A Bernstein-Markov Theorem for Normed Spaces

Lawrence A. Harris
Department of Mathematics, University of Kentucky
Lexington, Kentucky 40506-0027

Abstract

Let X and Y be real normed linear spaces and let $\phi : X \to \mathbb{R}$ be a non-negative function satisfying $\phi(x+y) \leq \phi(x) + \|y\|$ for all $x, y \in X$. We show that there exist optimal constants $c_{m,k}$ such that if $P : X \to Y$ is any polynomial satisfying $\|P(x)\| \leq \phi(x)^m$ for all $x \in X$, then $\|\hat{D}^k P(x)\| \leq c_{m,k}\phi(x)^{m-k}$ whenever $x \in X$ and $0 \leq k \leq m$. We obtain estimates for these constants and present applications to polynomials and multilinear mappings in normed spaces.

1. Introduction. This note considers the growth of the Fréchet derivatives of a polynomial on a normed linear space when the polynomial has restricted growth on the space. Our main concern is with real normed linear spaces. Here we obtain an estimate for the kth derivative of a polynomial bounded by an mth power where the constant $c_{m,k}$ in our estimate is best possible even when only the special case of the real line is considered. Moreover, we show that this inequality for the real line is equivalent to a Markov inequality for homogeneous polynomials. Our estimates are applied iteratively to obtain a bound for the values of a symmetric multilinear mapping where certain of its arguments are repeated. This bound is a constant multiple of the norm of the associated homogeneous polynomial.

Although we are unable to obtain a general formula for the constants $c_{m,k}$, we do establish elementary upper and lower bounds and provide a good estimate on their asymptotic growth. We determine the value of the constants in some low dimensional cases and find associated extremal polynomials with the aid of an interpolation formula for homogeneous polynomials.

In the case of complex normed linear spaces, we give a simple derivation of an estimate for the kth derivative which extends an inequality given in [2, Theorem 2] by allowing more general growth conditions. For comparison with the real case, we deduce an extension of an inequality of Bernstein and derive a bound for symmetric multilinear mappings where certain of its arguments are repeated. In both the real and complex cases, we show that equality holds in most of our estimates for some scalar-valued homogeneous polynomial defined on two dimensional ℓ^1 space.

2. Main results. Let m be a positive integer and let $0 \leq k \leq m$. We define $c_{m,k}$ to be the supremum of the values $|p^{(k)}(0)|$ where p varies through all polynomials satisfying

$$|p(t)| \leq (1+|t|)^m \tag{1}$$

for every $t \in \mathbb{R}$. Clearly any such polynomial p has degree at most m. Note that if p satisfies (1) then so does the polynomial $q(t) = t^m p(1/t)$. Hence $c_{m,k}/k! = c_{m,m-k}/(m-k)!$ for $0 \leq k \leq m$. In particular, $c_{m,0} = 1$ and $c_{m,m} = m!$.

Proposition 1.

$$\frac{m^m}{k^k(m-k)^{m-k}} \leq \frac{c_{m,k}}{k!} \leq \binom{m}{k} \frac{m^{m/2}}{k^{k/2}(m-k)^{(m-k)/2}} \tag{2}$$

for $0 \leq k \leq m$. *There exists an absolute constant* M *such that*

$$c_{m,k} \leq (Mm\log m)^k \tag{3}$$

for $0 \leq k \leq m$ *and* $m > 1$.

Estimates (2) and (3) follow from work of Sarantopoulos [9] and Nevai and Totik [8], respectively. See Section 5 for a proof and for values of some of the constants. For comparison with (3), note that (2) implies only that $c_{m,k} = O(m^{3k/2})$.

Theorem 2. *Let* X *and* Y *be normed linear spaces over* $\mathbb{F}$, *where* $\mathbb{F} = \mathbb{R}$ *or* $\mathbb{F} = \mathbb{C}$. *Let* $\phi : X \to \mathbb{R}$ *be a non-negative function satisfying*

$$\phi(x+y) \leq \phi(x) + \|y\| \tag{4}$$

for all $x, y \in X$ *and let* $P : X \to Y$ *be a polynomial satisfying*

$$\|P(x)\| \leq \phi(x)^m \tag{5}$$

for all $x \in X$. *If* $\mathbb{F} = \mathbb{R}$ *then*

$$\|\hat{D}^k P(x)\| \leq c_{m,k}\phi(x)^{m-k} \tag{6}$$

and if $\mathbb{F} = \mathbb{C}$ *then*

$$\|\hat{D}^k P(x)\| \le \frac{m^m k!}{k^k (m-k)^{m-k}} \phi(x)^{m-k} \tag{7}$$

whenever $x \in X$ *and* $0 \le k \le m$. *Moreover, in the case where* $X = \ell^1(\mathbb{F}^2)$, $Y = \mathbb{F}$ *and* $\phi(x) = \|x\|$, *for each inequality (6) and (7), there exists a homogeneous polynomial* P *as above and depending on* m *and* k *such that equality holds in the given inequality for some* x *with* $\phi(x)$ *any given non-negative number.*

Basic definitions and facts concerning polynomials on normed spaces can be found in [6] and [7]. Throughout, $\hat{D}^k P(x)$ denotes the homogeneous polynomial associated with the kth order Fréchet derivative $D^k P(x)$ and is given by

$$\hat{D}^k P(x) y = \frac{d^k}{dt^k} P(x+ty)\Big|_{t=0}.$$

In Theorem 2 and in all our other estimates we take $0^0 = 1$. The hypothesis (4) of Theorem 2 is satisfied, for example, when $\phi(x) = f(\|x\|)$, where $f : [0, \infty) \to \mathbb{R}$ is a continuous non-negative function satisfying $|f'(t)| \le 1$ for all $t > 0$. In particular, hypothesis (4) is satisfied when $\phi(x) = \|x\|$ and when $\phi(x) = (1 + \|x\|^p)^{1/p}$, where $p \ge 1$. It is easy to verify that this hypothesis is also satisfied when $\phi(x) = \max\{1, \|x\|\}$. The degree of any polynomial P satisfying (5) is at most m since

$$\|P(y)\| \le (M + \|y\|)^m, \qquad M = \phi(0),$$

for all $y \in X$ by (4).

Proof of Theorem 2. By composing P with a given linear functional and applying the Hahn-Banach theorem, we may suppose that $Y = \mathbb{F}$. Let $x, y \in X$ with $\|y\| \le 1$. Given $r > \phi(x)$, define

$$p(\alpha) = \frac{P(x + \alpha r y)}{r^m}.$$

Then $p(\alpha)$ is a polynomial with $p^{(k)}(0) = \hat{D}^k P(x)(ry)/r^m$ and

$$|p(\alpha)| \le \frac{\phi(x + \alpha r y)^m}{r^m} \le \left[\frac{\phi(x) + |\alpha| r}{r}\right]^m \le (1 + |\alpha|)^m$$

for all $\alpha \in \mathbb{F}$. If $\mathbb{F} = \mathbb{R}$ then $|p^{(k)}(0)| \le c_{m,k}$ by the definition of $c_{m,k}$. Hence,

$$\|\hat{D}^k P(x)\| \le c_{m,k} r^{m-k}$$

for all $r > \phi(x)$, and (6) follows.

If $\mathbb{F} = \mathbb{C}$, the function $p(\alpha)$ is entire and by the Cauchy estimates,

$$|p^{(k)}(0)| \leq \frac{k!(1+R)^m}{R^k},$$

for a given $R > 0$. Hence,

$$\|\hat{D}^k P(x)\| \leq \frac{k!(1+R)^m}{R^k} r^{m-k}$$

for all $r > \phi(x)$. Taking $R = k/(m-k)$ when $k \neq m$ and letting $R \to \infty$ otherwise, we obtain (7).

To prove the remainder of Theorem 2, recall that by definition X is $\mathbb{F}^2$ with the norm $\|x\| = |x_1| + |x_2|$, where $x = (x_1, x_2)$. Suppose $0 \leq k \leq m$. We first consider the case $\mathbb{F} = \mathbb{R}$. It follows from Proposition 1 that there exists a polynomial p satisfying (1) for which $p^{(k)}(0) = c_{m,k}$. Define a homogeneous polynomial $P : X \to \mathbb{R}$ of degree m by

$$P(x_1, x_2) = x_1^m p(\tfrac{x_2}{x_1}) \text{ for } x_1 \neq 0. \tag{8}$$

Then $|P(x)| \leq \|x\|^m$ for all $x \in X$ and

$$\hat{D}^k P(r,0)(0,1) = \frac{d^k}{dt^k} P(r,t)\Big|_{t=0} = r^{m-k} c_{m,k}.$$

Thus equality holds in (6) with $x = (r, 0)$ for $r \geq 0$.

We next consider the case $\mathbb{F} = \mathbb{C}$. Define

$$P(x_1, x_2) = M_k x_1^{m-k} x_2^k, \qquad M_k = \frac{m^m}{k^k (m-k)^{m-k}}. \tag{9}$$

Since the geometric mean is less than the arithmetic mean,

$$\left| \left(\frac{x_1}{m-k}\right)^{m-k} \left(\frac{x_2}{k}\right)^k \right|^{\frac{1}{m}} \leq \frac{|x_1| + |x_2|}{m},$$

so $|P(x)| \leq \|x\|^m$ for all $x \in X$. Moreover, $\hat{D}^k P(r,0)(0,1) = M_k k! r^{m-k}$. Thus equality holds in (7) with $x = (r, 0)$ for $r \geq 0$. $\square$

In the applications which follow, we use the notation

$$\begin{aligned} X_1 &= \{x \in X : \|x\| \leq 1\}, \\ \|P\| &= \sup\{\|P(x)\| : \ x \in X_1\} \end{aligned}$$

when $P : X \to Y$ is a polynomial. Since by definition a polynomial is a sum of continuous homogeneous polynomials, $\|P\|$ is finite and thus clearly a norm.

3. Applications to complex spaces. The complex case of Theorem 2 can be applied to obtain an extension of the Bernstein theorem given in [2, Corollary 2] to the case $\|x\| > 1$.

Corollary 3. *Let X and Y be complex normed linear spaces and let $P : X \to Y$ be a polynomial of degree at most m. Then*

$$\begin{aligned} \|\hat{D}^k P(x)\| &\leq \frac{m^m k!}{k^k (m-k)^{m-k}} \|P\|, & \|x\| \leq 1, \\ \|\hat{D}^k P(x)\| &\leq \frac{m^m k!}{k^k (m-k)^{m-k}} \|P\| \|x\|^{m-k}, & \|x\| > 1, \end{aligned}$$

for $0 \leq k \leq m$ and $x \in X$. Moreover, for each m and k there exists a non-trivial homogeneous polynomial P as above for which equality holds in the above inequalities (for some $x \in X$ with $\|x\|$ any given number ≥ 1) when $X = \ell^1(\mathbb{C}^2)$ and $Y = \mathbb{C}$.

Proof. As in the previous proof, we may suppose that $Y = \mathbb{C}$. Since the case where $P \equiv 0$ is obvious, we may also suppose that $\|P\| = 1$. By Theorem 2, all we need to establish is that (5) holds when $\phi(x) = \max\{1, \|x\|\}$. This is clear when $\|x\| \leq 1$. Let $x \in X$ with $\|x\| > 1$ and define $f(\lambda) = \lambda^m P(x/\lambda)$ for all complex $\lambda \neq 0$. Then f extends to a polynomial on $\mathbb{C}$ satisfying $|f(\lambda)| \leq \|x\|^m$ for all λ on the circle $|\lambda| = \|x\|$, so by the maximum principle, $|P(x)| = |f(1)| \leq \|x\|^m$. Thus (5) holds in all cases, as required. $\square$

One can apply Corollary 3 to obtain the case $p = 1$ of [2, Theorem 1] given next. See [3] for a discussion of this and related inequalities.

Corollary 4. *Let X and Y be complex normed linear spaces and let $F : X \times \cdots \times X \to Y$ be a continuous symmetric m-linear mapping with associated homogeneous polynomial $\hat{F}$ defined by $\hat{F}(x) = F(x, \ldots, x)$. If $x_1, \ldots, x_n$ are vectors in X_1, then*

$$\|F(x_1^{k_1} \cdots x_n^{k_n})\| \leq \frac{k_1! \cdots k_n!}{k_1^{k_1} \cdots k_n^{k_n}} \frac{m^m}{m!} \|\hat{F}\| \tag{10}$$

for all non-negative integers $k_1, \ldots, k_n$ with $k_1 + \cdots + k_n = m$. The constant in this inequality cannot be replaced by a smaller one.

Here and elsewhere,

$$F(x_1^{k_1} \cdots x_n^{k_n}) = F(\underbrace{x_1, \ldots, x_1}_{k_1}, \ldots, \underbrace{x_n, \ldots, x_n}_{k_n}).$$

Proof. Define $f(k) = k^k/k!$ for all non-negative integers k. By the binomial theorem for homogeneous polynomials [6, Th. 26.2.3],

$$\frac{1}{k!}\hat{D}^k\hat{F}(x)y = \binom{m}{k} F(x^{m-k}y^k)$$

and hence by Corollary 3 (or Theorem 2 with $\phi(x) = \|x\|$),

$$\|F(y^k x^{m-k})\| \leq \frac{f(m)}{f(k)f(m-k)}\|\hat{F}\| \tag{11}$$

for $x, y \in X_1$ and $0 \leq k \leq m$. Now $x \to F(x_1^{k_1}x^{m-k_1})$ is a homogeneous polynomial of degree $m - k_1$ and hence (11) applies again to show that

$$\|F(x_1^{k_1}x_2^{k_2}x^{m-k_1-k_2})\| \leq \frac{f(m)}{f(k_1)f(k_2)f(m-k_1-k_2)}\|\hat{F}\|$$

for $x \in X_1$. Continuing in this way, we obtain (10).

An example given in [2, p. 148] (which generalizes (9)) shows that the constant in (10) is best possible. □

4. Applications to real spaces. The following is an immediate consequence of Theorem 2 with $\phi(x) = 1 + \|x\|$.

Corollary 5. *Let X and Y be real normed linear spaces. If $P : X \to Y$ is a polynomial satisfying $\|P(x)\| \leq (1 + \|x\|)^m$ for all $x \in X$, then*

$$\|\hat{D}^k P(x)\| \leq c_{m,k}(1 + \|x\|)^{m-k}$$

whenever $x \in X$ and $0 \leq k \leq m$.

It is easy to see from Corollary 5 with $X = Y = \mathbb{R}$ that

$$c_{m,n} \leq c_{m,k}\, c_{m-k,n-k}$$

whenever $0 \leq n \leq m$ and $0 \leq k \leq n$.

Theorem 6. *Let X and Y be real normed linear spaces. If $P : X \to Y$ is a homogeneous polynomial of degree m, then*

$$\|\hat{D}^k P\| \leq c_{m,k}\|P\|$$

for $0 \leq k \leq m$. Moreover, for each m and k there exists a non-zero polynomial P as above for which equality holds when $X = \ell^1(\mathbb{R}^2)$ and $Y = \mathbb{R}$.

Corollary 7. *Let X and Y be real normed linear spaces and let $F : X \times \cdots \times X \to Y$ be a continuous symmetric m-linear mapping with associated homogeneous polynomial $\hat{F}$. If $x_1, \ldots, x_n$ are vectors in X_1, then*

$$\|F(x_1^{k_1} \cdots x_n^{k_n})\| \leq \sqrt{\frac{m^m}{k_1^{k_1} \cdots k_n^{k_n}}} \|\hat{F}\| \tag{12}$$

for all non-negative integers $k_1, \ldots, k_n$ with $k_1 + \cdots + k_n = m$.

The case $n = 2$ of the above corollary is essentially part (a) of the Corollary in [9]. The constants we give are rather far from being the best. For example, when $k_1 = 1, \ldots, k_n = 1$, the constant in (12) is $m^{m/2}$ while the best constant in this case (determined by R. S. Martin in 1932) is $m^m/m!$. The problem of determining the best constant in (12) is open. (See [3].)

Proof of Theorem 6 and Corollary 7. Theorem 6 follows from the fact that if $P \not\equiv 0$, Theorem 2 applies with P replaced by $P/\|P\|$ and $\phi(x) = \|x\|$. Thus (12) follows from the upper bound given in Proposition 1 and the proof of Corollary 4 with $f(k) = k^{k/2}$. □

Note that in Corollary 5, a bound on the derivative which holds for $X = Y = \mathbb{R}$ continues to hold when X and Y are any real normed linear spaces. According to a theorem of Sarantopoulos [9, Theorem 2], this is also the case with Markov's inequality for the first derivative. It is an open question whether Markov's inequality for the higher derivatives continues to hold for arbitrary real Banach spaces. However, in general, one cannot expect Bernstein-type estimates for polynomials on the real line to hold for arbitrary real (or even complex) Banach spaces.

For example, it was shown by S. N. Bernstein [1, p. 56] that

$$|q'(t)| \leq m(1 + t^2)^{\frac{m-1}{2}} \tag{13}$$

for any polynomial q satisfying $|q(t)| \leq (1 + t^2)^{m/2}$ for all $t \in \mathbb{R}$. Suppose $X = \ell^1(\mathbb{R}^2)$ and let $P : X \to \mathbb{R}$ be the homogeneous polynomial defined by (9) with $k = 1$. Then, $|P(x)| \leq (1 + \|x\|^2)^{m/2}$ for all $x \in X$ since $\|P\| \leq 1$. If the result analogous to (13) held for X, we would have

$$\|DP(x)\| \leq m(1 + \|x\|^2)^{\frac{m-1}{2}}$$

for all $x \in X$. Now $x \to DP(x)$ is a homogeneous polynomial of degree $m - 1$. Hence, replacing x by tu, where $u \in X$ with $\|u\| \leq 1$ and letting $t \to \infty$, we obtain $\|DP(u)\| \leq m$. Thus, $M_1 \leq m$, which is impossible when $m > 1$.

Inequality (13) and other classical polynomial inequalities are extended to Hilbert spaces in [2] and [4].

5. Estimating $c_{m,k}$.

Proof of Proposition 1. The lower bound for $c_{m,k}$ in (2) follows with $p(t) = at^k$, where a is the minimum of the function $(1+t)^m/t^k$ for $t > 0$. This minimum occurs at $t = k/(m-k)$. To obtain the upper bound in (2), suppose p satisfies (1) and put $q(t) = p(st)$, where $s > 0$. Then $q^{(k)}(0) = s^k p^{(k)}(0)$ and by the Cauchy-Schwarz inequality,

$$|q(t)| \le (1 + s|t|)^m \le (1+s^2)^{m/2}(1+t^2)^{m/2}.$$

Hence,

$$|p^{(k)}(0)| \le k! \binom{m}{k} \frac{(1+s^2)^{m/2}}{s^k}$$

by k applications of (13). The minimum of the right-hand side of the above occurs when $s^2 = k/(m-k)$ and this gives the asserted estimate. (Compare [9, p. 311].)

To prove (3), suppose p satisfies (1) and put $q(t) = p(t/m)$. Then q is a polynomial of degree at most m satisfying

$$|q(t)| \le \left(1 + \frac{|t|}{m}\right)^m \le e^{|t|}$$

for all $t \in \mathbb{R}$. By k applications of a Markov-Bernstein theorem given in [8, Theorem 3] with weight $\exp(-|t|)$, we obtain

$$\sup_{-\infty<t<\infty} |e^{-|t|}q^{(k)}(t)| \le (M \log m)^k \sup_{-\infty<t<\infty} |e^{-|t|}q(t)|,$$

where M is an absolute constant. Thus

$$|p^{(k)}(0)| = |m^k q^{(k)}(0)| \le (Mm \log m)^k,$$

completing the proof. (I am grateful to Prof. Tamás Erdélyi for providing the argument in the above paragraph.) □

In view of Theorems 2 and 6, it is important to know the value of $c_{m,k}$ as accurately as possible, especially for the case $k = 1$. Below is a table of a few values with a corresponding extremal polynomial which attains this value. Note that these extremal polynomials can be converted to extremal polynomials for Theorem 6 using (8).

m	$c_{m,k}$	Extremal polynomial $p(t)$
1	$c_{1,1} = 1$	$t + b,\ \|b\| \le 1$
2	$c_{2,1} = 4$	$c_{2,1}t$
3	$c_{3,1} = 6.976850$	$c_{3,1}t - t^3$
4	$c_{4,1} = 6\sqrt{3}$	$c_{4,1}t(1 - t^2)$
	$c_{4,2} = 36$	$\frac{1}{2}c_{4,2}t^2 - t^4 - 1$
6	$c_{6,1} = 17.61468$	$c_{6,1}t(1 - t^2)^2 - 64t^3$
	$c_{6,3} = 595.3761$	$\frac{1}{12}c_{6,3}t(1 - t^2)^2 - 32t(1 + t^4)$

The values of $c_{m,1}$ and the corresponding extremal polynomials given in the table can be deduced from the estimate below.

Lemma 8. *If $t_1, \ldots, t_m$ are any distinct real numbers, then*

$$c_{m,1} \le \sum_{i=1}^{m} \frac{(1 + |t_i|)^m}{\prod_{j \ne i} |t_i - t_j|} \tag{14}$$

To obtain the extremal polynomials of the table, choose the interpolation points $t_1, \ldots, t_m$ symmetric with respect to the origin (with the origin included for odd values of m) and select the positive points from t, $1/t$ and 1, in that order, where $t > 0$. The value of $c_{m,1}$ is obtained by minimizing over t.

For example, when $t_1 = t$, $t_2 = 1/t$, $t_3 = -t_2$, $t_4 = -t_1$, where $0 < t < 1$, the right-hand side of (14) reduces to

$$f(t) = \frac{t(1+t)^4}{1 - t^4} + \frac{(1+t)^4}{t(1 - t^4)} = \frac{(1+t)^4}{t(1 - t^2)} = \frac{(1+t)^3}{t(1-t)}.$$

Let $a = \min\{f(t) : 0 < t < 1\}$ and define $p(t) = at(1 - t^2)$. Then $c_{4,1} \le a$ by (14) and clearly $|p(t)| \le (1 + t)^4$ for $0 < t < 1$. It follows from the identities $p(-t) = -p(t)$ and $t^4 p(1/t) = -p(t)$, that $|p(t)| \le (1 + |t|)^4$ for all $t \in \mathbb{R}$. Since $p'(0) = a$, we have $c_{4,1} = a$. By calculus, $a = f(2 - \sqrt{3}) = 6\sqrt{3}$.

The entries in the table for $c_{4,2}$ and $c_{6,3}$ can be obtained from an interpolation formula for homogeneous polynomials given in [3, (4)]. (The oversize Γ given there is a misprint for the symbol denoting a product.) For the case of $c_{4,2}$, take the interpolation sets to be $-r, 0, r$ and $-s, 0, s$ and set $t = r/s$. For the case of $c_{6,3}$, take both interpolation sets to be $-s, -r, r, s$ and again set $t = r/s$. It is easy to deduce that $c_{6,3} = 12(c_{6,1} + 32)$ by comparison of the expressions being minimized; hence the last two extremal polynomials in the table are the same.

Proof of Lemma 8. One can deduce the estimate (14) easily from the interpolation formula given in [3, (4)], where $t_1, \ldots, t_m$ and $1, -1$ are the sets of interpolation points and where W is the homogeneous polynomial P defined by (8).

To give a direct proof, suppose p satisfies (1) and define

$$q(t) = t^m p_1\left(\frac{1}{t}\right), \quad \text{where } p_1(t) = \frac{p(t) - p(-t)}{2}.$$

Then q extends to a polynomial on $\mathbb{R}$ of degree at most $m-1$ satisfying $|q(t)| \leq (1+|t|)^m$ for all $t \in \mathbb{R}$. Moreover, the coefficient of t^{m-1} in q is the coefficient of t in p, i.e., $p'(0)$. By equating the coefficients of t^{m-1} in both sides of the Lagrange interpolation formula for q, we obtain

$$p'(0) = \sum_{i=1}^{m} \frac{q(t_i)}{\prod_{j \neq i}(t_i - t_j)}$$

and (14) follows. □

See [5] for further discussion of the determination of the values $c_{m,k}$ and related problems.

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