$\lim_{n\to\infty} \frac{1-|f(z_n)|}{1-|z_n|} = \alpha, \tag{0.0.6}$$

where α is finite. Let A be an operator on a Hilbert space $\mathcal{H}$ with $\|A\| < 1$. Then:

(a) We have

$$\begin{aligned} &\left\|\{I-f(A)\}\{I-f(A)^*f(A)\}^{-1}\{I-f(A)^*\}\right\| \\ &\qquad \le \alpha \left\|(I-A)(I-A^*A)^{-1}(I-A^*)\right\|. \end{aligned} \tag{0.0.7}$$

(b) If β is a positive number such that

$$(I-A^*)(I-A) < \beta(I-A^*A), \tag{0.0.8}$$

then

$$[I-f(A)^*]\,[I-f(A)] < \alpha\beta\,[I-f(A)^*f(A)]. \tag{0.0.9}$$

(c) If β is a positive number such that

$$\left\|A - \frac{1}{1+\beta}I\right\| < \frac{\beta}{1+\beta}, \tag{0.0.10}$$

then

$$\left\|f(A) - \frac{1}{1+\alpha\beta}I\right\| < \frac{\alpha\beta}{1+\alpha\beta}. \tag{0.0.11}$$

As a matter of fact, inequality (0.0.8) is equivalent to (0.0.10) and inequality (0.0.9) is equivalent to (0.0.11). Observe also that by the Julia–Carathéodory Theorem the number α in (0.0.6) is exactly the so-called angular derivative

$$\angle f'(1) = \lim_{z\to 1} \frac{1-f(z)}{1-z} \tag{0.0.12}$$

of f at the boundary point $w = 1$, where z tends to 1 in nontangential approach regions in Δ (see Section 3 below).

In addition, if $\alpha \leq 1$ and f is not the identity mapping, then the number w in Wolff's Theorem must be 1. This is the content of the so-called Julia–Wolff–Carathéodory Theorem (see, for example, [44, 45]).

All these results can be considered a boundary version of the **generalized Schwarz Lemma** (obtained by Ky Fan in [14]).

Theorem C *Let A be a proper contraction on a Hilbert space $\mathcal{H}$. Let $f, g, h \in \mathrm{Hol}(\Delta)$ be such that $f = gh$ and $|h(z)| \leq 1$ for $z \in \Delta$. Then*

$$g(A)^* g(A) \geq f(A)^* f(A) \tag{0.0.13}$$

and

$$\|g(A)\| \geq \|f(A)\|. \tag{0.0.14}$$

Strict inequality holds in (0.0.13) if and only if $g(A)^* g(A) > 0$ and h is not a constant function of modulus 1. Equality in (0.0.14) holds if and only if either $g(A) = 0$ or h is a constant function of modulus 1.

Using this operator analog of the Schwarz Lemma, Ky Fan (in the same work [14]) studied different geometric properties of analytic functions of a proper contraction.

In particular, he established the following result:

Theorem D *Let $g \in \mathrm{Hol}(\Delta)$ be univalent on Δ with $g(0) = 0$ and $g'(0) = 1$. If g is star-like (i.e., the image $g(\Delta)$ is a star-like set with respect to the origin), then the set of all $g(A)$, when A runs through all proper contractions on $\mathcal{H}$, is a star-like set of operators. More precisely: for every proper contraction A on $\mathcal{H}$ and any nonnegative number $r < 1$, there is a unique proper contraction B on $\mathcal{H}$ such that $g(B) = rg(A)$. Furthermore, $A^*A \geq B^*B$ with strict inequality in case $A^*A > 0$; and $\|A\| > \|B\|$ unless $A = 0$.*

The following problem is one of the classical issues in Analysis: Given a one-parameter semigroup defined on the half-axis, study its asymptotic behavior, and in particular, find an optimal rate of convergence to its stationary point (i.e., common fixed point) if it exists. For semigroups of holomorphic mappings this problem has been considered by many mathematicians for more than one hundred years in connection with classical geometric function theory (see, for example, [20]), stochastic branching processes [29, 43], the theory of composition operators on Hardy spaces [5, 7], optimization and control theory [31], and the theory of linear operators in indefinite metric spaces (Krein and Pontryagin spaces) [51]–[53].

In the one-dimensional case, the asymptotic behavior of a discrete-time semigroup defined by iterating a holomorphic self-mapping F of the open unit disk Δ is described by the classical Denjoy–Wolff Theorem: *If f is not the identity and is not an elliptic automorphism of Δ, then its iterates $f^{[n]}$ converge to a constant $w \in \overline{\Delta}$.* (This result also led to many investigations in higher dimensions.)

Note that even in this simplest situation, for the case where f has an interior null point in Δ, the optimal rate of convergence seems to be unknown.

At the same time, in the case where f has no null point in Δ, the best uniform rate of convergence of $f^{[n]}$ to a boundary point of Δ, in terms of the angular derivative of f at this point, can be given by Wolff's Theorem and the Julia–Carathéodory Theorem (when $\alpha < 1$).

In turn, Ky Fan's Theorems A and B give a complete description of the asymptotic behavior of the iterates of a fixed point free l-analytic self-mapping of the open unit ball in $L(\mathcal{H})$.

To discuss quantitative aspects of the asymptotic behavior of a continuous semigroup, one needs some basic data analogous to the given holomorphic mapping f in the case of a discrete semigroup $f^{[n]}$. In this case a fixed point of f is a common fixed point for $f^{[n]}$. This, however, is no longer true for a continuous semigroup $\{f_t : \ t \geq 0\}$ (even if it consists of fractional iterates of f).

To overcome this difficulty, one considers, instead of the fixed point set of a one-parameter semigroup, the null point set of its so-called generator, which are, in fact, one and the same. Namely, for a discrete-time one-parameter semigroup defined by iterating a mapping f, the generator h is just the mapping $h := I - f$, where I is the identity mapping on the underlying space. For a continuous semigroup $\{f_t : \ t \geq 0\}$ satisfying the condition $f_0 = I$, the (infinitesimal) generator h is defined as the limit

$$h := \lim_{t \to 0^+} \frac{1}{t} (I - f_t)$$

if it exists.

In this paper we will first establish infinitesimal versions of Ky Fan's results as well as of the classical Julia–Carathéodory and Wolff Theorems by developing the generation theory of continuous one-parameter semigroups of l-analytic functions.

More precisely, we will prove, inter alia, the following three assertions.

1. Let $\mathbf{D}$ *be the open unit ball of* $L(\mathcal{H})$*, and let* $\{\mathcal{S}(t) : \ t \geq 0\}$ *be a one-parameter continuous semigroup of l-analytic functions on* $\mathbf{D}$*. Then* $\mathcal{S}(t)$ *is right-differentiable at* 0 *and its derivative is a holomorphically anti-dissipative (accretive) l-analytic function on* $\mathbf{D}$*. Conversely, if* f *is a holomorphically anti-dissipative (accretive) l-analytic function on* $\mathbf{D}$*, then it generates a one-parameter semigroup via the Cauchy problem*

$$\begin{cases} \dfrac{d\mathcal{S}(t)}{dt} + f(\mathcal{S}(t)) = 0 \\ \\ \mathcal{S}(0) = I. \end{cases}$$

2. Let f *be an l-analytic anti-dissipative function on* $\mathbf{D}$*. Then* f *has no null points in* $\mathbf{D}$ *if and only if the following condition holds:*

For some $E \in \partial\mathbf{D}$ *such that* $E^*E = I$ *and* $aE > 0$ *for some* $a \in \mathbb{C}$*, there exists the limit*

$$\lim_{r \to 1^-} \frac{E^* f(rE)}{r-1} = B$$

with $\operatorname{Re} B \geq 0$*.*

Observe also, that if F is an l-analytic function on $\mathbf{D}$ with $F(\mathbf{D}) \subset \mathbf{D}$, then the function $f = I - F$ is an l-analytic anti-dissipative function on $\mathbf{D}$ (see, for example, [36, 37]). Thus the above assertion implies the following version of the Julia–Wolff–Carathéodory Theorem:

If the l-analytic function $F : \mathbf{D} \mapsto \mathbf{D}$ *is not the identity mapping, then* F *has no fixed point in* $\mathbf{D}$ *if and only if for some* $\tau \in \partial\Delta$ *the radial limit*

$$\lim_{r \to 1^-} \frac{F(r\tau I) - \tau I}{r-1}$$

exists and is equal to αI *with* $0 < \alpha \leq 1$*.*

Note that an interior common fixed point of a one-parameter semigroup is a null point of its generator. Therefore assertion 2 enables us to obtain information not only on fixed points of a single mapping, but also on the whole semigroup, as well as to study its asymptotic behavior.

3. Let $\{\mathcal{S}(t):\ t \geq 0\}$ be a semigroup of l-analytic functions on $\mathbf{D}$ which has no stationary point in $\mathbf{D}$. Then there is a unimodular point τ such that

$$\lim_{t\to\infty} \mathcal{S}(t) = \tau I.$$

In addition, if $f = -\dfrac{d\mathcal{S}}{dt}(0^+)$, then for $E = \tau I$ the limit in assertion 2 exists and for some positive number γ,

$$\begin{aligned}(\tau I - \mathcal{S}(t)(A)) \left[(I - \mathcal{S}(t)(A))^* \mathcal{S}(t)(A)\right]^{-1} (\bar{\tau} I - (\mathcal{S}(t)(A))^*) \\ \leq e^{-\gamma t} (\tau I - A) (I - A^*A)^{-1} (\bar{\tau} I - A^*) .\end{aligned}$$

Moreover, the maximal γ for which this inequality holds is exactly β.

It turns out that information on the asymptotic behavior of one-parameter semigroups of holomorphic mappings can be used to study geometric characteristics of biholomorphic functions in Banach spaces, and, in particular, star-like, spiral-like, convex and close-to-convex functions (see, for example, the review of Goodman [22] for the one-dimensional case and the book by Gong [21] for the finite-dimensional case). We also refer to Suffridge [47]–[49], Gurganus [23] and Fan [15] for infinite-dimensional approaches. Heath and Suffridge [30] considered geometric properties of analytic functions introduced by Lorch (L-analytic functions) on Banach algebras with identity.

As we mentioned above, Ky Fan [14] (see also Theorem D) has extended one-dimensional results for star-like functions to l-analytic functions on the open unit ball of $L(\mathcal{H})$, the space of bounded linear operators, by using the generalized Schwarz Lemma (Theorem C) and the so-called principle of subordination. Observe that although the classes of star-like and spiral-like functions were studied very extensively, little was known about functions that are star-like or spiral-like with respect to a boundary point. In fact, it was not until 1981 that Robertson [40] conjectured a condition which might characterize the star-likeness with respect to a boundary point. His conjecture was proved by Lyzzaik [34] in 1984 and completed by Silverman and Silvia [46] in 1990. However, the approaches used in their work have a crucially one-dimensional character (because of the Riemann mapping theorem and Carathéodory's theorem on kernel convergence). For the one-dimensional case, another (dynamic) approach was pointed out in [45] (see also [13]), which is also applicable to functions which are spiral-like with respect to a boundary point. In the present paper we introduce a new general approach to the study of geometric properties of univalent functions in Banach spaces by using the linear one-parameter semigroups defined on the space of holomorphic mappings. In addition, by applying the above-mentioned results on the asymptotic behavior of semigroups of l-analytic functions with no stationary point, we will also describe l-analytic functions which are star-like with respect to a boundary point.

Finally, we mention that all our considerations will be carried out in the framework of the so-called J^*-algebras with identity, which include, for example, C^*-algebras and some Cartan factors.

The paper is organized as follows. Section 1 is devoted to preliminaries involving vector fields of holomorphic mappings, semigroups, J^*-algebras, and l-analytic functions. Our main results are presented in Section 2. We begin with several results on l-analytic generators and semigroups (Theorems 2.1.1, 2.1.8 and 2.1.12) and continue with a result on a multiplication transformation in the spirit of Hummel (Theorem 2.2.2). We then investigate stationary points and the asymptotic behavior of semigroups of l-analytic functions (Theorems 2.3.1, 2.3.3, 2.4.1 and 2.4.4) and establish an infinitesimal version of the Julia-Wolff-Carathéodory Theorem (Theorem 2.4.3). In the third and last section we first present our general dynamic approach to star-like mappings in Banach spaces (Theorems 3.1.1 and 3.1.2) and then apply it to the study of star-like l-analytic functions on J^*-algebras (Theorems 3.2.1, 3.2.2 and 3.2.4).

1 Preliminaries

1.1 Vector fields of holomorphic mappings and semigroups

Throughout this paper, X denotes an arbitrary complex Banach space, I denotes the identity mapping on X, and Δ denotes the open unit disk of the complex plane $\mathbb{C}$.

Recall that a complex Banach space valued function h defined on a domain (open connected subset) $\mathbf{D}$ of X is said to be holomorphic in $\mathbf{D}$ if for each $x \in \mathbf{D}$, the Fréchet derivative of h at x (denoted by $Dh(x)$ or $h'(x)$) exists as a bounded complex-linear mapping of X into the Banach space containing the values of h.

If $\mathbf{D}$ and Ω are domains in complex Banach spaces X and Y, respectively, then we denote the set of holomorphic mappings from $\mathbf{D}$ into Ω by $\mathrm{Hol}(\mathbf{D},\Omega)$.

For $f \in \mathrm{Hol}(\mathbf{D},\Omega)$ and a ball K strictly inside $\mathbf{D}$ (we write $K \Subset \mathbf{D}$), we set

$$\|f(x)\|_K = \sup_{x\in K} \|f(x)\|. \tag{1.1.1}$$

We will mostly consider the subspace $\widetilde{\mathrm{Hol}}(\mathbf{D},\Omega)$ of $\mathrm{Hol}(\mathbf{D},\Omega)$ consisting of all those holomorphic mappings which are bounded on each ball strictly inside $\mathbf{D}$. The system of semi-norms (1.1.1) induces on $\widetilde{\mathrm{Hol}}(\mathbf{D},\Omega)$ the topology of local uniform convergence over $\mathbf{D}$ (or briefly, T-convergence).

Thus if a net $\{f_j\}_{j\in\mathcal{A}} \subset \widetilde{\mathrm{Hol}}(\mathbf{D},Y)$ T-converges to $f \in \widetilde{\mathrm{Hol}}(\mathbf{D},Y)$, we write

$$f = T\text{-}\lim_{j\in\mathcal{A}} f_j$$

(For more details see [33] and [18]).

For our purpose we need the following concept.

Definition 1.1.1 *A holomorphic vector field*

$$T_g = g(x)\frac{\partial}{\partial x} \tag{1.1.2}$$

on a domain $\mathbf{D}$ *is determined by a holomorphic mapping* $g \in \mathrm{Hol}(\mathbf{D},X)$ *and can be regarded as a linear operator mapping* $\mathrm{Hol}(\mathbf{D},X)$ *into itself, where* $T_g f \in \mathrm{Hol}(\mathbf{D},X)$ *is defined by*

$$\left(T_g f\right)(x) = Df(x)g(x), \quad x \in \mathbf{D}. \tag{1.1.3}$$

The set of all holomorphic vector fields on $\mathbf{D}$ is a Lie algebra under the commutator brackets

$$[T_g, T_h] = \left[g(x)\frac{\partial}{\partial x}, h(x)\frac{\partial}{\partial x}\right] := \left(Dg(x)h(x) - Dh(x)g(x)\right)\frac{\partial}{\partial x} \tag{1.1.4}$$

(see, for example, [4, 33, 9, 50, 6]).

It follows by the Cauchy inequalities that $\widetilde{\mathrm{Hol}}(\mathbf{D},X)$ is invariant under T_g whenever $g \in \widetilde{\mathrm{Hol}}(\mathbf{D},X)$.

Furthermore, each vector field (1.1.2) is locally integrable in the following sense: For each $x \in \mathbf{D}$ there exist a neighborhood Ω of x and $\delta > 0$ such that the Cauchy problem

$$\begin{cases} \dfrac{\partial u(t,x)}{\partial t} + g(u(t,x)) = 0 \\ \\ u(0,x) = x \end{cases} \tag{1.1.5}$$

has a unique solution $\{u(t,x)\} \subset \mathbf{D}$ defined on the set $\{|t| < \delta\} \times \Omega \subset \mathbb{R} \times \mathbf{D}$.

Definition 1.1.2 *The holomorphic vector field T_g defined by (1.1.2) and (1.1.3) is said to be (right) semi-complete (respectively, complete) on $\mathbf{D}$ if the solution of the Cauchy problem (1.1.5) is well defined on all of $\mathbb{R}^+ \times \mathbf{D}$ (respectively, $\mathbb{R} \times \mathbf{D}$), where $\mathbb{R}^+ = [0, \infty)$ (respectively, $\mathbb{R} = (-\infty, \infty)$).*

Now we relate these concepts to semigroup theory.

Definition 1.1.3 *A family $\{\mathcal{S}(t)\} \subset \mathrm{Hol}(\mathbf{D}, \mathbf{D})$, where either $t \in \mathbb{R}^+$ or $t \in \mathbb{N}$ $(= \{0, 1, 2, \ldots\})$ of holomorphic self-mappings of $\mathbf{D}$ is called a (one-parameter) semigroup if*

$$\mathcal{S}(s+t) = \mathcal{S}(s) \circ \mathcal{S}(t), \quad s, t \in \mathbb{R}^+ \ (s, t \in \mathbb{N}), \tag{1.1.6}$$

and

$$\mathcal{S}(0) = I_{\mathbf{D}},$$

where $I_{\mathbf{D}}$ is the restriction of the identity operator I to $\mathbf{D}$.

A semigroup $\{\mathcal{S}(t) : \ t \in \mathbb{R}^+\}$ is said to be (strongly) continuous if the vector-valued function $\mathcal{S}(t)x : \mathbb{R}^+ \mapsto \mathbf{D}$ is continuous in t for each $x \in \mathbf{D}$.

It is known that $\{\mathcal{S}(t) : \ t \in \mathbb{R}^+\}$ is strongly continuous if and only if

$$\lim_{t \to 0^+} \mathcal{S}(t)(x) = x \tag{1.1.7}$$

for each $x \in \mathbf{D}$ (see, for example, [36]).

When $t \in \mathbb{N}$ we say that the semigroup is discrete. In other words, a discrete semigroup $\{\mathcal{S}(t) : \ t \in \mathbb{N}\}$ is the family of iterates of the self-mapping $\mathcal{S}(1) \in \mathrm{Hol}(\mathbf{D}, \mathbf{D})$.

Definition 1.1.4 *Let $\{\mathcal{S}(t):\ t\in\mathbb{R}^+\}$ be a continuous semigroup defined on $\mathbf{D}$. If the strong limit*

$$g(x)=\lim_{t\to 0^+}\frac{1}{t}\left(x-\mathcal{S}(t)(x)\right) \tag{1.1.8}$$

exists for each $x\in\mathbf{D}$, then g will be called the (infinitesimal) generator of the semigroup $\{\mathcal{S}(t)\}$.

In this case we will say that $\{\mathcal{S}(t):\ t\in\mathbb{R}^+\}$ is a differentiable semigroup.

In fact, it can be shown (see, for example, [37]) that if $\mathbf{D}$ is hyperbolic (for example, bounded), then a continuous semigroup has a generator if and only if the convergence in (1.1.7) is locally uniform on each ball strictly inside $\mathbf{D}$, i.e.,

$$T\text{-}\lim_{t\to 0^+}\mathcal{S}(t)=I_{\mathbf{D}}. \tag{1.1.9}$$

In other words, a continuous semigroup is differentiable if and only if it is locally uniformly continuous (briefly, T-continuous). Moreover, the mapping defined by

$$u(t,x)=\mathcal{S}(t)(x) \tag{1.1.10}$$

is the solution of the Cauchy problem (1.1.5) (see, for example, [36]–[39]).

If a semigroup $\{\mathcal{S}(t):\ t\in\mathbb{R}^+\}$ has a continuous extension to all of $\mathbb{R}$, then $\{\mathcal{S}(t):\ t\in\mathbb{R}\}$ is actually a (one-parameter) group of automorphisms of $\mathbf{D}$.

The converse is also true: If an element $\mathcal{S}(t_0)$, $t_0>0$, of a semigroup $\{\mathcal{S}(t):\ t\in\mathbb{R}^+\}$ is an automorphism of $\mathbf{D}$, then so is each $\mathcal{S}(t)$ and this semigroup can be continuously extended to a (one-parameter) group. Thus, a holomorphic vector field T_g, defined by (1.1.2), is semi-complete (complete) on $\mathbf{D}$ if and only if g is the generator of a one-parameter semigroup (respectively, group) of holomorphic self-mappings of $\mathbf{D}$. It can also be shown (see [37]) that g actually belongs to $\widetilde{\mathrm{Hol}}(\mathbf{D},X)$, i.e., it is holomorphic on $\mathbf{D}$ and bounded on each ball strictly inside $\mathbf{D}$. *The subset of $\widetilde{\mathrm{Hol}}(\mathbf{D},X)$ consisting of all holomorphic mappings which induce (right) semi-complete vector fields will be denoted by $\mathcal{G}_+(\mathbf{D})$. ($\mathcal{G}_-(\mathbf{D})$ can be defined similarly by using the limit $\lim\limits_{t\to 0^-}\dfrac{1}{t}(x-\mathcal{S}(t)(x))$). The set $\mathcal{G}_\pm(\mathbf{D})=\mathcal{G}_+(\mathbf{D})\bigcap\mathcal{G}_-(\mathbf{D})$ consists of all holomorphic mappings which induce through (1.1.2) the set of complete vector fields. Usually this set is denoted by* $\mathrm{aut}(\mathbf{D})$.

On the other hand, if $\mathbf{D}$ is bounded, then a semigroup (group) $\{\mathcal{S}(t)\}$, $t \in \mathbb{R}^+$, (respectively, $t \in \mathbb{R}$), induces the linear semigroup (group) $\{\mathcal{L}(t)\}$ of linear mappings $\mathcal{L}(t) : \widetilde{\mathrm{Hol}}(\mathbf{D}, X) \mapsto \widetilde{\mathrm{Hol}}(\mathbf{D}, X)$ defined by

$$(\mathcal{L}(t)f)(x) := f(\mathcal{S}(t)(x)), \ t \in \mathbb{R}^+ \ (t \in \mathbb{R}), \ x \in \mathbf{D}. \tag{1.1.11}$$

This semigroup is called the semigroup of composition operators on $\widetilde{\mathrm{Hol}}(\mathbf{D}, X)$. If $\{\mathcal{S}(t)\}$, $t \in \mathbb{R}^+$ $(t \in \mathbb{R})$, is T-continuous, (that is, differentiable), then $\{\mathcal{L}(t)\}$, $t \in \mathbb{R}^+$ $(t \in \mathbb{R})$, is also differentiable and

$$\begin{cases} \dfrac{\partial \mathcal{L}(t)f}{\partial t} + T_g(\mathcal{L}(t)f) = 0 \\ \\ \mathcal{L}(0)f = f \end{cases} \tag{1.1.12}$$

for all $f \in \widetilde{\mathrm{Hol}}(\mathbf{D}, X)$, where $g = -\left.\dfrac{d\mathcal{S}(t)}{dt}\right|_{t=0}$.

In other words, the holomorphic vector field T_g, defined by (1.1.2) and (1.1.3), and considered as a linear operator on $\widetilde{\mathrm{Hol}}(\mathbf{D}, X)$, is the infinitesimal generator of the semigroup $\{\mathcal{L}(t)\}$. We will call it the Lie generator. Thus a holomorphic vector field T_g is semi-complete (respectively, complete) if and only if it is the Lie generator of a linear semigroup (respectively, group) of composition operators on $\widetilde{\mathrm{Hol}}(\mathbf{D}, X)$. This follows from the observation that

$$\mathcal{L}(t)I_{\mathbf{D}} = \mathcal{S}(t) \tag{1.1.13}$$

and

$$T_g I_{\mathbf{D}} = g. \tag{1.1.14}$$

Moreover, using the exponential formula representation for the linear semigroup:

$$\mathcal{L}(t)f = \sum_{k=0}^{\infty} \frac{(-1)^k t^k}{k!} \cdot T_g^k f = \exp[-tT_g]f, \tag{1.1.15}$$

(see, for example, [54, 36]), (1.1.13) and (1.1.14) we have

$$\mathcal{S}(t) = \sum_{k=0}^{\infty} \frac{(-1)^k t^k}{k!} T_g^k I_{\mathbf{D}} = \exp[-tT_g]I_{\mathbf{D}}. \tag{1.1.16}$$

In other words, a T-continuous semigroup of holomorphic self-mappings on a bounded domain can be represented in exponential form by the holomorphic vector field corresponding to its generator.

Another exponential representation on a hyperbolic convex domain can be given by using the so-called nonlinear resolvent of g.

More precisely, let $\mathbf{D}$ be a bounded (or more generally, hyperbolic) convex domain. Then it was shown in [36]–[38] that $g \in \mathrm{Hol}(\mathbf{D}, X)$ belongs to $\mathcal{G}_+(\mathbf{D})$ if and only if for each $r \geq 0$ the mapping $(I + rg)^{-1} = \mathcal{J}_r$ is a well-defined holomorphic self-mapping of $\mathbf{D}$.

Furthermore, if $\{G(r) : \ r \geq 0\}$ is any continuous family of holomorphic self-mappings of $\mathbf{D}$ such that the limit

$$g(x) = \lim_{r \to 0^+} \frac{1}{r} \left(x - G(r)x \right), \quad x \in \mathbf{D},$$

exists, then $g \in \mathcal{G}_+(\mathbf{D})$ and the semigroup generated by g can be represented by the product formula

$$\mathcal{S}(t) = \lim_{n \to \infty} G^n \left(\frac{t}{n} \right). \tag{1.1.17}$$

In particular,

$$\mathcal{S}(t) = \lim_{n \to \infty} \left(I + \frac{t}{n} g \right)^{-n} \tag{1.1.18}$$

(exponential formula), where the limits in (1.1.17) and (1.1.18) are taken with respect to the T-topology on $\mathrm{Hol}(\mathbf{D}, X)$.

A characterization of semi-complete vector fields can be given also in terms of the numerical range of holomorphic functions [24]. For our further needs it is sufficient to consider the case of the open unit ball $\mathbf{D}$ of X. For $x \in \partial\mathbf{D}$, let $J(x)$ be the set of all continuous linear functionals which are tangent to $\mathbf{D}$ at x, i.e.,

$$J(x) = \{ l \in X^* : \ l(x) = 1, \ \mathrm{Re}\, l(y) \leq 1, \quad y \in \mathbf{D} \}.$$

If $h \in \mathrm{Hol}(\mathbf{D}, X)$ has a continuous extension to $\overline{\mathbf{D}}$, the closure of $\mathbf{D}$, then we define the set

$$V(h) = \{ l(h(x)) : \ l \in J(x), \ x \in \partial\mathbf{D} \}$$

to be the (total) numerical range of h.

Definition 1.1.5 *We say that a mapping $h \in \mathrm{Hol}(\mathbf{D}, X)$ is holomorphically dissipative if*

$$L(h) := \lim_{s \to 1^-} [\sup \mathrm{Re}\, V(h_s)] \leq 0,$$

where $h_s(x) = h(sx)$.

A mapping h is said to be holomorphically anti-dissipative (or accretive) if $-h$ is dissipative.

It follows by results in [28] and [37] that $\mathcal{G}_-(\mathbf{D})$ (respectively, $\mathcal{G}_+(\mathbf{D})$) consists exactly of the holomorphically dissipative (respectively, accretive) mappings.

1.2 Some $*$-algebras

Let $\mathcal{A}$ be a Banach algebra over the complex field $\mathbb{C}$, i.e., $\mathcal{A}$ is a complex Banach space with the multiplication operation satisfying $\|x \cdot y\| \leq \|x\| \cdot \|y\|$.

Recall that $\mathcal{A}$ is called an algebra with involution if there is an anti-linear mapping $x \mapsto x^*$ of $\mathcal{A}$ into itself such that $(x^*)^* = x$ and $(x \cdot y)^* = y^* x^*$. The element x^* is called the element adjoint to x.

Definition 1.2.1 *A complex Banach algebra $\mathcal{A}$ with involution is called a C^*-algebra if $\|x\|^2 = \|x \cdot x^*\|$ for all $x \in \mathcal{A}$.*

Standard examples of C^*-algebras are $\mathcal{A} = \mathbb{C}$, the complex plane with the conjugation $z \mapsto \bar{z}$, which is obviously an involution; $\mathcal{A} = C_0(S)$, the algebra of all complex continuous functions vanishing at infinity on a locally compact Hausdorff space S; $\mathcal{A} = L(\mathcal{H})$, the algebra of bounded linear operators on a complex Hilbert space $\mathcal{H}$ with the inner product $(\cdot, \cdot)$. For an element $A \in L(\mathcal{H})$, the adjoint operator A^* is defined by the equality $(Ax, y) = (x, A^* y)$.

At the same time, by the Gelfand–Naimark theorem (see, for example, [41]) each C^*-algebra can be realized as a closed subalgebra of $L(\mathcal{H})$ with a suitable $\mathcal{H}$. Actually, for our purpose we can restrict ourselves to these subalgebras.

An element $x \in \mathcal{A}$ is said to be Hermitian (self-adjoint) if $x^* = x$. The set of all Hermitian elements will be denoted by $\mathcal{A}_H$. In particular, the elements

$$\operatorname{Re} x := \frac{1}{2}(x + x^*) \quad \text{and} \quad \operatorname{Im} x := \frac{1}{2i}(x - x^*) \tag{1.2.1}$$

are Hermitian.

Definition 1.2.2 *An element $x \in \mathcal{A}$ is said to be positive if $x \in \mathcal{A}_H$ and there is $y \in \mathcal{A}_H$, such that $y^2 = x$.*

It is known (see, for example, [8]) that $x \in \mathcal{A}_H$ is positive if and only if there is $y \in \mathcal{A}$ such that

$$x = y^* y. \tag{1.2.2}$$

If $A \in L(\mathcal{H})$, then (1.2.2) implies in turn that $A \in \mathcal{A}$ is positive if and only if

$$(A\zeta, \zeta) \geq 0 \quad \text{for all} \quad \zeta \in \mathcal{H}, \tag{1.2.3}$$

or if and only if the spectrum of A

$$\sigma(A) \subset \mathbb{R}^+ = [0, \infty). \tag{1.2.4}$$

Now we turn to the class of J^*-algebras which were introduced by L. A. Harris [25, 27]. This class consists of Banach spaces of operators mapping one Hilbert space into another.

Definition 1.2.3 *Let $\mathcal{H}$ and $\mathcal{K}$ be complex Hilbert spaces and let $L(\mathcal{H}, \mathcal{K})$ be the space of bounded linear operators from $\mathcal{H}$ into $\mathcal{K}$. A closed subspace $\mathfrak{A}$ of $L(\mathcal{H}, \mathcal{K})$ is called a J^*-algebra if $AA^*A \in \mathfrak{A}$ whenever $A \in \mathfrak{A}$.*

Of course, J^*-algebras are not algebras in the ordinary sense. However, from the point of view of operator theory they may be considered a generalization of C^*-algebras; see [25]. Any open unit ball of a J^*-algebra is a natural generalization of the open unit disk of the complex plane. In particular, any Hilbert space $\mathcal{H}$ may be thought of as a J^*-algebra identified with $L(\mathcal{H}, \mathbb{C})$. Also, any C^*-algebra in $L(\mathcal{H})$ is a J^*-algebra. Other important examples of J^*-algebras are the so-called Cartan factors of type I, II, III, IV, which are the sets $\mathfrak{A}_1$, $\mathfrak{A}_2$, $\mathfrak{A}_3$, $\mathfrak{A}_4$, respectively, where $\mathfrak{A}_1 = L(\mathcal{H}, \mathcal{K})$, $\mathfrak{A}_2 = \{A \in L(\mathcal{H}) : \ A^t = A\}$, $\mathfrak{A}_3 = \{A \in L(\mathcal{H}) : \ A^t = -A\}$ (where $A^t x = \overline{A^* \bar{x}}$ for a given conjugation $x \mapsto \bar{x}$ in $\mathcal{H}$) and $\mathfrak{A}_4$ is any closed complex subspace $\mathfrak{A}$ of $L(\mathcal{H})$ such that both $A^* \in \mathfrak{A}$ and $A^2 = \lambda I$ for some complex number λ whenever $A \in \mathfrak{A}$. (Cartan factors of type IV are variants of the spin factors.)

Thus, the four basic types of the classical Cartan domains and their infinite dimensional analogues are the open unit balls of J^*-algebras, and the same holds for any finite and infinite product of these domains.

A crucial property of J^*-algebras is that they have a kind of Jordan triple product structure and contain certain symmetrically formed products of their elements. In particular, for all elements A, B, C in a J^*-algebra $\mathfrak{A}$,

$$AB^*C + CB^*A \in \mathfrak{A}. \tag{1.2.5}$$

Also, we observe that every J^*-algebra is isometrically J^*-isomorphic (see [25, 27]) to a J^*-algebra in $L(\mathcal{H})$ for a suitable Hilbert space $\mathcal{H}$. Our further considerations will be restricted to this case.

Definition 1.2.4 *Call a J^*-algebra unital if the underlying Hilbert spaces are the same and if it contains the identity operator.*

Note that a closed subspace of $L(\mathcal{H})$ which contains the identity operator is a unital J^*-algebra if and only if it contains the squares and adjoints of each of its elements (see the identities (1) in [25]).

When a unital J^*-algebra contains an operator, it contains any polynomial in that operator.

Finally, we observe that if $\mathbf{D}$ is the open unit ball of a J^*-algebra (which is known to be a bounded symmetric domain) and $g \in \mathrm{Hol}(\mathbf{D}, \mathfrak{A})$, then the holomorphic vector field $T_g = g \dfrac{\partial}{\partial A}$ on $\mathrm{Hol}(\mathbf{D}, \mathfrak{A})$ induced by g is complete if and only if g is a polynomial of degree at most 2 of the form

$$g(A) = B - AC^*A + \Phi(A), \tag{1.2.6}$$

where B is an element of $\mathfrak{A}$ and $\Phi : \mathfrak{A} \mapsto \mathfrak{A}$ is a conservative linear operator on $\mathfrak{A}$, i.e., for each $A \in \mathfrak{A}$ with $\|A\| = 1$ and $l \in \mathfrak{A}^*$ with $\|l\| = l(A) = 1$, we have $\mathrm{Re}\, l(\Phi(A)) = 0$ (see, for example, [50, 4, 9]).

In other words, (1.2.6) gives a representation of the class

$$\mathcal{G}_{\pm}(\mathbf{D}) = \mathcal{G}_{+}(\mathbf{D}) \bigcap \mathcal{G}_{-}(\mathbf{D}).$$

1.3 l-analytic functions on unital J^*-algebras

Let $\mathfrak{A}$ be a unital J^*-algebra and let Ω be a domain in the complex plane $\mathbb{C}$. Consider the set

$$\mathbf{D}_{\Omega} = \{A \in \mathfrak{A} : \ \sigma(A) \subset \Omega\}\,. \tag{1.3.1}$$

Since $\mathfrak{A}$ is a closed subset of $L(\mathcal{H})$, $\mathbf{D}_\Omega$ is an open set in the topology of $\mathfrak{A}$ induced by the sup-norm of $L(\mathcal{H})$.

For a function $f \in \mathrm{Hol}(\Omega, \mathbb{C})$ we define the function $\hat{f}: \mathbf{D}_\Omega \mapsto L(\mathcal{H})$ by using the Riesz–Dunford integral:

$$\hat{f}(A) = \frac{1}{2\pi i} \int\limits_\Gamma f(\lambda)\,(\lambda I - A)^{-1} d\lambda \tag{1.3.2}$$

(see, for example, [10, p. 568]), where $\Gamma \subset \mathbf{D}_\Omega$ is a positively oriented simple closed rectifiable contour such that the interior domain of Γ contains $\sigma(A)$.

Remark. *Since $\mathfrak{A}$ is closed, the equality*

$$A^n = \frac{1}{2\pi i} \int\limits_\Gamma \lambda^n\,(\lambda I - A)^{-1} d\lambda \tag{1.3.3}$$

and (1.2.5) imply that $\hat{f}: \mathbf{D}_\Omega \mapsto \mathfrak{A}$, i.e., the values of $\hat{f}(A)$ for $A \in \mathbf{D}_\Omega \subset \mathfrak{A}$ are also in $\mathfrak{A}$.

It is clear that $\hat{f}$ belongs to $\mathrm{Hol}(\mathbf{D}_\Omega, \mathfrak{A})$.

Definition 1.3.1 *Let $\mathbf{D}$ be a domain in a unital J^*-algebra $\mathfrak{A} \subset L(\mathcal{H})$. A holomorphic mapping $F: \mathbf{D} \mapsto \mathfrak{A}$ is said to be an l-analytic function if*

(i) there is a domain $\Omega \subset \mathbb{C}$ such that $\mathbf{D} \subset \mathbf{D}_\Omega$, where $\mathbf{D}_\Omega$ is defined by (1.3.1);

(ii) there is a holomorphic function $f \in \mathrm{Hol}(\Omega, \mathbb{C})$ such that $F(A) = \hat{f}(A)$ for all $A \in \mathbf{D}_\Omega$, where $\hat{f}$ is defined by (1.3.2).

This function will be called the producing function of F $(= \hat{f})$. The set of all l-analytic functions on $\mathbf{D}$ will be denoted by $\widehat{\mathrm{Hol}}(\mathbf{D}, \mathfrak{A})$.

It is well known (see, for example, [10, p. 568] and [41]) that the "lifting" mapping $f \mapsto \hat{f}$ defined by (1.3.2) is multiplicative, i.e., if f and g belong to $\mathrm{Hol}(\Omega, \mathbb{C})$ and $h = f \cdot g$, then

$$\hat{f}(A)\hat{g}(A) = \hat{h}(A). \tag{1.3.4}$$

Remark. *Although in general $\mathfrak{A}$ is not an algebra in the ordinary sense, i.e., for each pair of elements, $\mathfrak{A}$ does not necessarily contain their product, condition (1.3.4) shows that the product of values of l-analytic functions of the same element is commutative and is also an element of $\mathfrak{A}$.*

Also, if $f \in \mathrm{Hol}(\Omega, \mathbb{C})$, Ω_1 is a domain in $\mathbb{C}$ such that $f(\Omega) \subset \Omega_1$, and $g \in \mathrm{Hol}(\Omega_1, \mathbb{C})$, then the equality $h(\lambda) = g(f(\lambda))$ implies that

$$\hat{h}(A) = \hat{g}(\hat{f}(A)) \tag{1.3.5}$$

for all $A \in \mathbf{D}_\Omega$.

Thus the lifting mapping $f \mapsto \hat{f}$ preserves the composition operation.

Remark. If $f : \Omega \mapsto \mathbb{C}$ is univalent, then it is clear that $f(\Omega)$ is open and that f has a holomorphic inverse g which maps $f(\Omega)$ onto Ω. In this situation, the composition law (1.3.5) shows that *the l-analytic function $\hat{f} : \mathbf{D}_\Omega \mapsto \hat{f}(\mathbf{D}_\Omega)$ is biholomorphic and, in particular, that $\hat{f}(\mathbf{D}_\Omega)$ is open.*

In addition, the lifting mapping is continuous in the following sense: If $\{f_n\} \subset \mathrm{Hol}(\Omega, \mathbb{C})$ converges to f uniformly on compact subsets of Ω, then

$$\hat{f}(A) = \lim_{n \to \infty} \hat{f}_n(A), \quad A \in \mathbf{D}_\Omega \tag{1.3.6}$$

(see [10, p. 571]).

Finally, we mention a simple but very useful property of l-analytic functions, namely

$$\hat{f}(\lambda I) = f(\lambda) I \tag{1.3.7}$$

for all $\lambda \in \Omega$ (note that $\lambda I \in \mathbf{D}_\Omega$).

2 Main results

2.1 l-analytic generators and semigroups

Let now $\mathbf{D}$ be the open unit ball of a unital J^*-algebra $\mathfrak{A}$, and let $\Omega = \Delta$ be the open unit disk of the complex plane $\mathbb{C}$. It is clear that the set $\mathbf{D}_\Delta$ defined by (1.3.1) contains $\mathbf{D}$.

So, we can define l-analytic functions on $\mathbf{D}$ the producing functions of which are holomorphic in Δ. In other words, for each $f \in \mathrm{Hol}(\Delta, \mathbb{C})$ one can consider the lifting mapping $f \mapsto \hat{f} \in \widehat{\mathrm{Hol}}(\mathbf{D}, \mathfrak{A})$ defined by (1.3.2).

We will study the set $\widehat{\mathcal{G}_+}(\mathbf{D}) = \mathcal{G}_+(\mathbf{D}) \cap \widehat{\mathrm{Hol}}(\mathbf{D}, \mathfrak{A})$ (respectively, $\widehat{\mathcal{G}_\pm}(\mathbf{D}) = \mathcal{G}_\pm(\mathbf{D}) \cap \widehat{\mathrm{Hol}}(\mathbf{D}, \mathfrak{A})$) which consists of all those l-analytic functions on $\mathbf{D}$ the associated vector fields of which are (right) semi-complete (respectively, complete) (see Section 1.1).

Theorem 2.1.1 *The lifting mapping $f \mapsto \hat{f}$ is a bijective correspondence between $\mathcal{G}_+(\Delta)$ (respectively, $\mathcal{G}_\pm(\Delta)$) and $\widehat{\mathcal{G}_+}(\mathbf{D})$ (respectively, $\widehat{\mathcal{G}_\pm}(\mathbf{D})$), i.e., an l-analytic function $\hat{f} \in \widehat{\mathrm{Hol}}(\mathbf{D}, \mathfrak{A})$ is a generator of a one-parameter semigroup (group) $\mathcal{S}_{\hat{f}}$ of holomorphic mappings acting on $\mathbf{D}$, if and only if its producing function $f \in \mathrm{Hol}(\Delta, \mathbb{C})$ is a generator of a semigroup (group) $\mathcal{S}_f$ acting on Δ.*

Moreover, the semigroup (group) $\mathcal{S}_{\hat{f}}$ consists of the l-analytic functions produced by the functions of $\mathcal{S}_f$.

Proof. Let $f \in \mathcal{G}_+(\Delta)$ and let $\mathcal{S}_f = \{v(t,\cdot)\}_{t\geq 0}$ be the semigroup of holomorphic self-mappings $v(t,\cdot)$ of Δ generated by f. Using (1.3.2), one can define a family $\{\hat{v}(t,\cdot)\}_{t\geq 0}$ of l-analytic functions on $\mathbf{D}$, i.e.,

$$\hat{v}(t,A) = \frac{1}{2\pi i}\int\limits_{\Gamma} v(t,\lambda)(\lambda I - A)^{-1} d\lambda, \quad t \geq 0,\ A \in \mathbf{D}. \tag{2.1.1}$$

It follows from a result of Ky Fan (see [14, Theorem 1, p. 276]) that for each $t \geq 0$, $\hat{v}(t,A) \in \mathbf{D}$, i.e., $\|\hat{v}(t,A)\| < 1$.

In addition, the property of the composition operation (1.3.5) implies that the family $\{\hat{v}(t,\cdot)\}_{t\geq 0}$ is actually a semigroup. Since the integral in (2.1.1) is a smooth function of $t \geq 0$ (see, for example, [42, Theorem IV.155]), we have

$$\begin{aligned} \left.\frac{\partial \hat{v}(t,A)}{\partial t}\right|_{t=0^+} &= \frac{1}{2\pi i}\int\limits_{\Gamma} \left.\frac{\partial v(t,\lambda)}{\partial t}\right|_{t=0^+} (\lambda I - A)^{-1} d\lambda \\ &= -\frac{1}{2\pi i}\int\limits_{\Gamma} f(\lambda)(\lambda I - A)^{-1} d\lambda = -\hat{f}(A). \end{aligned} \tag{2.1.2}$$

Thus $\hat{f}$ is the generator of the semigroup $\{\hat{v}(t,\cdot)\}_{t\geq 0}$, i.e., $\hat{f} \in \widehat{\mathcal{G}_+}(\mathbf{D})$.

Conversely, let $\hat{f}$ be an l-analytic function on $\mathbf{D}$ such that $\hat{f}$ generates a semigroup $\mathcal{S}_{\hat{f}} = \{u(t,\cdot)\}_{t\geq 0}$ of holomorphic self-mappings of $\mathbf{D}$. All we need to show is that for each $t \geq 0$ the mapping $u(t,\cdot)$ is also an l-analytic function.

To this end, take a continuous linear functional $l \in \mathfrak{A}^*$ such that $\|l\| = l(I) = 1$, and consider the family of functions $\{w(t,\cdot)\}_{t\geq 0}$ defined by

$$w(t,\lambda) = l(u(t,\lambda I)). \tag{2.1.3}$$

By direct calculations and (1.3.6) we get

$$\begin{aligned}\left.\frac{\partial w(t,\lambda)}{\partial t}\right|_{t=0^+} &= \lim_{t\to 0^+}\frac{1}{t}\,[w(t,\lambda)-w(0,\lambda)]\\ &= l\left(\left.\frac{\partial u(t,\lambda I)}{\partial t}\right|_{t=0^+}\right) = -l(\hat{f}(\lambda I))\\ &= -l(f(\lambda)I) = -f(\lambda),\end{aligned}$$

where f is the producing function of $\hat{f}$. Hence it follows by (1.1.17) that the family

$$v(t,\lambda) = \lim_{n\to\infty} w^{[n]}(\frac{t}{n},\lambda) \tag{2.1.4}$$

is the semigroup of holomorphic self-mappings of Δ generated by f. Here $w^{[n]}$ denotes the n-th iterate of w.

Now defining $\hat{v}(t,\cdot)$ by (2.1.1) and using (2.1.2) and (1.3.5), we obtain that $\{\hat{v}(t,\cdot)\}_{t\geq 0}$ is also a semigroup generated by $\hat{f}$. It follows by the uniqueness of the solution of the Cauchy problem (1.1.5) that $\hat{v}(t,\lambda) = u(t,\lambda)$, i.e., $\mathcal{S}_{\hat{f}} = \{\hat{v}(t,\cdot)\}_{t\geq 0}$.

We conclude our proof by replacing the semigroups with groups in the above considerations. □

Remark. *Thus we have proved that $\hat{f} \in \widehat{\mathrm{Hol}}(\mathbf{D},\mathfrak{A})$ is holomorphically accretive on $\mathbf{D}$ (i.e., $\hat{f}$ generates a one-parameter semigroup $\{\widehat{\mathcal{S}}(t) : t \geq 0\} \subset$ $\mathrm{Hol}(\mathbf{D},\mathbf{D})$) if and only if its producing function f is holomorphically accretive on Δ, i.e.,*

$$\lim_{s\to 1^-}\inf_{\lambda\in\partial\Delta} \mathrm{Re}(f(s\lambda)\bar{\lambda}) \geq 0$$

(see Definition 1.1.5).

It follows that $\widehat{\mathcal{G}_+}(\mathbf{D})$ is a closed real subcone of $\mathcal{G}_+(\mathbf{D})$.

Now we turn to the notion of holomorphic vector fields (see Section 1.1).

Since each l-analytic function $\hat{g} \in \widehat{\mathrm{Hol}}(\mathbf{D},\mathfrak{A})$ holomorphically maps $\mathbf{D}$ into $\mathfrak{A}$, it defines the holomorphic vector field

$$T_{\hat{g}} = \hat{g}(x)\,\frac{\partial}{\partial x} \tag{2.1.5}$$

on $\mathrm{Hol}(\mathbf{D},\mathfrak{A})$.

Theorem 2.1.1 asserts that the vector field (2.1.5) is a semi-complete (complete) vector field if and only if so is the vector field

$$T_g = g(\lambda)\,\frac{\partial}{\partial\lambda} \tag{2.1.6}$$

defined on $\mathrm{Hol}(\Delta, \mathbb{C})$ by the producing function $g \in \mathrm{Hol}(\Delta, \mathbb{C})$ of $\hat{g}$.

We are interested in a representation of the holomorphic vector field (2.1.5) restricted to the subspace $\widehat{\mathrm{Hol}}(\mathbf{D}, \mathfrak{A})$ of $\mathrm{Hol}(\mathbf{D}, \mathfrak{A})$.

Corollary 2.1.2 *Let $\hat{g}$ be an l-analytic function on $\mathfrak{A}$ and let $T_{\hat{g}}$ be defined by (2.1.5). Then*

(i) $\widehat{\mathrm{Hol}}(\mathbf{D}, \mathfrak{A})$ is an invariant subspace of $\mathrm{Hol}(\mathbf{D}, \mathfrak{A})$ *for the operator $T_{\hat{g}}$.*

(ii) The restriction $T_{\hat{g}}\Big|_{\widehat{\mathrm{Hol}}(\mathbf{D},\,\mathfrak{A})}$ of $T_{\hat{g}}$ to $\widehat{\mathrm{Hol}}(\mathbf{D}, \mathfrak{A})$ is a commutative operation of multiplication:

$$\left(T_{\hat{g}}\hat{f}\right)(A) = \hat{f}'(A)\hat{g}(A) = \hat{g}(A)\hat{f}'(A) \tag{2.1.7}$$

for each $\hat{f} \in \widehat{\mathrm{Hol}}(\mathbf{D}, \mathfrak{A})$ and $A \in \mathbf{D}$.

(iii) The following formula of representation holds:

$$\left(T_{\hat{g}}\hat{f}\right)(A) = \widehat{T_g f}(A) = \frac{1}{2\pi i}\int\limits_{\Gamma} f'(\lambda)g(\lambda)(\lambda I - A)^{-1}d\lambda, \tag{2.1.8}$$

where $A \in \mathfrak{A}$, $\hat{f} \in \widehat{\mathrm{Hol}}(\mathbf{D}, \mathfrak{A})$, and $f \in \mathrm{Hol}(\Delta, \mathbb{C})$ is the producing function of $\hat{f}$.

Proof. Obviously (iii) implies (i), because (as we have mentioned) the Riesz–Dunford integral acts from $\mathfrak{A}$ into itself and defines an l-analytic function on $\mathbf{D}$.

Also (ii) implies (iii) because of (1.3.4). So, we need to prove formula (2.1.7). Indeed, by definition (see 1.1.3),

$$\begin{aligned}\left(T_{\hat{g}}\hat{f}\right)(A) &= \lim_{t\to 0^+}\frac{1}{t}\left[\hat{f}(A + t\hat{g}(A)) - \hat{f}(A)\right]\\ &= \lim_{t\to 0^+}\frac{1}{t}\left[\frac{1}{2\pi i}\int\limits_{\Gamma} f(\lambda)\left[(\lambda I - (A + t\hat{g}(A)))^{-1} - (\lambda I - A)^{-1}\right]d\lambda\right],\end{aligned}$$

where $\Gamma \subset \Delta$ is a suitable contour such that the interior domain of Γ contains the spectra of A and $A + t\hat{g}(A)$ for small enough t.

Since A and $\hat{g}(A)$ commute (as l-analytic functions of the same element), using the identity

$$\begin{aligned}&(\lambda I - (A + t\hat{g}(A)))^{-1} - (\lambda I - A)^{-1}\\&= t\,(\lambda I - (A + t\hat{g}(A)))^{-1} \cdot \hat{g}(A) \cdot (\lambda I - A)^{-1}\end{aligned}$$

and letting t tend to zero, we get

$$\left(T_{\hat{g}}\hat{f}\right)(A) = \hat{g}(A) \cdot \left[\frac{1}{2\pi i}\int_{\Gamma} f(\lambda)(\lambda I - A)^{-2} d\lambda\right] = \hat{g}(A)\widehat{f'}(A).$$

This concludes the proof. $\square$

Corollary 2.1.3 *If the holomorphic vector field $T_{\hat{g}}$ (2.1.5) is semi-complete (complete), then its restriction to $\widehat{\mathrm{Hol}}(\mathbf{D}, \mathfrak{A})$ is the generator of a semigroup (group) $\mathcal{L}$ of linear operators on $\widehat{\mathrm{Hol}}(\mathbf{D}, \mathfrak{A})$ defined by*

$$\left(\mathcal{L}(t)\hat{f}\right)(A) = \hat{f}(\hat{v}(t, A)) = f(\widehat{v(t, \cdot)})(A),$$

where f is the producing function of an element $\hat{f} \in \widehat{\mathrm{Hol}}(\mathbf{D}, \mathfrak{A})$ and $\{v(t, \cdot)\}_{t \geq 0}$ is the semigroup (group) generated by g, where g is the producing function of $\hat{g}$.

In addition, the function $F(t, \hat{f}) = \mathcal{L}(t)\hat{f} : \mathbb{R}^+ \times \widehat{\mathrm{Hol}}(\mathbf{D}, \mathfrak{A}) \mapsto \widehat{\mathrm{Hol}}(\mathbf{D}, \mathfrak{A})$ (respectively, $\mathbb{R} \times \widehat{\mathrm{Hol}}(\mathbf{D}, \mathfrak{A}) \mapsto \widehat{\mathrm{Hol}}(\mathbf{D}, \mathfrak{A})$) is the solution of the Cauchy problem

$$\begin{cases} \dfrac{\partial F(t, \hat{f})}{\partial t} + \hat{f}'(F(t, f))\hat{g}(F(t, f)) = 0 \\ F(0, \hat{f}) = \hat{f} \in \widehat{\mathrm{Hol}}(\mathbf{D}, \mathfrak{A}). \end{cases} \tag{2.1.9}$$

Now we will characterize the set $\widehat{\mathcal{G}_+}(\mathbf{D})$, the set of all l-analytic generators on $\mathbf{D}$, the open unit ball of the unital J^*-algebra $\mathfrak{A}$. To do this, we will first establish some sufficient conditions for a holomorphic mapping on $\mathbf{D}$ to be holomorphically dissipative (respectively, accretive) on $\mathbf{D}$.

Theorem 2.1.4 *Let* $\mathbf{D}$ *be the open unit ball of* $\mathfrak{A}$ *and let* $h : \mathbf{D} \mapsto \mathfrak{A}$ *be a holomorphic function with a continuous extension to* $\overline{\mathbf{D}}$*. Then the numerical range* $V(h)$ *is contained in the closed convex hull of the set*

$$S(h) = \Big\{ (A^* h(A)x, x) : \ \|A\| = 1, \ \|x\| = 1, \ A \in L(\mathcal{H}), \ x \in \mathcal{H} \Big\}.$$

Corollary 2.1.5 *If* $\sup \operatorname{Re} S(h) \le 0$*, then* h *is dissipative.*

In the proof of Theorem 2.1.4 we are going to use the following two lemmas.

Lemma 2.1.6 *Let* S *and* T *be subsets of the complex plane satisfying*

$$\sup \operatorname{Re} \lambda S \le \sup \operatorname{Re} \lambda T$$

for all $\lambda \in \mathbb{C}$*. Then* S *is contained in the closed convex hull of* T*.*

Lemma 2.1.7 *Let* $A \in L(\mathcal{H})$ *with* $\|A\| = 1$ *and let* $\phi \in L(\mathcal{H})^*$ *with* $\|\phi\| = \phi(A) = 1$*. Given* $W \in L(\mathcal{H})$*, there exist unit vectors* x*,* $y \in \mathcal{H}$ *such that* $\operatorname{Re} \phi(W) \le \operatorname{Re}(Wx, y) + \epsilon$ *and* $|1 - (Ax, y)| < \epsilon$*.*

Lemma 2.1.6 is a consequence of the Hahn–Banach separation theorem for the case of the complex plane (where all complex linear functionals are multiplication by a scalar λ). Lemma 2.1.7 is a consequence of [26, Lemma 1].

Proof of Theorem 2.1.4. Let $A \in \mathfrak{A}$ with $\|A\| = 1$. Suppose $\phi \in L(\mathcal{H})^*$ with $\|\phi\| = \phi(A) = 1$. Applying Lemma 2.1.7 with $W = \lambda h(A)$ and $\lambda \in \mathbb{C}$ we have that for each $\epsilon > 0$ there exist unit vectors x, $y \in \mathcal{H}$ such that

$$\operatorname{Re}\Big\{ \lambda \phi(h(A)) \Big\} \le \operatorname{Re}\Big\{ \lambda (h(A)x, y) \Big\} + \epsilon$$

and

$$|1 - (Ax, y)| < \epsilon.$$

It follows that $\operatorname{Re}[1 - (Ax, y)] < \epsilon$, so

$$\begin{aligned} \|y - Ax\|^2 &= \|y\|^2 - 2 \operatorname{Re}(Ax, y) + \|Ax\|^2 \\ &\le 1 - 2 \operatorname{Re}(Ax, y) + 1 \le 2\epsilon. \end{aligned}$$

Hence,

$$\begin{aligned}\operatorname{Re}\{\lambda\phi(h(A))\} &\leq \operatorname{Re}\{\lambda(h(A)x, Ax)\} + \operatorname{Re}\{\lambda(h(A)x, y - Ax)\} + \epsilon \\ &\leq \operatorname{Re}\{\lambda(A^*h(A)x, x)\} + |\lambda| \cdot \|h(A)\|\sqrt{2\epsilon} + \epsilon.\end{aligned}$$

Since $\epsilon > 0$ was arbitrary, it follows that $\operatorname{Re}\{\lambda\phi(h(A))\} \leq \operatorname{Re}\{\lambda S(h)\}$. Thus the theorem follows from Lemma 2.1.6. □

Theorem 2.1.8 *Let $p : \mathbf{D} \mapsto \mathfrak{A}$ be a holomorphic mapping on $\mathbf{D}$ such that*

$$\operatorname{Re} p(A) \geq 0 \tag{2.1.10}$$

for each $A \in \mathbf{D}$. Then for a given $B \in \mathfrak{A}$, the mapping $h : \mathbf{D} \mapsto \mathfrak{A}$, defined by the formula

$$h(A) = p(A)A + B - AB^*A, \tag{2.1.11}$$

is holomorphically accretive. In other words, $h \in \mathcal{G}_+(\mathbf{D})$ induces a semi-complete vector field $T_h = h\,\dfrac{\partial}{\partial A}$ on $\operatorname{Hol}(\mathbf{D}, \mathfrak{A})$.

Proof. Note that it follows by Theorem 2.1.4 and Lemma 2.1.9 below that the mapping $h_1(\cdot)$, defined by the first term $h_1(A) = p(A)A$ in the representation of h, is a holomorphically accretive mapping on $\mathbf{D}$, i.e., $h_1 \in \mathcal{G}_+(\mathbf{D})$, while the second term $h_2(A) = B - AB^*A$ induces a complete vector field on every J^*-algebra, i.e., $h_2 \in \mathcal{G}_\pm(\mathbf{D}) = \mathcal{G}_+(\mathbf{D}) \cap \mathcal{G}_-(\mathbf{D})$ (see Section 1). Since $\mathcal{G}_+(\mathbf{D})$ is a real cone, the result follows. □

In fact, we will show below that for the class of l-analytic functions on $\mathbf{D}$ the converse assertion also holds, that is, each element of $\widehat{\mathcal{G}_+}(\mathbf{D})$ admits the representation (2.1.11) with (2.1.10). Then we will establish some necessary and sufficient conditions for the global solvability of the Cauchy problem (2.1.9). To this end, we need some simple auxiliary assertions.

As above, for $B \in L(\mathcal{H})$ we write $B \geq 0$ to denote that B is positive and $B > 0$ to denote that B is positive and invertible.

Lemma 2.1.9 *If $B \geq 0$ for some $B \in L(\mathcal{H})$, then $A^*BA \geq 0$ for all $A \in L(\mathcal{H})$; if $B > 0$, then $A^*BA > 0$ for all those $A \in L(\mathcal{H})$ which are invertible.*

Proof. By Definition 1.2.2 there is $C \in L(\mathcal{H})$ with $B = C^2$ and $C^* = C$. Hence, $A^*BA = A^*C^*CA = (CA)^*CA \geq 0$. The proof for $B > 0$ is analogous since in that case C is invertible. □

Lemma 2.1.10 *Let A and B be elements of a J^*-algebra $\mathfrak{A} \subset L(\mathcal{H})$. Then $C = B - AB^*A$ is also an element of $\mathfrak{A}$ and* $\mathrm{Re}(A^*C) = \mathrm{Re}[A^*B(I - A^*A)]$.

Proof. The first assertion follows immediately from (1.2.5). The second assertion is the result of the following direct calculations:

$$\begin{aligned}\mathrm{Re}(A^*C) &= \mathrm{Re}(A^*B) - \mathrm{Re}(A^*AB^*A) = \mathrm{Re}(A^*B) - \mathrm{Re}[(A^*AB^*A)^*] \\ &= \mathrm{Re}(A^*B) - \mathrm{Re}(A^*BA^*A) = \mathrm{Re}[A^*B(I - A^*A)]. \quad \square\end{aligned}$$

Lemma 2.1.11 *Let $\mathbf{D}$ be the open unit ball of a unital J^*-algebra $\mathfrak{A}$ and let $\hat{p} \in \widehat{\mathrm{Hol}}(\mathbf{D}, \mathfrak{A})$ be an l-analytic function with the producing function $p \in$* $\mathrm{Hol}(\Delta, \mathbb{C})$. *Then the following assertions are equivalent:*

(a) $\mathrm{Re}\, p(\lambda) \geq 0$ *(respectively,* $\mathrm{Re}\, p(\lambda) > 0$*) for all $\lambda \in \Delta$;*

(b) $\mathrm{Re}\, \hat{p}(A) \geq 0$ *(respectively,* $\mathrm{Re}\, \hat{p}(A) > 0$*) for all $A \in \mathbf{D}$;*

(c) $\mathrm{Re}[A^*\hat{p}(A)A] \geq 0$ *(respectively,* $\mathrm{Re}[A^*\hat{p}(A)A] > 0$*) for all invertible $A \in \mathbf{D}$.*

Proof. The implication (a)$\Rightarrow$(b) follows from [14, Theorem 1, p. 275]. The implication (b)$\Rightarrow$(c) is a consequence of Lemma 2.1.9 applied to the operator $B = \mathrm{Re}\, \hat{p}(A)$.

To show that (c) implies (a), set $A = \lambda I$. Then $A^* = \bar{\lambda} I$ and $A^*\hat{p}(A)A = |\lambda|^2 p(\lambda) I$.

Now the spectrum of the element $\mathrm{Re}[A^*\hat{p}(A)A]$ contains exactly one point, namely $|\lambda|^2 \,\mathrm{Re}\, p(\lambda)$, which must lie on the half-axis $[0, \infty)$ (respectively, $(0, \infty)$). This implies that $\mathrm{Re}\, p(\lambda) \geq 0$ (respectively, $\mathrm{Re}\, p(\lambda) > 0$) for all $\lambda \in \Delta$. We are done. $\square$

Now we formulate the main result of this subsection.

Theorem 2.1.12 *Let $\mathfrak{A}$ be a unital J^*-algebra and let $\hat{g} \in \widehat{\mathrm{Hol}}(\mathbf{D}, \mathfrak{A})$. The following assertions are equivalent:*

(i) $\hat{g} \in \widehat{\mathcal{G}_+}(\mathbf{D})$ (the set of holomorphically accretive l-analytic functions), i.e., the vector field

$$T_{\hat{g}} = \hat{g}(A) \frac{\partial}{\partial A} \tag{2.1.12}$$

is semi-complete;

(ii) $\hat{g}(A) = A\hat{p}(A) + \hat{g}(0) - [\hat{g}(0)]^* A^2$ *for some* $\hat{p} \in \widehat{\mathrm{Hol}}(\mathbf{D}, \mathfrak{A})$ *with*

$$\operatorname{Re} \hat{p}(A) \geq 0, \quad A \in \mathbf{D}; \tag{2.1.13}$$

(iii)

$$\operatorname{Re}[A^* \hat{g}(A)] \geq \operatorname{Re}\Big[A^* \hat{g}(0)(I - AA^*)\Big], \quad A \in \mathbf{D}. \tag{2.1.14}$$

Furthermore, if the inequality in either (2.1.13) or (2.1.14) is not strict for some $A \in \mathbf{D}$*, then the vector field (2.1.12) is complete, and consequently we have equalities in both (2.1.13) and (2.1.14) for all* $A \in \mathbf{D}$*.*

Proof. We first show that (i)⇒(ii). If $\hat{g} \in \widehat{\mathcal{G}_+}(\mathbf{D})$ and g is the producing function of $\hat{g}$, then it is easy to check (see the Remark after Theorem 2.1.1) that the vector field

$$\Big[-g(0) + \overline{g(0)}\lambda^2\Big] \frac{\partial}{\partial \lambda} \tag{2.1.15}$$

is complete on Δ (see also [50]).

Since $\hat{g}(0) = \dfrac{1}{2\pi i} \displaystyle\int_\Gamma \frac{g(\lambda) d\lambda}{\lambda} I = g(0)I$, the l-analytic function

$$\big[-\hat{g}(0) + [\hat{g}(0)]^* A^2\big]$$

produced by the function in the square brackets in (2.1.15) is a generator of a one-parameter group of l-analytic functions on $\mathbf{D}$. Note also that the set $\widehat{\mathcal{G}_+}(\mathbf{D})$ is a real cone, hence the l-analytic function $\hat{h} \in \widehat{\mathrm{Hol}}(\mathbf{D}, \mathfrak{A})$ defined by

$$\hat{h}(A) = \hat{g}(A) + \big[-\hat{g}(0) + [\hat{g}(0)]^* A^2\big] \tag{2.1.16}$$

belongs to $\widehat{\mathcal{G}_+}(\mathbf{D})$, and

$$\hat{h}(0) = h(0)I = 0. \tag{2.1.17}$$

Hence

$$h(0) = 0, \tag{2.1.18}$$

where h is the producing function of $\hat{h}$. Since h is holomorphically accretive, this implies that

$$\operatorname{Re} \bar{\lambda} h(\lambda) \geq 0, \quad \lambda \in \Delta,$$

or

$$h(\lambda) = \lambda p(\lambda)$$

with $\operatorname{Re} p(\lambda) \geq 0$.

Applying (1.3.4) and Lemma 2.1.11, we get that $\hat{h}$ has the form

$$\hat{h}(A) = A\hat{p}(A) \tag{2.1.19}$$

with $\operatorname{Re} \hat{p}(A) \geq 0,\ A \in \mathbf{D}$.

Now we obtain (ii) from (2.1.16) and (2.1.19). The reverse implication (ii)$\Rightarrow$(i) is a consequence of Theorem 2.1.8.

Now if we rewrite (ii) in the form

$$\hat{g}(A) = \hat{p}(A)A + \hat{g}(0) - A[\hat{g}(0)]^*A$$

with $\operatorname{Re} \hat{p}(A) \geq 0$, then we get

$$\operatorname{Re}[A^*\hat{g}(A)] = \operatorname{Re}[A^*\hat{p}(A)A] + \operatorname{Re}\{A^*\left[\hat{g}(0) - A[\hat{g}(0)]^*A\right]\}.$$

By Lemma 2.1.9 we have that the first term of this equality is positive, while by Lemma 2.1.10 the second term is equal to the expression

$$\operatorname{Re}\left\{A^*\hat{g}(0)\left(I - A^*A\right)\right\}.$$

This proves the implication (ii)$\Rightarrow$(iii).

Finally, if (iii) holds, then setting $A = \lambda I$ we obtain

$$\operatorname{Re}[\bar{\lambda}\hat{g}(\lambda I)] \geq \operatorname{Re}\left\{\bar{\lambda}\hat{g}(0)(I - |\lambda|^2 I)\right\}$$

or

$$\operatorname{Re}(\bar{\lambda}g(\lambda)) \geq \operatorname{Re}[\bar{\lambda}g(0)(1 - |\lambda|^2)],$$

where g is the producing function of $\hat{g}$. By the Remark after Theorem 2.1.1, the last inequality implies that the vector field $g(\lambda)\dfrac{\partial}{\partial\lambda}$ is semi-complete on Δ. Thus by Theorem 2.1.1 we have (i).

The above considerations show that if for some $A \in \mathbf{D}$, strict inequality holds in one of the conditions (ii) or (iii), then it also holds in both of them.

If, for example, it does not hold in (2.1.13), then it follows by [14, Theorem 1], that $\operatorname{Re} p(\lambda) = 0$ for some $\lambda \in \Delta$. By the maximum principle, $p(\lambda) = b$ with $\operatorname{Re} b = 0$. But in this case, $g(\lambda) = \lambda b + g(0) - \overline{g(0)}\lambda^2$ (the producing function of $\hat{g}$) defines a complete vector field, hence so does $\hat{g}$. The theorem is proved. $\square$

2.2 Involution operators and Hummel's multiplication transformation of semi-complete vector fields

In our previous considerations we represented holomorphically accretive l-analytic functions on $\mathbf{D}$. However, by using this representation one cannot recognize the location of null points of such functions (if they do exist). The only fact we know is that $\hat{g} \in \mathcal{G}_+(\mathbf{D})$ satisfies the condition

$$\mathrm{Re}[A^* \hat{g}(A)] \geq 0$$

if and only if $\hat{g}(0) = 0$.

Indeed, if $\hat{g}(0) = 0$, then $\mathrm{Re}[A^* \hat{g}(A)] \geq 0$ by Theorem 2.1.12. Conversely, fix any $r \in (0,1)$ and let $\mathbf{D}_r$ be the open ball centered at the origin with radius r. Then the condition $\mathrm{Re}[A^* \hat{g}(A)] \geq 0$ for all $A \in \partial \mathbf{D}_r$, the boundary of $\mathbf{D}_r$, and Corollary 2.1.5 imply that $\hat{g}$ is holomorphically accretive on $\mathbf{D}_r$. Now it follows by [28, Theorem 1] that the mapping $(I+\hat{g})^{-1}$ is a well-defined holomorphic self-mapping of $\mathbf{D}_r$. Since r is arbitrary it follows that the origin is the unique fixed point of this mapping. Hence $\hat{g}(0) = 0$.

It turns out that one can define a transformation on the set $\mathcal{G}_+(\mathbf{D})$ of holomorphically accretive l-analytic functions on $\mathbf{D}$ which is analogous to the Möbius transformation on the set $\mathrm{Hol}(\mathbf{D}, \mathbf{D})$ of self-mappings of $\mathbf{D}$. More precisely, it is well known that for each J^*-algebra $\mathfrak{A}$, given $B \in \mathbf{D}$, the open unit ball of $\mathfrak{A}$, one can define the generalized Möbius transformation M_B by

$$M_B(A) = (I - BB^*)^{-\frac{1}{2}}(B - A)(I - B^*A)^{-1}(I - B^*B)^{\frac{1}{2}},$$

which is a fractional-linear automorphism on $\mathbf{D}$ taking 0 into B and B into 0 (see, for example, [25]).

Now we recall that for a given $h \in \mathrm{Hol}(\mathbf{D}, \mathfrak{A})$, a holomorphic vector field $T_h = h \dfrac{\partial}{dA}$ is defined as the linear operator acting on $\mathrm{Hol}(\mathbf{D}, \mathfrak{A})$ by the formula

$$(T_h f)(A) = Df(A)h(A).$$

Substituting here $M_B(A)$ for A we have

$$(T_h f)(M_B(A)) = Df(M_B(A))h(M_B(A)).$$

Changing now the point of view and fixing some $f \in \mathrm{Hol}(\mathbf{D}, \mathfrak{A})$, one can consider the latter expression as the linear action L_f on an element $h \in \mathrm{Hol}(\mathbf{D}, \mathfrak{A})$.

It follows easily from the identity $M_B^2 = I$ that the operator $\Gamma := L_{M_B} : \mathrm{Hol}(\mathbf{D}, \mathfrak{A}) \mapsto \mathrm{Hol}(\mathbf{D}, \mathfrak{A})$ is an involution operator, i.e., $\Gamma^2 = I$.

Now we denote by E_B the set of $h \in \mathcal{G}_+(\mathbf{D})$ such that $h(B) = 0,\ B \in \mathbf{D}$.

Proposition 2.2.1 *The following relations hold:* $\Gamma(E_B) = E_0$ *and* $\Gamma(E_0) = E_B$.

Proof. Let $h \in E_B$. Then B is a stationary point of the semigroup $\{\mathcal{S}(t) : t \geq 0\}$ generated by h. Consider the family $\{\mathcal{F}(t)\},\ t \geq 0$, defined by

$$\mathcal{F}(t) = M_B \circ \mathcal{S}(t) \circ M_B, \quad t \geq 0.$$

It is obvious that this family is also a semigroup of holomorphic self-mappings of $\mathbf{D}$, and that the origin is its stationary point.

Let f be the generator of this semigroup, i.e.,

$$f(A) := -\left.\frac{\partial \mathcal{F}(t)(A)}{\partial t}\right|_{t=0^+}.$$

By direct calculations we obtain

$$f(A) = DM_B\left(M_B(A)\right) h\left(M_B(A)\right)$$

and

$$f(0) = 0.$$

Since $M_B^{-1} = M_B$ we have also

$$h(A) = DM_B\left(M_B(A)\right) f\left(M_B(A)\right).$$

This concludes the proof of Proposition 2.2.1. $\square$

A deficiency of the above considerations is the presence of the Möbius transformation in the definition of the operator Γ. Thus one cannot extend this construction to all of $\mathcal{G}_+(\mathbf{D})$, since this set contains functions with no null point in $\mathbf{D}$.

Nevertheless, for l-analytic functions on a unital J^*-algebra this can be done by using a multiplication transformation in the spirit of Hummel.

Let Ω be the subset of all $A \in \mathbf{D}$ such that $0 \notin \sigma(A)$. Given an operator $B \in \overline{\mathbf{D}}$, let us define a function $\Psi_B : \Omega \mapsto \mathfrak{A}$ by the formula

$$\Psi_B(A) = (A - B)(I - AB^*)A^{-1}.$$

For the one-dimensional case this (meromorphic) function was introduced by J. A. Hummel [32] for the study of star-like functions on the unit disk of the complex plane.

Now for a given $\hat{g} \in \widehat{\mathcal{G}_+}(\mathbf{D})$ such that $\hat{g}(0) = 0$ and $\tau \in \overline{\Delta}$, define the transformation W_τ by using the multiplication operation:

$$(W_\tau \hat{g})(A) = \Psi_{\tau I}(A)\hat{g}(A). \tag{2.2.1}$$

Theorem 2.2.2 *The transformation W_τ defined by (2.2.1) takes $\widehat{\mathcal{G}_+}(\mathbf{D})$ into itself. Moreover, for each $\hat{h} \in \widehat{\mathcal{G}_+}(\mathbf{D})$ there exists $\hat{g} \in \widehat{\mathcal{G}_+}(\mathbf{D})$ with $\hat{g}(0) = 0$ and $\tau \in \overline{\Delta}$ such that*

$$\hat{h} = W_\tau \hat{g}. \tag{2.2.2}$$

Proof. First we note that $\hat{h}$ can be holomorphically extended to all of $\mathbf{D}$. Indeed, since $\hat{g} \in \mathcal{G}_+(\mathbf{D})$ and $\hat{g}(0) = 0$ we have that $\hat{g}(A) = A\hat{q}(A)$ with $\operatorname{Re}\hat{q}(A) \geq 0$. Hence

$$\hat{h}(A) = (A - \tau I)(I - \bar{\tau}A)\hat{q}(A), \quad A \in \mathbf{D}. \tag{2.2.3}$$

Since Ω is an open subset of $\mathbf{D}$ this extension is unique and $\hat{h} \in \widehat{\mathrm{Hol}}(\mathbf{D}, \mathfrak{A})$.

Now let h be the producing function of $\hat{h}$. By Theorem 2.1.1 it is enough to show that $h \in \mathcal{G}_+(\Delta)$.

By (2.2.3) we have that h admits the representation

$$h(\lambda) = (\lambda - \tau)(1 - \bar{\tau}\lambda)q(\lambda) \tag{2.2.4}$$

with $\operatorname{Re} q(\lambda) \geq 0$.

Let us write h in the form

$$h(\lambda) = a - \bar{a}\lambda^2 + \lambda p(\lambda), \tag{2.2.5}$$

where $a = h(0)$ and $p \in \mathrm{Hol}(\Delta, \mathbb{C})$.

By Theorem 2.1 in [2], we have that $\operatorname{Re} p(\lambda) \geq 0$ for all $\lambda \in \Delta$. Thus by Theorem 2.1.12 we have $h \in \mathcal{G}_+(\Delta)$.

Let now $\hat{h} \in \mathcal{G}_+(\mathbf{D})$, respectively, $h \in \mathcal{G}_+(\Delta)$. Assume first that h has a null point τ inside Δ. Then by Proposition 2.2.1 the function f defined by

$$f(\lambda) := M_\tau'\left(M_\tau(\lambda)\right) h\left(M_\tau(\lambda)\right)$$

is an element of $\mathcal{G}_+(\Delta)$ with $f(0) = 0$, where M_τ is the Möbius involution transformation of Δ defined by

$$M_\tau(\lambda) = \frac{\tau - \lambda}{1 - \lambda\bar{\tau}}$$

and

$$h(\lambda) = M_\tau'\left(M_\tau(\lambda)\right) f\left(M_\tau(\lambda)\right).$$

Since $f(0) = 0$ we can write $f(\lambda) = \lambda p(\lambda)$ with $\operatorname{Re} p(\lambda) \geq 0$. Therefore, by direct calculations we obtain

$$\begin{aligned} h(\lambda) &= M_\tau'\left(M_\tau(\lambda)\right) \cdot M_\tau(\lambda) p\left(M_\tau(\lambda)\right) \\ &= \frac{1}{1 - |\tau|^2}(1 - \lambda\bar{\tau})(\lambda - \tau) p\left(M_\tau(\lambda)\right). \end{aligned}$$

Setting $q(\lambda) = \dfrac{1}{1 - |\tau|^2} p(M_\tau(\lambda))$ and $g(\lambda) = \lambda q(\lambda)$, we have $g \in \mathcal{G}_+(\Delta)$ and

$$h(\lambda) = (1 - \lambda\bar{\tau})(\lambda - \tau)\lambda^{-1} g(\lambda). \tag{2.2.6}$$

Since the lifting mapping $h \mapsto \hat{h}$ preserves the multiplication operation, the last equality when applied to $\hat{h}$ yields the representation (2.2.3).

Finally, let us suppose that $\hat{h}$ (respectively, h) has no null point in $\mathbf{D}$ (respectively, Δ). Then it follows by Corollary 1.6 in [37] that there is a unimodular point $\tau \in \partial\Delta$ such that for each $w \in \Delta$, the net $\{z_r(w)\}$, $r \geq 0$, defined as the solution of the equation

$$z_r(w) + r h(z_r(w)) = w \tag{2.2.7}$$

converges to τ as $r \to \infty$.

Fix $\varepsilon > 0$ and consider the mapping h_ε defined by

$$h_\varepsilon(\lambda) = \varepsilon\lambda + h(\lambda).$$

Since $\mathcal{G}_+(\Delta)$ is a real cone we have $h_\varepsilon \in \mathcal{G}_+(\Delta)$ for each $\varepsilon > 0$.

In addition, the equation $h_\varepsilon(\lambda) = 0$ is a particular case of (2.2.7) with $r = \dfrac{1}{\varepsilon}$ and $w = 0$. Hence, h_ε has a unique null point $\tau_\varepsilon \in \Delta$ and the net $\{\tau_\varepsilon\}$ converges to τ as ε goes to zero. By the previous case we have that h_ε admits the representation

$$h_\varepsilon(\lambda) = (1 - \lambda\bar{\tau}_\varepsilon)(\lambda - \tau_\varepsilon)\lambda^{-1} g_\varepsilon(\lambda),$$

where $g_\varepsilon \in \mathcal{G}_+(\Delta)$ with $g_\varepsilon(0) = 0$.

It is clear that h_ε converges to h as ε tends to 0. Hence g_ε converges to $g \in \mathrm{Hol}(\Delta, \mathbb{C})$ defined by

$$g(\lambda) = \frac{\lambda h(\lambda)}{(1 - \lambda\bar{\tau})(\lambda - \tau)}.$$

Since $g \in \mathcal{G}_+(\Delta)$ and $g(0) = 0$, this equality implies (2.2.6) (hence (2.2.2)), and we are done. Theorem 2.2.2 is proved. □

Now let B, once again, belong to $\mathbf{D}$. Then one can define the transformation $\tilde{W}_B$ on the cone $\mathcal{G}_+(\mathbf{D})$ by

$$(\tilde{W}_B \hat{g})(A) = \tilde{\Psi}_B(A)\hat{g}(A),$$

where

$$\tilde{\Psi}_B(A) = \frac{1}{1 - \|B\|^2}(A - B)(I - AB^*)A^{-1}.$$

Calculations show that when $B = \tau I$ the function $\tilde{\Psi}_B : \mathbf{D} \mapsto \mathfrak{A}$ has the following property:

$$\tilde{\Psi}_B(A)\tilde{\Psi}_B\left(M_B(A)\right) = I.$$

Using this fact and Theorem 2.2.2, we obtain the following assertion.

Corollary 2.2.3 *Each element $\hat{h} \in \widehat{\mathcal{G}_+}(\mathbf{D})$ which does not vanish identically has at most one null point in $\mathbf{D}$. Moreover, if $B \in \mathbf{D}$ is a null point of $\hat{h}$, then there is a complex number $\tau \in \Delta$ such that $B = \tau I$ and*

$$\hat{h} = \tilde{W}_{\tau I}\hat{g} = \frac{1}{1 - |\tau|^2} W_\tau \hat{g},$$

where $\hat{g} \in \widehat{\mathcal{G}_+}(\mathbf{D})$ with $\hat{g}(0) = 0$, and

$$\hat{g}(A) = \tilde{\Psi}_{\tau I}\left(M_{\tau I}(A)\right)\hat{h}(A).$$

Proof. By Theorem 2.2.2 there is $\tau \in \overline{\Delta}$ such that (2.2.3) holds. If $\tau \in \Delta$, then the second factor $(I - \bar{\tau}A)$ in (2.2.3) is invertible. A priori two cases are possible: (a) the third factor $\hat{q}(A)$ is also invertible and we have $\hat{h}(A) = 0$ if and only if $A = \tau I$; (b) if there is $A \in \mathbf{D}$ such that $\hat{q}(A)$ is not invertible, then $\mathrm{Re}\, q(\lambda) = 0,\ \lambda \in \Delta$, by Lemma 2.1.11. So, $q(\lambda) = a \in \mathbb{C}$ with $\mathrm{Re}\, a = 0$

and $\hat{q}(A) = aI$. Since $a \neq 0$ (otherwise $\hat{g} \equiv 0$), $\hat{q}(A)$ turns out to be invertible after all, a contradiction.

Let now $\tau \in \partial\Delta$, the boundary of Δ. Then for $\lambda \in \Delta$,

$$h(\lambda)I = \widehat{h}(\lambda I) = (\lambda I - \tau I)(I - \bar{\tau}\lambda I)\hat{q}(\lambda I) = (\lambda - \tau)(1 - \bar{\tau}\lambda)q(\lambda)I,$$
$$\tau \in \partial\Delta, \quad \operatorname{Re} q(\lambda) \geq 0.$$

It is easy to see that this implies that $h \in \operatorname{Hol}(\Delta, \mathbb{C})$ has no null point in Δ. On the other hand, if $\hat{h}(A)$ were not invertible for some $A \in \mathbf{D}$, then the spectral mapping theorem (see, for example, [41]) would imply that $h(\lambda)$ vanishes for some $\lambda \in \sigma(A) \subset \Delta$. Thus $\hat{h}(A)$ is invertible for all $A \in \mathbf{D}$, and a fortiori, does not vanish. $\square$

Remark 1. Actually, the above assertions extend the one-dimensional parametric representation of holomorphically accretive mappings due to E. Berkson and H. Porta [5].

Namely, one can say that $\hat{g} \in \widehat{\mathcal{G}_+}(\mathbf{D})$ if and only if there exist $\tau \in \overline{\Delta}$ and $\hat{q} \in \widehat{\operatorname{Hol}}(\mathbf{D}, \mathfrak{A})$ with

$$\operatorname{Re} \hat{q}(A) \geq 0, \quad A \in \mathbf{D}, \tag{2.2.8}$$

such that $\hat{g}$ admits the representation

$$\hat{g}(A) = (A - \tau I)(I - \bar{\tau}A)\hat{q}(A). \tag{2.2.9}$$

Remark 2. In general, the lifting mapping $f \mapsto \hat{f}$ does not preserve the uniqueness property of null points of f. Take, for example, $f(\lambda) = \lambda^2$ which has a unique null point in Δ, while $\hat{f}(A) = A^2$ has infinitely many null points in $\mathbf{D}$. However, the above assertions show that this property is preserved for the restriction

$$h \in \mathcal{G}_+(\Delta) \mapsto \hat{h} \in \widehat{\mathcal{G}_+}(\mathbf{D}).$$

2.3 Stationary points and the asymptotic behavior of one-parameter semigroups of l-analytic functions

As above, let $\mathbf{D}$ be the open unit ball in a unital J^*-algebra $\mathfrak{A}$ and let a family $\{\mathcal{S}(t)\}$, $t \in \mathbb{R}^+$ ($t \in \mathbb{R}$), be a one-parameter continuous semigroup (group) mapping $\mathbf{D}$ into itself.

Recall that a point $a \in \mathbf{D}$ is said to be a stationary point of $\{\mathcal{S}(t)\},\ t \in \mathbb{R}^+\,(t \in \mathbb{R})$, if it is a common fixed point of all the mappings $\mathcal{S}(t),\ t \in \mathbb{R}^+\,(t \in \mathbb{R})$, i.e.,

$$\mathcal{S}(t)(a) = a, \quad t \in \mathbb{R}^+\,(t \in \mathbb{R}). \tag{2.3.1}$$

A stationary point a is called (locally) asymptotically stable if there is a neighborhood $\mathcal{V}$ of a such that $\mathcal{S}(t)x$ converges to a for all $x \in \mathcal{V}$.

We begin with the classical questions of existence, uniqueness and stability of stationary points of a semigroup of l-analytic functions.

Theorem 2.3.1 *Let $\{\widehat{\mathcal{S}}(t):\ t \geq 0\} \subset \widehat{\mathrm{Hol}}(\mathbf{D},\mathbf{D})$ be a strongly continuous semigroup of l-analytic functions mapping $\mathbf{D}$ into itself. Then*

(i) $\widehat{\mathcal{S}}(t)$ converges to I as $t \to 0^+$, uniformly on each subset strictly inside $\mathbf{D}$, i.e.,

$$T\text{-}\lim_{t\to 0^+} \widehat{\mathcal{S}}(t) = I, \tag{2.3.2}$$

or, which is equivalent, $\widehat{\mathcal{S}}(t)$ is strongly differentiable at $t = 0^+$;

(ii) $\widehat{\mathcal{S}}(t)$ has at most one stationary point in $\mathbf{D}$;

(iii) If $\{\widehat{\mathcal{S}}(t)\}_{t\geq 0}$ has a stationary point $A \in \mathbf{D}$, then it is asymptotically stable if and only if $\{\widehat{\mathcal{S}}(t)\}$ does not contain an automorphism of $\mathbf{D}$.

Moreover, local stability implies global stability in the sense of strong convergence on all of $\mathbf{D}$.

Proof. Assertion (i) is a direct consequence of a result of Berkson–Porta [5] and Theorem 2.1.1 (see also Section 1.1).

Now, if $\hat{g} \in \widehat{\mathrm{Hol}}(\mathbf{D},\mathfrak{A})$ is the generator of $\{\widehat{\mathcal{S}}(t)\}$, i.e., $\hat{g} = \lim\limits_{t\to 0} \dfrac{1}{t}\,(I - \widehat{\mathcal{S}}(t))$, then $\widehat{\mathcal{S}}(t)(A) =: u(t,A)$ is the solution of the Cauchy problem

$$\begin{cases} \dfrac{\partial u(t,A)}{\partial t} + \hat{g}(u(t,A)) = 0 \\[2ex] u(0,A) = A \in \mathbf{D}. \end{cases} \tag{2.3.3}$$

It follows from the uniqueness of this solution that the stationary points of $\widehat{\mathcal{S}}(t)$ inside $\mathbf{D}$ are exactly the null points of the l-analytic function $\hat{g}$. Thus (ii) is a consequence of Corollary 2.2.3. This Corollary also asserts that if $A \in \mathbf{D}$ is a stationary point of $\{\widehat{\mathcal{S}}(t)\}$, then it must be τI, where τ is the null point of the producing function $g \in \mathcal{G}_+(\Delta)$ of $\hat{g}$. Consequently, τ is a stationary point of the semigroup $\{\mathcal{S}(t) : \Delta \mapsto \Delta\},\ t \geq 0$, where $\mathcal{S}(t)$ is the producing function of $\widehat{\mathcal{S}}(t)$. Thus (iii) follows from the continuous version of the Denjoy–Wolff theorem (see, for example, [1, 37, 38]) and (1.3.5). $\square$

When combined with this continuous version, Theorem 2.3.1 can be rephrased, in terms of generators, as follows.

Corollary 2.3.2 *Let $\{\widehat{\mathcal{S}}(t) :\ t \geq 0\}$ be a semigroup of l-analytic functions and let its generator $\hat{g}$ admit the representation*

$$\lim_{t\to 0^+} \frac{A - \widehat{\mathcal{S}}(t)(A)}{t} = \hat{g}(A) = (A - \tau I)(I - \bar{\tau} A)\hat{q}(A)$$

with $\operatorname{Re}\hat{q}(A) \geq 0$. Then

(i) The semigroup $\widehat{\mathcal{S}}(t)$ has a stationary point A in $\mathbf{D}$ if and only if $\tau \in \Delta$;

(ii) this point A $(= \tau I)$ is asymptotically stable if and only if $\operatorname{Re}\hat{q}(A) > 0$;

(iii) if $\tau \in \partial\Delta$, the boundary of Δ, then $\{\widehat{\mathcal{S}}(t) :\ t \geq 0\}$ has no stationary points in $\mathbf{D}$, and for all $A \in \mathbf{D}$, the net $\{\widehat{\mathcal{S}}(t)(A) :\ t \geq 0\}$ converges to the boundary point $\tau I \in \partial\mathbf{D}$, as t tends to infinity.

Remark. It can be shown by using Theorem 6.3 in [36] that the convergence in (iii) of Theorem 2.3.1 is actually T-convergence. As a matter of fact, this also follows from Theorem 2.3.3 below.

We will now derive a rate of convergence of the semigroup to its stationary point. We recall first that the hyperbolic metric ρ on $\mathbf{D}$ can be defined as

$$\rho(A, B) = \tanh^{-1} \|M_B(A)\|,$$

where

$$M_B(A) = (I - BB^*)^{-\frac{1}{2}}(B - A)(I - B^*A)^{-1}(I - B^*B)^{\frac{1}{2}}$$

(see, for example, [25]) is the fractional-linear automorphism on $\mathbf{D}$ taking 0 into B and B into 0.

Theorem 2.3.3 *Let $\hat{g} \in \widehat{\mathcal{G}_+}(\mathbf{D})$ have the representation*

$$\hat{g}(A) = (A - \tau I)(I - \bar{\tau} A)\hat{q}(A)$$

with $\tau \in \Delta$ and $\operatorname{Re}\hat{q}(A) > 0$. Suppose that $\{\widehat{\mathcal{S}}(t) :\ t \geq 0\}$ is the semigroup generated by $\hat{g}$. Then for all $A \in \mathbf{D}$ the following estimate holds:

$$\rho(\widehat{\mathcal{S}}(t)A, \tau I) \leq k(t, A)\rho(A, \tau I),$$

where

$$k(t,A)=\exp\left\{t(|\ \tau\ |^2-1)\,\frac{1-\|M_{\tau I}(A)\|}{1+\|M_{\tau I}(A)\|}\,\mathrm{Re}\,\hat{q}(\tau I)\right\}$$

tends to zero as t tends to ∞.

Proof. Let $\{\widehat{\mathcal{S}}(t)\}$ be the semigroup generated by $\hat{g}$. Consider the family $\{\widehat{U}(t)\}$ defined by

$$\widehat{U}(t)=M_{\tau I}\widehat{\mathcal{S}}(t)M_{\tau I},\quad t\geq 0.$$

It is easy to see that $\{\widehat{U}(t)\}$ is also a semigroup of l-analytic functions generated by the mapping

$$\hat{h}:=-\frac{d\widehat{U}(0^+)}{dt}=DM_{\tau I}(M_{\tau I})\,[\hat{g}(M_{\tau I})]$$

which is l-analytic too. This formula can be rewritten explicitly as follows:

$$\hat{h}(A)=(1-|\tau|^2)\hat{q}\left(M_{\tau I}(A)\right)A=:\hat{q}_1(A)A$$

with $\mathrm{Re}\,\hat{q}_1(A)\geq 0$.

Now it follows from Ky Fan's generalization of Harnack's inequality (see [17]) that

$$\mathrm{Re}\,\hat{q}_1(A)\geq\mathrm{Re}\,\hat{q}_1(0)\,\frac{1-\|A\|}{1+\|A\|}=\mathrm{Re}\,q_1(0)\,\frac{1-\|A\|}{1+\|A\|}I.$$

Consequently, by Lemma 2.1.9 we obtain

$$\mathrm{Re}\,A^*\hat{h}(A)\geq\mathrm{Re}\,q_1(0)\,\frac{1-\|A\|}{1+\|A\|}A^*A.$$

Now, exactly as in the proof of Theorem 2.1.4, it can be shown that this inequality implies that for each $s\in(0,1)$,

$$\inf\mathrm{Re}\,V\left(\hat{h}_s\right)\geq\mathrm{Re}\,q_1(0)\frac{1-s}{1+s}s,$$

where $\hat{h}_s(A):=\hat{h}(sA)$ and $V(\hat{h}_s)$ is the (total) numerical range of $\hat{h}_s$ (see Section 1.1).

Next, it is easy to verify that for $w\in\Delta$ and $t\geq 0$ the equation

$$z+t\,[\mathrm{Re}\,q_1(0)]\,z\,\frac{1-z}{1+z}=w$$

has a unique solution $z = z(t, w)$ in Δ. (See [28, Theorem 1] for a more general situation.) In addition, for each fixed $t \geq 0$ this solution depends holomorphically on $w \in \Delta$. Since $z(t, 0) = 0$, we have by the Schwarz Lemma

$$|z(t, w)| \leq |w|.$$

In particular, if w is a real number in $[0, 1)$, then $z(t, w)$ is real too, and we get by the monotonicity of the function $\dfrac{1-s}{1+s}$ on $[0, 1)$ that

$$z(t, w) \leq \frac{w}{1 + t \operatorname{Re} q_1(0) \frac{1-w}{1+w}}$$

for all w in $[0, 1)$ and $t \geq 0$.

Take now any operator $B \in \mathbf{D}$ and consider the function $f_s : \mathbf{D} \mapsto X$ defined by

$$f_s(A) = sA + t\hat{h}(sA) - B, \quad s \in [0, 1),\ t \geq 0.$$

Then for $1 > s > z(t, \|B\|)$ we obtain

$$\inf \operatorname{Re} V(f_s) \geq s + t \operatorname{Re} q_1(0) \frac{1-s}{1+s} s - \|B\| > 0.$$

Now using Corollary 2 of [28] we get that the equation

$$A + t\hat{h}(A) = B$$

has a unique solution $W(t, B) = (I + t\hat{h})^{-1}(B)$ in the ball $\mathbf{D}_s$ centered at the origin with radius s. Letting s tend to $z(t, \|B\|)$ we obtain

$$\|W(t, B)\| \leq z(t, \|B\|) \leq \frac{\|B\|}{1 + t \operatorname{Re} q_1(0) \frac{1-\|B\|}{1+\|B\|}}.$$

Furthermore, using induction we estimate

$$\left\| \left[W\left(\frac{t}{n}, B\right) \right]^{[n]} \right\| \leq \frac{\|B\|}{\left(1 + \frac{t}{n} \operatorname{Re} q_1(0) \frac{1 - \|B\|}{1 + \|B\|}\right)^n}.$$

This inequality and the exponential formula

$$\widehat{U}(t)(\cdot) = \lim_{n \to \infty} \left[W\left(\frac{t}{n}, \cdot\right) \right]^{[n]}$$

imply that

$$\left\|\widehat{U}(t)(B)\right\| \le \exp\left\{-t\,\mathrm{Re}\,q_1(0)\,\frac{1-\|B\|}{1+\|B\|}\right\}\|B\|.$$

Finally, we conclude our proof by substituting $M_{\tau I}(A)$ for B and applying the following properties of the hyperbolic metric:

$$\rho(0,kA) \le k\rho(0,A) \quad \text{and} \quad \rho(M_B(A), M_B(C)) = \rho(A,C)$$

(cf. [19]). $\square$

2.4 Flow-invariance sets and an infinitesimal version of the Julia–Wolff–Carathéodory Theorem

In this section we are interested in the following problem: Given an l-analytic function $\hat{g}$ on $\mathbf{D}$ such that $T_{\hat{g}} = \hat{g}\,\dfrac{\partial}{\partial A}$ is a semi-complete vector field on $\mathbf{D}$, find a family $\{\Omega_\alpha\}_{\alpha\in\mathcal{A}}$ of subsets of $\mathbf{D}$ such that

(a) $\Omega_\alpha \subset \Omega_\beta$ when $\alpha < \beta$;

(b) $\bigcup\limits_{\alpha\in\mathcal{A}} \Omega_\alpha = \mathbf{D}$;

(c) $T_{\hat{g}}$ is a semi-complete vector field on Ω_α for each $\alpha \in \mathcal{A}$.

In other words, each Ω_α is an invariant subset of $\mathbf{D}$ for the semigroup (flow) generated by $\hat{g}$.

If $\hat{g} \in \widehat{\mathcal{G}_+}(\mathbf{D})$ has a null point at the origin, $\hat{g}(0) = 0$, then it follows by the Schwarz Lemma that each ball $\mathbf{D}_r = \{a \in \mathbf{D} : \|A\| < r\}$ is a flow-invariance set for the semigroup $\{\widehat{\mathcal{S}}_t(A)\}$ generated by $\hat{g}$. It is clear that the family $\{\mathbf{D}_r\}$, $r \in (0,1)$, satisfies conditions (a)–(c).

If $\hat{g}$ has another null point $A_0 \in \mathbf{D}$, $A_0 \ne 0$, then one can use the fractional-linear Möbius transformation

$$M_{A_0}(A) = (I - A_0{A_0}^*)^{-\frac{1}{2}}(A_0 - A)(I - {A_0}^*A)^{-1}(I - {A_0}^*A_0)^{\frac{1}{2}} \qquad (2.4.1)$$

(see, for example, [25]) to translate A_0 into zero and find flow-invariance sets about the point A_0. Thus the sets

$$\Omega_r = [M_{A_0}]^{-1}(\mathbf{D}_r), \quad r \in (0,1), \qquad (2.4.2)$$

are flow-invariance subsets of $\mathbf{D}$ satisfying (a)–(c). In fact, these sets are hyperbolic balls (with respect to the hyperbolic metric on $\mathbf{D}$) centered at A_0.

Since in our situation $A_0 = \tau I$ for some $\tau \in \Delta$, we can write (2.4.1) in the form

$$M_{\tau I}(A) = (\tau I - A)(I - \bar{\tau} A)^{-1} \tag{2.4.3}$$

and get the following assertion.

Theorem 2.4.1 *Let $\hat{g} \in \widehat{\mathcal{G}_+}(\mathbf{D})$ have a null point $A_0 = \tau I \in \mathbf{D}$ for some $\tau \in \Delta$. Then the sets*

$$\Omega_r = \left\{ A \in \mathbf{D} : \left\| (\tau I - A)(I - \bar{\tau} A)^{-1} \right\| < r \right\} \subset \mathbf{D}, \; r \in (0,1), \tag{2.4.4}$$

satisfy conditions (a)–(c).

The situation becomes more complicated if $\hat{g} \in \widehat{\mathcal{G}_+}(\mathbf{D})$ has no null point in $\mathbf{D}$, i.e., if the complex number τ in the representation (2.2.9) is unimodular. In this case we already know that the semigroup $\{\widehat{\mathcal{S}}(t)\}$ converges strongly to the point τI.

A question which arises at this juncture is: What is the rate of convergence of the semigroup $\{\widehat{\mathcal{S}}(t)\}$? To answer this question we will establish an infinitesimal version of the Julia–Wolff–Carathéodory (JWC) Theorem in terms of generators. Moreover, since each mapping of the form $I - F$, where F is an l-analytic function on $\mathbf{D}$, is a generator, this version will include some standard formulations of the JWC Theorem.

We define the subset Γ of $\overline{\mathbf{D}}$ by

$$\Gamma := \Big\{ E \in \partial \mathbf{D} : \; E^* E = I \Big\}.$$

In the finite-dimensional case, Γ is precisely the Bergman–Shilov boundary of $\mathbf{D}$ (see [25, Corollary 9]). We also set

$$\Gamma_+ := \Big\{ E \in \Gamma : \text{ there exists } a \in \mathbb{C} \text{ such that } aE > 0 \Big\}.$$

Proposition 2.4.2 *Let $g \in \mathrm{Hol}(\mathbf{D}, \mathfrak{A})$ admit the representation*

$$g(A) = (A - \tau I) q(A)(I - \bar{\tau} A) \tag{2.4.5}$$

with some $\tau \in \overline{\Delta}$ *and* $q \in \mathrm{Hol}(\mathbf{D}, \mathfrak{A})$ *such that*

$$\operatorname{Re} q \geq 0 \tag{2.4.6}$$

everywhere.

Suppose that for some element $E \in \Gamma$,

$$\lim_{r \to 1^-} \frac{1}{r-1} E^* g(rE) = B, \tag{2.4.7}$$

where

$$\operatorname{Re} B > 0. \tag{2.4.8}$$

Then $\tau \in \partial\Delta$.

Proof. If $\tau \in \Delta$, then $C = (I - \bar{\tau}E)^*$ is invertible, and by (2.4.5), (2.4.7) and (2.4.8) we have

$$\begin{aligned} \operatorname{Re} B &= \operatorname{Re}\left\{ \lim_{r \to 1^-} \frac{1}{r-1}\, E^*(rE - \tau I) q(rE)(I - \bar{\tau} rE) \right\} \\ &= \operatorname{Re}\left\{ \lim_{r \to 1^-} \frac{1}{r-1} (rI - \tau E^*) q(rE)(I - \bar{\tau} rE) \right\} \\ &= \operatorname{Re}\left\{ (I - \bar{\tau}E)^* [\lim_{r \to 1^-} \frac{1}{r-1} q(rE)](I - \bar{\tau}E) \right\} > 0. \end{aligned} \tag{2.4.9}$$

At the same time, we get by (2.4.6) and Lemma 2.1.9,

$$\begin{aligned} \operatorname{Re}[(Cq(rE)C^*)] &= \frac{1}{2}\Big(Cq(rE)C^* + C(q(rE))^* C^* \Big) \\ &= C(\operatorname{Re} q(rE))C^* \geq 0. \end{aligned}$$

Since $r < 1$, this contradicts (2.4.9). Thus τ must lie on the boundary of Δ. □

Theorem 2.4.3 *Let* $\hat{g} \in \widehat{\mathcal{G}_+}(\mathbf{D})$. *Then* $\hat{g}$ *has no null point in* $\mathbf{D}$ *if and only if there is a point* E *in the subset* Γ_+ *of* $\partial\mathbf{D}$ *such that the limit*

$$\lim_{r \to 1^-} \frac{1}{r-1} E^* \hat{g}(rE) = B \tag{2.4.10}$$

exists with

$$\operatorname{Re} B \geq 0. \tag{2.4.11}$$

Proof. *Sufficiency.* We suppose first that (2.4.11) holds with

$$\operatorname{Re} B > 0.$$

If we assume that $\hat{g}$ has a null point $A_0 \in \mathbf{D}$, then $\hat{g}$ admits the representation (2.2.9)

$$\hat{g}(A) = (A - \tau I)(I - \bar{\tau} A)\hat{q}(A)$$

with some $\tau \in \Delta$ and $\hat{q} \in \widehat{\operatorname{Hol}}(\mathbf{D}, \mathfrak{A})$ such that $\operatorname{Re} \hat{q}(A) \geq 0$, and thus $A_0 = \tau I$ (see Corollary 2.2.3).

Note that $\hat{q}(A)$ and $(I - \bar{\tau} A)$ commute as two l-analytic functions evaluated at the same element. Thus Proposition 2.4.2 shows that $\tau \in \partial \Delta$. The contradiction we have reached shows that $\hat{g}$ cannot have a null point in $\mathbf{D}$.

In the general case, when $\operatorname{Re} B \geq 0$, we consider the holomorphic mapping $g_\varepsilon : \mathbf{D} \mapsto \mathfrak{A}$ defined by

$$\hat{g}_\varepsilon(A) = \hat{g}(A) + \varepsilon(aA^2 - \bar{a} I),$$

where $a \in \mathbb{C}$ satisfies the condition $aE > 0$.

Observe that for each $\varepsilon \geq 0$, the l-analytic function $\hat{g}_\varepsilon$ is anti-dissipative on $\mathbf{D}$, i.e., $\hat{g}_\varepsilon \in \widehat{\mathcal{G}_+}(\mathbf{D})$. Indeed, it is well known that the vector field induced by $AX^*A - X$ is complete on $\operatorname{Hol}(\mathbf{D}, \mathfrak{A})$ for each $X \in \mathfrak{A}$, whenever $\mathfrak{A}$ is a J^*-algebra. Since $\mathcal{G}_+(\mathbf{D})$ is a real cone the result follows.

In addition,

$$\begin{aligned}
\operatorname{Re}\left(\lim_{r \to 1^-} \frac{1}{r-1} E^* \hat{g}_\varepsilon(rE)\right) &= \\
= \operatorname{Re}\left(\lim_{r \to 1^-} \frac{1}{r-1} E^* \hat{g}(rE)\right) &+ \varepsilon \operatorname{Re} \lim_{r \to 1^-} \frac{1}{r-1}\left[E^*(ar^2E^2 - \bar{a} I)\right] = \\
&= \operatorname{Re} B + 2\varepsilon a E > 0.
\end{aligned}$$

Thus it follows by the previous case that $\hat{g}_\varepsilon$ (hence its producing function g_ε) has no null point in $\mathbf{D}$ (in Δ). On the other hand, g_ε converges to g on Δ when $\varepsilon \to 0$, where g is the producing function of $\hat{g}$. Since by Rouché's Theorem g cannot have a null point in Δ, $\hat{g}$ has no null point in $\mathbf{D}$ and we are done.

Necessity. Suppose now that $\hat{g}$ has no null point in $\mathbf{D}$. Then we already know that $\hat{g}$ has the representation

$$\hat{g}(A) = (A - \tau I)\hat{q}(A)(I - \bar{\tau} A) \tag{2.4.12}$$

for some unimodular $\tau \in \partial\Delta$ and $\hat{q} \in \widehat{\mathrm{Hol}}(\mathbf{D}, \mathfrak{A})$ such that

$$\operatorname{Re} \hat{q}(A) \geq 0.$$

Setting $\tau = 1$ without loss of generality, we can rewrite (2.4.12) as

$$\hat{g}(A) = -\hat{q}(A)(I - A)^2$$

or

$$g(\lambda) = -q(\lambda)(1 - \lambda)^2,$$

where $g(\lambda)$ and $q(\lambda)$ are the producing functions of $\hat{g}(A)$ and $\hat{q}(A)$, respectively.

First we want to show that $g(r)$ tends to zero when r tends to 1. To this end, we will prove somewhat more. Namely, that

$$\frac{g(\lambda)}{\lambda - 1} = (1 - \lambda) q(\lambda)$$

converges to a real number when λ approaches 1 nontangentially:

$$\lambda \in \Lambda(1, \alpha) = \Big\{ \lambda \in \Delta : \ |\lambda - 1| < \alpha(1 - |\lambda|) \Big\}, \quad \alpha > 1.$$

Indeed, since $\operatorname{Re} q \geq 0$ everywhere, we have by the Riesz–Herglotz formula

$$\frac{g(\lambda)}{\lambda - 1} = \int\limits_{\partial\Delta} (1 - \lambda) \, \frac{1 + \lambda\bar{\zeta}}{1 - \lambda\bar{\zeta}} \, d\mu(\zeta) + i(1 - \lambda) b,$$

where $\mu(\zeta)$ is a positive measure on $\partial\Delta$ and b is a real number.

Since for $\lambda \in \Lambda(1, \alpha)$,

$$\left| (1 - \lambda) \frac{1 + \lambda\bar{\zeta}}{1 - \lambda\bar{\zeta}} \right| \leq 2\alpha,$$

we have by Lebesgue's Dominated Convergence Theorem (see, for example, [42]) that

$$\lim_{\lambda \to 1, \lambda \in \Lambda} \frac{g(\lambda)}{\lambda - 1} = 2\mu(1) =: \beta \geq 0.$$

Since $\hat{g}(rI) = g(r)I$, we finish the proof by setting $E = I$ and $B = \beta I$. $\square$

Remark. In fact we have proved the following assertion:

A mapping $\hat{g} \in \widehat{\mathcal{G}_+}(\mathbf{D})$ has no null point in $\mathbf{D}$ if and only if there is a point $\tau \in \partial\Delta$ such that for $E = \tau I$ the limit (2.4.10) exists with $B = \beta I$, where β is a real non-negative number.

Now we are able to give a quantitative description of the behavior of one-parameter semigroups with no stationary point.

For operators A and B in $\mathbf{D}$ such that $A^*A < B^*B$ we define the "distance" operator

$$\Phi(A, B) := (B - A)(B^*B - A^*A)^{-1}(B^* - A^*).$$

Theorem 2.4.4 *Let $\{\widehat{\mathcal{S}}(t) : \ t \geq 0\}$ be a one-parameter continuous semigroup of l-analytic proper contractions of $\mathbf{D}$ with no stationary point in $\mathbf{D}$. Then there is a unimodular point $\tau \in \partial\Delta$ and a positive number γ such that*

$$\Phi(\widehat{\mathcal{S}}(t)A, \tau I) \leq \exp(-\gamma t)\Phi(A, \tau I) \tag{2.4.13}$$

and

$$\left\|\Phi(\widehat{\mathcal{S}}(t)A, \tau I)\right\| \leq \exp(-\gamma t)\left\|\Phi(A, \tau I)\right\| \tag{2.4.14}$$

for each $A \in \mathbf{D}$ and $t \geq 0$.

Moreover, the maximal γ for which (2.4.13) (or (2.4.14)) holds is the number

$$\beta = -\lim_{r \to 1^-} \left.\frac{\partial^2 u(t, \lambda)}{\partial t \partial \lambda}\right|_{t=0^+,\, \lambda = r\tau}, \tag{2.4.15}$$

where $u(t, \lambda) := \mathcal{S}(t)(\lambda)$ is the producing function of $\widehat{\mathcal{S}}(t)$.

Remark. By Lemma 3 in [15], condition (2.4.14) is equivalent to the following one:

$$\left\|\widehat{\mathcal{S}}(t)A - \frac{\tau}{1 + \exp(-t\gamma)k} I\right\| \leq \frac{k}{k + \exp(t\gamma)} \tag{2.4.16}$$

for each $k > 0$, whenever

$$\left\|A - \frac{\tau}{1 + k} I\right\| \leq \frac{k}{1 + k}. \tag{2.4.17}$$

Thus the sets defined by (2.4.17) are $\widehat{\mathcal{S}}(t)$-invariant.

Proof. If $\{\widehat{\mathcal{S}}(t)\}$ has no stationary point in $\mathbf{D}$, then the producing semigroup $\{\mathcal{S}(t)\}$ has no stationary point in Δ, and its generator

$$g = -\left.\frac{d\mathcal{S}(t)}{dt}\right|_{t=0^+}$$

has the form

$$g(\lambda) = (\lambda - \tau)(1 - \bar{\tau}\lambda)q(\lambda)$$

for some $\tau \in \partial\Delta$ and $q \in \mathrm{Hol}(\Delta, \mathbb{C})$ with $\mathrm{Re}\, q \geq 0$ everywhere. In what follows we assume without loss of generality that $\tau = 1$. By the proof of Theorem 2.4.3, the limit

$$\lim_{\lambda \to 1,\, \lambda \in \Lambda} \frac{g(\lambda)}{\lambda - 1} =: \beta \tag{2.4.18}$$

exists, where λ approaches τ nontangentially and β is a nonnegative real number.

In addition, it follows by Lindelöf's theorem (see, for example, [11, p. 6] or [35, p. 79]) that

$$\beta = \lim_{\lambda \to 1} g'(\lambda) \tag{2.4.19}$$

where λ approaches 1 nontangentially. So, the limit in (2.4.15) exists and it is a real nonnegative number. We want to show that this implies (2.4.13) and (2.4.14) with $\gamma = \beta$. Consider now the resolvent $\mathcal{J}_s : \Delta \mapsto \Delta$ of g defined by the equation

$$\mathcal{J}_s(\lambda) + sg(\mathcal{J}_s(\lambda)) = \lambda, \quad \lambda \in \Delta, \tag{2.4.20}$$

(see, for example, [37]).

Since g has no null point in Δ, for each $s > 0$ the mapping $\mathcal{J}_s$ has no fixed point in Δ. Moreover, it was shown in [37] that for all s, the resolvent $\mathcal{J}_s$ has the same sink point, say $w \in \partial\Delta$, i.e.,

$$\frac{|w - \mathcal{J}_s(\lambda)|^2}{1 - |\mathcal{J}_s(\lambda)|^2} \leq \frac{|w - \lambda|^2}{1 - |\lambda|^2} =: \phi_w(\lambda) \tag{2.4.21}$$

and

$$\mathcal{J}_s(\lambda) \to w \tag{2.4.22}$$

as $s \to \infty$.

On the other hand, fixing $s>0$ and applying to $\mathcal{J}_s$ the Julia–Carathéodory and Wolff Theorems (see, for example, [44, pp. 57 and 81]) we get that for each $s>0$ there exists the nontangential limit

$$\lim_{\lambda\to w}(\mathcal{J}_s)'(\lambda)=\lim_{\lambda\to w}\frac{w-\mathcal{J}_s(\lambda)}{w-\lambda}=\alpha_s. \tag{2.4.23}$$

Moreover,

$$1\geq\alpha_s=\liminf_{\lambda\to w}\frac{1-|\mathcal{J}_s(\lambda)|}{1-|\lambda|}>0 \tag{2.4.24}$$

as λ tends to w unrestrictedly.

Now observe that it follows by (2.4.21) and (2.4.22) that the two sequences $\lambda_n=\mathcal{J}_n(0)$ and $\lambda_{n,s}=\mathcal{J}_s(\lambda_n)$ (here $s>0$ is fixed) converge nontangentially to w as $n\to\infty$.

Hence we have by (2.4.20) and (2.4.23),

$$\lim_{\lambda_{n,s}\to w}\frac{g(\lambda_{n,s})}{\lambda_{n,s}-w}=\frac{1-\alpha_s}{s\alpha_s}\geq 0. \tag{2.4.25}$$

Setting $\psi(s):=\phi_w(\mathcal{J}_s(\lambda))$, we see by (2.4.21) that

$$\psi(s)\leq\psi(0)\quad\text{for all}\quad s\geq 0.$$

Therefore $\psi'(0)\leq 0$. A computation shows that this is equivalent to the inequality

$$\operatorname{Re}\frac{g(\lambda)}{(\lambda-w)(1-\lambda\bar{w})}\geq 0.$$

In other words,

$$g(\lambda)=(\lambda-w)(1-\lambda\bar{w})\,p(\lambda),$$

where $p:\ \Delta\mapsto\mathbb{C}$ with $\operatorname{Re}p(\lambda)\geq 0$ for all $\lambda\in\Delta$. By the uniqueness of the Berkson–Porta representation [5, 2] (see also [45]), it now follows that $w=1$.

Thus (2.4.18) and (2.4.25) imply that

$$\alpha_s=\frac{1}{1+\beta s}. \tag{2.4.26}$$

If now $\widehat{\mathcal{J}_s}:\ \mathbf{D}\mapsto\mathbf{D}$ is the l-analytic function on $\mathbf{D}$ induced by $\mathcal{J}_s,\ s>0$, then we have by the Ando–Fan extension of the Pick–Julia Theorem [3, Theorem 2 and Corollary, p. 31], (2.4.23) and (2.4.26),

$$\Phi(\widehat{\mathcal{J}_s}(A),I)\leq\frac{1}{1+s\beta}\,\Phi(A,I), \tag{2.4.27}$$

while by [15, Lemma 3] we get also

$$\|\Phi(\widehat{\mathcal{J}_s}(A), I)\| \leq \frac{1}{1+s\beta}\|\Phi(A, I)\|, \tag{2.4.28}$$

where Φ is defined just before Theorem 2.4.4.

Finally, note that equation (2.4.20) implies that

$$\lim_{s\to 0^+}\frac{1}{s}(\lambda - \mathcal{J}_s(\lambda)) = g(\lambda).$$

Hence

$$\lim_{s\to 0^+}\frac{1}{s}\left(A - \widehat{\mathcal{J}_s}(A)\right) = \hat{g}(A).$$

Therefore it follows by the product formula (see, for example, [37]) that for each $t \geq 0$,

$$\lim_{n\to\infty}[\widehat{\mathcal{J}}_{t/n}]^{[n]}(A) = \widehat{\mathcal{S}}(t)A.$$

This equality, (2.4.27) and (2.4.28) immediately imply (2.4.13) and (2.4.14) with $\gamma = \beta$, since $\lim\limits_{n\to\infty}(1+\dfrac{t}{n}\beta)^{-n} = \exp(-t\beta)$.

To prove the second part of our theorem, we note that (2.4.13) (as well as (2.4.14)) implies that

$$\phi_1(\mathcal{S}(t)(\lambda)) \leq \exp(-t\gamma)\phi_1(\lambda), \tag{2.4.29}$$

where ϕ_1 is defined in (2.4.21) with $w = 1$, when we set in these inequalities $A = \lambda I$.

Since both sides of (2.4.29) are equal at $t = 0^+$, differentiating them at $t = 0^+$ and setting $\lambda = r$ in (2.4.18), we obtain $\gamma \leq \beta$. The proof is complete. $\square$

3 Univalent star-like functions

3.1 General dynamic approach to star-like mappings in Banach spaces

Let M be a subset of a complex Banach space X.

Definition 3.1.1 *A subset M of X is said to be star-shaped if for each $w \in M$ and $t \geq 0$, the point $e^{-t}w$ also belongs to M.*

Definition 3.1.2 *If $\mathbf{D}$ is a domain in X, then a biholomorphic mapping $f \in \mathrm{Hol}(\mathbf{D}, X)$ is said to be a star-like mapping on $\mathbf{D}$ if its image $\Omega = f(\mathbf{D})$ is a star-shaped set.*

In addition, if $0 \in \Omega$ we will say that f is star-like with respect to an interior point; if $0 \in \partial\Omega$, the boundary of Ω, then we will say that f is star-like with respect to a boundary point.

The following result was announced in [12].

Theorem 3.1.1 *Let $\mathbf{D}$ be a domain in a complex Banach space X and let $g \in \mathcal{G}_+(\mathbf{D})$, that is, g is a holomorphic mapping on $\mathbf{D}$ such that the vector field*

$$T_g = g\,\frac{\partial}{\partial x} : \mathrm{Hol}(\mathbf{D}, X) \mapsto \mathrm{Hol}(\mathbf{D}, X) \tag{3.1.1}$$

is semi-complete. Then for each element f of $\mathrm{Ker}(I - T_g) \subset \mathrm{Hol}(\mathbf{D}, X)$, the set $f(\mathbf{D})$ is a star-shaped set.

Proof. Since T_g is semi-complete, it follows that it is the generator of a linear semigroup $\{\mathcal{L}(t)\}$ of composition operators defined on $\mathbf{D}$, i.e.,

$$\mathcal{L}(t)f = \exp\left[-tT_g\right]f = f(\mathcal{S}(t)), \qquad t \geq 0, \tag{3.1.2}$$

where $\{\mathcal{S}(t) : t \geq 0\}$ is the semigroup of holomorphic self-mappings of $\mathbf{D}$ generated by g.

Thus, if $f \in \mathrm{Ker}(I - T_g)$ we get by the exponential formula (1.1.15) and (3.1.2),

$$f(\mathcal{S}(t)) = \sum_{k=0}^{\infty} \frac{(-1)^k t^k}{k!} \cdot I^k f = e^{-t} f, \tag{3.1.3}$$

so, for each $x \in \mathbf{D}$, the point $e^{-t}f(x)$ belongs to $f(\mathbf{D})$, as required by Definition 3.1.1. $\square$

If $f \in \mathrm{Hol}(\mathbf{D}, X)$ is biholomorphic on $\mathbf{D}$, then the converse assertion is also true.

Theorem 3.1.2 *Let f be a biholomorphic mapping on a domain $\mathbf{D}$ in X such that $f(\mathbf{D})$ is star-shaped. Then there is a semi-complete holomorphic vector field T_g such that $f \in \mathrm{Ker}(I - T_g)$.*

Proof. Since f is biholomorphic, we may define $\mathcal{S}(t) = f^{-1}(e^{-t}f)$ so that $\{\mathcal{S}(t) : t \geq 0\}$ is a semigroup of holomorphic self-mappings on $\mathbf{D}$. In addition,

$$\left.\frac{d\mathcal{S}(t)x}{dt}\right|_{t=0^+} = -[Df^{-1}](f(x))f(x) = -[Df(x)]^{-1}(f(x))f(x) =: -g(x). \tag{3.1.4}$$

Thus for this g the vector field T_g is semi-complete, and it follows by the right equality in (3.1.4) that

$$T_g f = f, \tag{3.1.5}$$

i.e., $f \in \operatorname{Ker}(I - T_g)$. $\square$

Corollary 3.1.3 *A biholomorphic mapping f on a domain $\mathbf{D} \subset X$ is starlike if and only if it satisfies the differential equation*

$$f(x) = Df(x)g(x), \tag{3.1.6}$$

where g is the generator of a one-parameter semigroup of holomorphic self-mappings of $\mathbf{D}$.

Proof. Equation (3.1.6) is another form of the equation (3.1.5). $\square$

Corollary 3.1.4 *Let $\{\mathcal{S}(t) : t \geq 0\}$ be a one-parameter T-continuous semigroup of holomorphic self-mappings of a domain $\mathbf{D}$. If for some $F \in \operatorname{Hol}(\mathbf{D}, X)$ the strong limit*

$$f = \lim_{t\to\infty} e^t F(\mathcal{S}(t)) \tag{3.1.7}$$

exists, then $f(\mathbf{D})$ is a star-shaped set.

Proof. Since $\{\mathcal{S}(t)\}$ is T-continuous, it is differentiable with respect to $t \in [0, \infty)$ [39]. Hence the vector field $T_g = g\dfrac{\partial}{\partial x}$ is semi-complete for $g = -\left.\dfrac{d\mathcal{S}(t)}{dt}\right|_{t=0^+}$. Define the linear semigroup $\mathcal{B}(t) = e^t\mathcal{L}(t)$ on $\operatorname{Hol}(\mathbf{D}, X)$, where $\mathcal{L}(t)$ is the semigroup of composition operators generated by T_g.

Then (3.1.7) can be rewritten as

$$f = \lim_{t\to\infty} \mathcal{B}(t)F. \tag{3.1.8}$$

It follows by the semigroup property that f is a stationary point of the semigroup $\{\mathcal{B}(t)\}$, i.e.,

$$f = \mathcal{B}(t)f, \quad t \geq 0. \tag{3.1.9}$$

This equality can be written as

$$f = e^t f(\mathcal{S}(t)) \tag{3.1.10}$$

or

$$e^{-t} f = f(\mathcal{S}(t)), \tag{3.1.11}$$

and we are done. $\square$

Remark. It can be shown (see, for example, [13]) that if $g \in \mathcal{G}_+(\mathbf{D})$ is bounded and satisfies the conditions

$$g(a) = 0, \quad a \in \mathbf{D}, \tag{3.1.12}$$

and

$$Dg(a) = I_X, \tag{3.1.13}$$

then for F defined by $F(x) = x - a$, the limit f in (3.1.7) exists, and by (3.1.11), $f(\mathbf{D})$ is a star-shaped domain (with respect to $0 = f(a)$). In other words, under conditions (3.1.12) and (3.1.13), the equation

$$f(x) = Df(x)g(x)$$

can be solved, and for each $x \in \mathbf{D}$,

$$f(x) = \lim_{t\to\infty} e^t \left(\mathcal{S}(t)(x) - a\right), \tag{3.1.14}$$

where $\{\mathcal{S}(t) : t \geq 0\}$ is the semigroup generated by g.

3.2 Star-like l-analytic functions on J^*-algebras

Let $X = \mathfrak{A}$ be a unital J^*-algebra and let $\mathbf{D}$ be the open unit ball in $\mathfrak{A}$. The following assertion is a direct consequence of Theorems 2.1.8 and 3.1.1.

Theorem 3.2.1 *Let f be a holomorphic mapping on $\mathbf{D}$ which satisfies the following condition:*

$$f(A) = \Big[Df(A)\Big]\Big(p(A)A + B - AB^*A\Big)$$

for some $B \in \mathfrak{A}$ and $p \in \operatorname{Hol}(\mathbf{D}, \mathfrak{A})$ with $\operatorname{Re} p(A) \geq 0$. Then $f(\mathbf{D})$ is a star-shaped set.

For l-analytic star-like functions the converse assertion is also true.

Theorem 3.2.2 *Let $\mathfrak{A}$ be a unital J^*-algebra, let $\hat{f}$ be a univalent l-analytic function on $\mathbf{D}$, the open unit ball of $\mathfrak{A}$, and let $f \in \operatorname{Hol}(\Delta, \mathbb{C})$ be its producing function. Then the following conditions are equivalent:*

(a) $\hat{f}(\mathbf{D})$ is a star-shaped set;

(b) $f(\Delta)$ is a star-shaped set;

(c) there exists $g \in \mathcal{G}_+(\Delta)$ (respectively, $\hat{g} \in \widehat{\mathcal{G}_+}(\mathbf{D})$) such that

$$f(\lambda) = f'(\lambda)g(\lambda), \quad \lambda \in \Delta,$$

(respectively, $\hat{f}(A) = \hat{f}'(A)\hat{g}(A)$, $A \in \mathbf{D}$).

Proof. First we note that $f \in \operatorname{Hol}(\Delta, \mathbb{C})$ is univalent on Δ. Indeed, the equality $f(\lambda_1) = f(\lambda_2)$ for some $\lambda_1 \neq \lambda_2$ in Δ implies the equality $\hat{f}(\lambda_1 I) = \hat{f}(\lambda_2 I)$, which is impossible because $\hat{f}$ is univalent.

Now let (a) hold. Suppose that there is $\lambda \in \Delta$ and $t_0 \in (0, 1)$ such that the point $t_0 f(\lambda)$ does not belong to $f(\Delta)$. Increasing t_0 if necessary, we can assume that $t_0 f(\lambda) \in \partial\,(f(\Delta))$ while $tf(\lambda) \in f(\Delta)$ for all $t \in (t_0, 1]$.

On the other hand, $t_0\hat{f}(\lambda I) \in \hat{f}(\mathbf{D})$. Hence there is $A_0 \in \mathbf{D}$ such that $\hat{f}(A_0) = t_0\hat{f}(\lambda I)$. It is clear that $A_0 \in \mathbf{D} \setminus \{\lambda I : \ \lambda \in \Delta\}$. Denote

$$\rho = \inf_{\lambda \in \mathbb{C}} \|A_0 - \lambda I\| > 0$$

and define the set

$$\mathcal{V} = \left\{ A \in \mathbf{D} : \ \|A_0 - A\| < \frac{1}{2}\min(\rho, 1 - \|A\|) \right\}$$

which lies in $\mathbf{D} \setminus \{\lambda I : \ \lambda \in \Delta\}$.

Since $\mathcal{V}$ is open, so is $\hat{f}(\mathcal{V})$ by the last Remark in Section 1.3. Therefore the points $t\hat{f}(\lambda I)$ $(= tf(\lambda)I)$ belong to $\hat{f}(\mathcal{V})$ for all t close enough to t_0. On the other hand, for all $t_0 < t < 1$, $tf(\lambda) \in f(\Delta)$, hence $t\hat{f}(\lambda I) \in \hat{f}(\mathbf{D} \bigcap \{\lambda I : \ \lambda \in \Delta\})$. This contradicts the univalence of $\hat{f}$. Thus (a) implies (b).

Now suppose (b) holds. Then it follows by Corollary 3.1.3 that there is $g \in \mathcal{G}_+(\Delta)$ such that

$$f(\lambda) = f'(\lambda)g(\lambda)$$

(see equation (3.1.6)). Since the lifting mapping $f \mapsto \hat{f} : \mathrm{Hol}(\Delta, \mathbb{C}) \mapsto \widehat{\mathrm{Hol}}(\mathbf{D}, \mathfrak{A})$ is multiplicative,

$$\hat{f}(A) = \hat{f}'(A)\hat{g}(A),$$

and $\hat{g}$ belongs to $\widehat{\mathcal{G}_+}(\mathbf{D})$ by Theorem 2.1.1. Thus (b) implies (c).

Applying Corollary 2.1.2 and Theorem 3.1.1 (or again Corollary 3.1.3), we get the implication (c)$\Longrightarrow$(a). The proof is complete. $\square$

Corollary 3.2.3 *Let $\hat{f} \in \widehat{\mathrm{Hol}}(\mathbf{D}, \mathfrak{A})$ be a univalent l-analytic function. Then the following assertions are equivalent:*

(i) the set $\hat{f}(\mathbf{D})$ is star-shaped;

(ii) for each $A \in \mathbf{D}$ the following inequality holds:

$$\mathrm{Re}\left[A^* \left[\hat{f}'(A)\right]^{-1} \hat{f}(A)\right] \geq \mathrm{Re}\left[A^* \left[\hat{f}'(0)\right]^{-1} \hat{f}(0)(I - AA^*)\right];$$

(iii) there exists a unique $\tau \in \overline{\Delta}$ such that

$$\hat{f}(A) = (A - \tau I)(I - \bar{\tau} A)\hat{f}'(A)\hat{q}(A) \tag{3.2.1}$$

with $\mathrm{Re}\,\hat{q}(A) \geq 0, \ A \in \mathbf{D}$.

Moreover, if $\tau \in \Delta$, then $A = \tau I$ is the unique null point of $\hat{f}$ in $\mathbf{D}$ and $\hat{f}$ is star-like with respect to an interior point. If $\tau \in \partial\Delta$, then $\lim\limits_{\lambda \to \tau} \hat{f}(\lambda I) = 0$ and $\hat{f}$ is star-like with respect to a boundary point.

Proof. Let $\hat{f} \in \widehat{\mathrm{Hol}}(\mathbf{D}, \mathfrak{A})$ be a univalent l-analytic function. By Theorem 3.2.2, assertion (i) ($\hat{f}(\mathbf{D})$ is a star-shaped set) is equivalent to the requirement that

$$\hat{g}(A) := [\hat{f}'(A)]^{-1} \hat{f}(A)$$

belongs to the class $\widehat{\mathcal{G}_+}(\mathbf{D})$ of holomorphically accretive l-analytic functions.

In turn, this fact is equivalent to assertion (ii) by Theorem 2.1.12 (see inequality (2.1.14)).

Furthermore, the Berkson–Porta generalized representation of the class $\widehat{\mathcal{G}_+}(\mathbf{D})$ (see Remark 1 after Corollary 2.2.3, formula (2.2.9)) shows that (i) is equivalent to (iii).

Finally, since $\hat{f}'(A)$ is invertible, $\hat{f}$ vanishes in $\mathbf{D}$ if and only if $\hat{g}$ does. Thus, it follows by (3.2.1) that if $\tau \in \Delta$, then $\hat{f}(\tau I) = 0$ and $\hat{f}$ is star-like with respect to an interior point. Otherwise ($\tau \in \partial\Delta$), $\hat{f}$ is star-like with respect to a boundary point. □

Now using this Corollary, Theorem 2.4.1 and Proposition 2.4.2, we obtain the following general assertion.

Theorem 3.2.4 *Let $\hat{f} \in \widehat{\mathrm{Hol}}(\mathbf{D}, \mathfrak{A})$ be univalent on $\mathbf{D}$ and satisfy equation (3.2.1).*

(I) If $\tau \in \Delta$, then for each set of the form

$$\Omega_l = \left\{ A \in \mathbf{D} : \ \|(A - \tau I)(I - \bar{\tau} A)^{-1}\| < l \right\}, \quad l \in (0,1),$$

the set $\hat{f}(\Omega_l)$ is star-shaped (with respect to $0 = \hat{f}(\tau I)$);

(II) if $\tau \in \partial\Delta$, then for each set of the form

$$\Omega_\beta = \left\{ A \in \mathbf{D} : \ \left\| A - \frac{\tau}{1+\beta} I \right\| < \frac{\beta}{\beta+1} \right\}, \quad \beta > 0,$$

the set $\hat{f}(\Omega_\beta)$ is star-shaped with respect to a boundary point. (In this case there is a sequence $\lambda_n \to \tau$ such that $f(\lambda_n) \to 0$).

Proof. If τ in (3.2.1) belongs to Δ, then $A_0 = \tau I$ is a null point of the function $\hat{g} \in \widehat{\mathcal{G}_+}(\mathbf{D})$, where $\hat{g}(A) := (A - \tau I)(I - \bar{\tau} A)\hat{q}(A)$.

Now it follows by Theorem 2.4.1 that for each $l \in (0,1)$, the vector field

$$T_{\hat{g}} = \hat{g} \frac{\partial}{\partial A}$$

is semi-complete on $\Omega_l = \{A \in \mathbf{D} : \|(A - \tau I)(I - \bar{\tau} A)^{-1}\| < l\}$. Since $\hat{f} = T_{\hat{g}} \hat{f}$ (by (3.2.1)), it follows by Theorem 3.1.1 that $\hat{f}(\Omega_l)$ is a star-shaped set.

Similarly, by using Theorem 2.4.2 (instead of Theorem 2.4.1), one proves that if $\tau \in \partial\Delta$, then the sets $\hat{f}(\Omega_\beta)$, where

$$\Omega_\beta = \left\{ A \in \mathbf{D} : \left\| A - \frac{\tau}{1+\beta} I \right\| < \frac{\beta}{1+\beta} \right\},$$

are star-shaped for each $\beta > 0$.

Moreover, in this case the semigroup $\{\mathcal{S}(t) : t \geq 0\}$ converges to τ, uniformly on each compact subset of Δ. Hence, if we set $\lambda_n = \mathcal{S}(n)(0)$, then we get by (3.1.3) that $f(\lambda_n) = e^{-n} f(0)$ converges to zero as n tends to ∞. Theorem 3.2.4 is proved. □

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References

[1] M. Abate, Converging semigroups of holomorphic maps, *Atti Accad. Naz. Lincei Rend. Cl. Sci. Fis. Mat. Natur.* **82** (1988), 223–227.

[2] D. Aharonov, M. Elin, S. Reich and D. Shoikhet, Parametric representations of semi-complete vector fields on the unit balls in $\mathbb{C}^n$ and in Hilbert space, *Atti Accad. Naz. Lincei Rend. Cl. Sci. Fis. Mat. Natur.* **10** (1999), 229–253.

[3] T. Ando and Ky Fan, Pick–Julia theorems for operators, *Math. Z.* **168** (1979), 23–34.

[4] J. Arazy, An application of infinite dimensional holomorphy to the geometry of Banach spaces, *Lecture Notes in Math.* **1267** (1987), 122–150.

[5] E. Berkson and H. Porta, Semigroups of analytic functions and composition operators, *Michigan Math. J.* **25** (1978), 101–115.

[6] C.-H. Chu and P. Mellon, Jordan structures in Banach spaces and symmetric manifolds, *Exp. Math.* **16** (1998), 157–180.

[7] C. C. Cowen and B. D. MacCluer, *Composition Operators on Spaces of Analytic Functions*, CRC Press, Boca Raton, FL, 1995.

[8] J. Dieudonné, *Foundations of Modern Analysis*, Academic Press, New York and London, 1960.

[9] S. Dineen, *The Schwarz Lemma*, Clarendon Press, Oxford, 1989.

[10] N. Dunford and J. T. Schwartz, *Linear Operators*, Interscience, New York, 1958.

[11] P. L. Duren, *Theory of H^p Spaces*, Academic Press, New York, 1970.

[12] M. Elin, S. Reich and D. Shoikhet, A semigroup approach to the geometry of domains in complex Banach spaces, *Nonlinear Analysis* **47** (2001), 3271–3280.

[13] M. Elin, S. Reich and D. Shoikhet, Complex dynamical systems and the geometry of domains in Banach spaces, Technion Preprint Series No. 1059 (2001).

[14] Ky Fan, Analytic functions of a proper contraction, *Math. Z.* **160** (1978), 275–290.

[15] Ky Fan, Julia's lemma for operators, *Math. Ann.* **239** (1979), 241–245.

[16] Ky Fan, Iterations of analytic functions of operators, *Math. Z.* **179** (1982), 293–298.

[17] Ky Fan, Inequalities for proper contractions and strictly dissipative operators, *Linear Algebra Appl.* **105** (1988), 237–248.

[18] T. Franzoni and E. Vesentini, *Holomorphic Maps and Invariant Distances*, North-Holland, Amsterdam, 1980.

[19] K. Goebel and S. Reich, *Uniform Convexity, Hyperbolic Geometry and Nonexpansive Mappings*, Marcel Dekker, New York and Basel, 1984.

[20] G. M. Golusin, *Geometric Theory of Functions of a Complex Variable*, Amer. Math. Soc., Providence, RI, 1969.

[21] S. Gong, *Convex and Starlike Mappings in Several Complex Variables*, Science Press, Beijing–New York & Kluwer Acad. Publ., Dordrecht–Boston–London, 1998.

[22] A. W. Goodman, *Univalent Functions*, Vols. I, II, Mariner Publ. Co., Tampa, FL, 1983.

[23] K. R. Gurganus, Φ-like holomorphic functions in $\mathbb{C}^n$ and Banach space, *Trans. Amer. Math. Soc.* **205** (1975), 389–406.

[24] L. A. Harris, The numerical range of holomorphic functions in Banach spaces, *Amer. J. Math.* **93** (1971), 1005–1019.

[25] L. A. Harris, Bounded symmetric homogeneous domains in infinite-dimensional spaces, *Lecture Notes in Math.* **364** (1974), 13–40.

[26] L. A. Harris, The numerical range of functions and best approximation, *Proc. Camb. Phil. Soc.* **76** (1974), 133–141.

[27] L. A. Harris, A generalization of C^*-algebras, *Proc. London Math. Soc.* **42** (1981), 331–361.

[28] L. A. Harris, S. Reich and D. Shoikhet, Dissipative holomorphic functions, Bloch radii, and the Schwarz Lemma, *J. Analyse Math.* **82** (2000), 221–232.

[29] T. E. Harris, *The Theory of Branching Processes*, Springer, Berlin, 1963.

[30] L. F. Heath and T. J. Suffridge, Starlike, convex, close-to-convex, spirallike and ϕ-like maps in a commutative Banach algebra with identity, *Trans. Amer. Math. Soc.* **250** (1979), 195–212.

[31] U. Helmke and J. B. Moore, *Optimization and Dynamical Systems*, Springer, London, 1994.

[32] J. A. Hummel, Multivalent starlike functions, *J. Analyse Math.* **18** (1967), 133–160.

[33] J. M. Isidro and L. L. Stacho, *Holomorphic Automorphism Groups in Banach Spaces: An Elementary Introduction*, North Holland, Amsterdam, 1984.

[34] A. Lyzzaik, On a conjecture of M. S. Robertson, *Proc. Amer. Math. Soc.* **91** (1984), 108–110.

[35] Ch. Pommerenke, *Boundary Behavior of Conformal Maps*, Springer, Berlin, 1992.

[36] S. Reich and D. Shoikhet, Generation theory for semigroups of holomorphic mappings in Banach spaces, *Abstr. Appl. Anal.* **1** (1996), 1–44.

[37] S. Reich and D. Shoikhet, Semigroups and generators on convex domains with the hyperbolic metric, *Atti Accad. Naz. Lincei Rend. Cl. Sci. Fis. Mat. Natur.* **8** (1997), 231–250.

[38] S. Reich and D. Shoikhet, The Denjoy–Wolff theorem, *Ann. Univ. Mariae Curie–Sklodowska* **51** (1997), 219–240.

[39] S. Reich and D. Shoikhet, Metric domains, holomorphic mappings and nonlinear semigroups, *Abstr. Appl. Anal.* **3** (1998), 203–228.

[40] M. S. Robertson, Univalent functions starlike with respect to a boundary point, *J. Math. Anal. Appl.* **81** (1981), 327–345.

[41] W. Rudin, *Functional Analysis*, McGraw-Hill, New York, 1973.

[42] L. Schwartz, *Analyse Mathématique*, Hermann, Paris, 1967.

[43] B. A. Sevastyanov, *Branching Processes*, Nauka, Moscow, 1971.

[44] J. H. Shapiro, *Composition Operators and Classical Function Theory*, Springer, Berlin, 1993.

[45] D. Shoikhet, *Semigroups in Geometrical Function Theory*, Kluwer, Dordrecht, 2001.

[46] M. Silverman and E. M. Silvia, Subclasses of univalent functions starlike with respect to a boundary point, *Houston J. Math.* **16** (1990), 289–299.

[47] T. J. Suffridge, The principle of subordination applied to functions of several variables, *Pacific J. Math.* **33** (1970), 241–248.

[48] T. J. Suffridge, Starlike and convex maps in Banach space, *Pacific J. Math.* **46** (1973), 575–589.

[49] T. J. Suffridge, Starlikeness, convexity and other geometric properties of holomorphic maps in higher dimensions, *Complex Analysis (Proc. Conf. Univ. Kentucky, Lexington, KY, 1976), Lecture Notes in Math.* **599** (1977), 146–159.

[50] H. Upmeier, *Jordan Algebras in Analysis, Operator Theory and Quantum Mechanics*, CBMS Reg. Conf. Ser. in Math., **67**, Amer. Math. Soc., Providence, RI, 1987.

[51] E. Vesentini, Iterates of holomorphic mappings, *Uspekhi Mat. Nauk* **40** (1985), 13–16.

[52] E. Vesentini, Krein spaces and holomorphic isometries of Cartan domains, in *Geometry and Complex Variables*, Marcel Dekker, New York, 1991, 409–413.

[53] E. Vesentini, Semigroups of holomorphic isometries, in *Complex Potential Theory*, Kluwer, Dordrecht, 1994, 475–548.

[54] K. Yosida, *Functional Analysis*, Springer, Berlin, 1968.