Dissipative Holomorphic Functions, Bloch Radii, and the Schwarz Lemma

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1. Introduction

The Hille-Yosida and Lumer-Phillips theorems play an important role in the theory of linear operators and its applications to evolution equations, probability and ergodic theory. (See, for example, [17] and [9].) Different nonlinear generalizations and analogues of these theorems can be found, for instance, in [13] and [2].

We are interested in establishing analogues of these theorems for the class of holomorphic functions which we view as a natural extension of the class of continuous linear operators.

An analogue of the Hille-Yosida theorem for holomorphic functions was given in [14]. See also [15, 1]. Also, certain versions of the Hille-Yosida theorem as well as the Lumer-Phillips theorem were presented for nonlinear continuous operators in the works of R. H. Martin [10, 11]. His results are based on a nonlinear analogue of dissipativeness and the so-called flow-invariance condition for nonlinear functions.

In the present paper, we define the notion of dissipativeness for holomorphic functions in a somewhat different way that generalizes the usual definition for the linear case and remains equivalent to the flow invariance condition. Our definition follows naturally from the extension of the numerical range to holomorphic functions given in [7].

For the case of bounded linear operators, the Lumer-Phillips theorem can be formulated as follows:

Let X be a complex Banach space and let $A : X \to X$ be a bounded linear operator on X. Then the numerical range of A lies in the closed left half-plane (i.e., A is dissipative [17, p. 250]) if and only if its resolvent $(I - tA)^{-1}$ is well defined on X for all $t > 0$ and satisfies the condition

$$\|(I - tA)^{-1}\| \leq 1, \quad t > 0.$$

In other words, for all $t \geq 0$, the operator $(I - tA)^{-1} : X \to X$ is a contraction on X.

Our object is to prove an analogue of this fact for holomorphic functions and to use it to find new estimates for Bloch radii, i.e., the radii of balls centered at the origin where a normalized holomorphic function has a well-defined inverse that maps one ball into the other. We also prove a distortion form of the Schwarz lemma, which is used in our arguments.

2. Dissipative holomorphic functions

Let D be a convex domain in a complex Banach space X and suppose D contains the origin. Let $h : D \to X$ be a holomorphic function. Define

$$h_s(x) = h(sx), \quad 0 < s < 1,$$

and note that h_s is holomorphic in the enveloping domain $\frac{1}{s}D$. Clearly this domain contains $\mathrm{Cl}(D)$, the closure of D.

Let ∂D denote the boundary of D. For each $x \in \partial D$, let $J(x)$ be the set of all continuous linear functionals on X which are tangent to D at x, i.e.,

$$J(x) = \{\ell \in X^* : \ \ell(x) = 1, \ \mathrm{Re}\, \ell(y) \leq 1 \text{ for all } y \in D\}.$$

Thus J maps ∂D into 2^{X^*}. For each $x \in \partial D$, it is known [3, Cor. 6, p. 449] that $J(x) \neq \emptyset$. Let $Q(x)$ be a non-empty subset of $J(x)$.

Definition 1. If h has a continuous extension to $\mathrm{Cl}(D)$, the *numerical range* of h (taken with respect to Q) is the set

$$W(h) = \{\ell(h(x)) : \ \ell \in Q(x), \quad x \in \partial D\}.$$

We write V in place of W in this definition when the numerical range is taken with respect to $J(x)$. Note that in the case where D is the open unit ball

$$B = \{x \in X : \ \|x\| < 1\},$$

Definition 1 agrees with that given by L. A. Harris in [7].

Definition 2. A holomorphic function $h : D \to X$ is *dissipative* if

$$L(h) := \lim_{s \to 1^-} \sup \mathrm{Re}\, V(h_s) \leq 0.$$

When D is balanced, one can show using the maximum principle that $\sup \mathrm{Re}\, V(h_s)$ is an increasing function of s for $s > 0$ and hence the limit in the above definition exists.

Let $h : B \to X$ be a holomorphic function and put

$$\|h\| = \sup\{\|h(x)\| : \ x \in B\},$$

when this is finite. Suppose h has a uniformly continuous extension to $\mathrm{Cl}(B)$. Then h is dissipative if and only if $\sup \mathrm{Re}\, V(h) \leq 0$ since $\|h - h_s\| \to 0$ as $s \to 1^-$. Moreover, as shown in [7, Theorem 2],

$$\lim_{t \to 0^+} \frac{\|I + th\| - 1}{t} = \sup \mathrm{Re}\, W(h).$$

Thus, even though the numerical range may differ in general when taken with respect to different choices of Q, the number sup $\mathrm{Re}\, W(h)$ remains fixed. Hence we can replace it by sup $\mathrm{Re}\, V(h)$. In fact, this remains true for functions that do not necessarily have a continuous extension to $\mathrm{Cl}(B)$. (See the end of this section.) For the case of the open unit ball, the only properties of $J(x)$ our arguments use are that $J(x) \neq \emptyset$ and

$$\overline{\lambda} J(\lambda x) \subseteq J(x) \tag{1}$$

whenever $|\lambda| = 1$ and $\|x\| = 1$.

Our main result is the following extension of the Lumer-Phillips theorem.

Theorem 1. *Let B be the open unit ball of a complex Banach space X and let $h : B \to X$ be holomorphic. Then h is dissipative if and only if $(I - th)(B) \supseteq B$ and $(I - th)^{-1}$ is a well-defined holomorphic mapping of B into itself for each $t \geq 0$.*

For $r > 0$ define B_r to be the ball of radius r in X with center at the origin. Thus,

$$B_r = \{x \in X : \ \|x\| < r\},$$

$B_1 = B$ and $B_r = rB$.

Corollary 2. *Let $h : B \to X$ be holomorphic and suppose $L(h) < 0$. Then h has a unique null point in B. More generally, the equation*

$$h(x) = y$$

has a unique solution $x \in B$ for each $y \in B_\delta$, where $\delta = -L(h)$.

Proof. Since

$$L(h + \delta I) \leq L(h) + \delta = 0,$$

the function $h + \delta I$ is dissipative. Also,

$$I - t(h + \delta I) = -th$$

when $t = 1/\delta$. Hence, by part (b) of Theorem 1, $h(B) \supseteq B_\delta$ and $h^{-1} : B_\delta \to B$ is well defined. □

Clearly Theorem 1 is a consequence of Corollary 9 (below) and the following result.

Theorem 3. *Let D be a convex domain in X containing the origin and let $h : D \to X$ be holomorphic. Label assertions as follows:*
(a) h is dissipative.
(b) For each $t \geq 0$, $(I - th)(D) \supseteq D$ *and* $(I - th)^{-1}$ *is a well-defined holomorphic mapping of D into itself.*
If h_s is bounded on D for each s with $0 < s < 1$ *and if D is bounded, then* $(a) \Rightarrow (b)$*.*
If D is balanced, then $(b) \Rightarrow (a)$*.*

Proof. To show that (a) implies (b), let h be as given. Fix $y \in D$ and $t > 0$. Define (suspending our subscript convention)

$$g_s(x) = \frac{1}{s}\,(y + th(sx))$$

for $x \in D$ and $0 < s < 1$. Given $x \in \partial D$ and $\ell \in J(x)$, we have

$$\operatorname{Re}\ell\,(g_s(x)) \leq \frac{1}{s}\,(\operatorname{Re}\ell(y) + t\alpha_s)\,,$$

where $\alpha_s = \sup \operatorname{Re} V(h_s)$. Let p be the Minkowski functional for D. Since $p(y) < 1$, by (a) there exists a constant $k < 1$ and a number $\delta > 0$ such that

$$\frac{1}{s}\,(p(y) + t\alpha_s) \leq k$$

whenever $1 - \delta < s < 1$. Then $f := I - rg_s$ satisfies

$$p(f(x)) \geq \operatorname{Re}\ell\,(x - rg_s(x)) \geq 1 - rk \tag{2}$$

whenever $r \geq 0$.

The space X is an F-space with respect to p since D is open and bounded. Moreover, by hypothesis and the mean value theorem (see [7, Lemma 2]), there exists a number M_s satisfying

$$p(g_s(x) - g_s(y)) \leq M_s p(x - y), \quad x, y \in D$$

for each s with $0 < s < 1$. Fix s with $1 - \delta < s < 1$. Then by the inverse function theorem (see [16, p. 14]), for all small $r > 0$, the function f maps $\mathrm{Cl}(D)$ homeomorphically onto a subset of X containing 0. Hence

$$f(D) \supseteq (1 - rk)D$$

by (2). Therefore the function

$$F := f^{-1}\left((1 - r)I\right)$$

is a holomorphic mapping of cD into D, where

$$c = (1 - rk)/(1 - r).$$

Clearly $c > 1$.

Since D lies strictly inside cD, the Earle-Hamilton theorem [5] implies that F has a unique fixed point x^* in D. This is also a fixed point for g_s and hence $z = sx^*$ satisfies

$$(I - th)(z) = y. \tag{3}$$

Suppose now that there is another solution z_1 of (3). Choose $s_1 < 1$ so that $s_1 > s$ and $s_1 > p(z_1)$. Clearly $x_1 = z_1/s_1$ is in D and satisfies $g_{s_1}(x_1) = x_1$. One can show as above that x_1 is the unique fixed point of g_{s_1} in D. On the other hand, if we set $x_2 = z/s_1$, then $x_2 \in D$ and $g_{s_1}(x_2) = x_2$ by (3). Hence, $x_2 = x_1$, so $z_1 = z$.

It can be shown (as, for example, in [8, Theorem 5]) that the solution z of (3) depends holomorphically on $y \in D$. Thus (b) holds.

To prove that (b) implies (a), let $x \in \partial D$ and let $\ell \in J(x)$. Since D is balanced, $\lambda x \in D$ whenever $|\lambda| < 1$ and $|\ell(y)| \leq 1$ whenever $y \in D$. Define

$$f(\lambda, t) = \ell((I - th)^{-1}(\lambda x)).$$

Then f satisfies the hypotheses of Lemma 4 (below). By definition,

$$(I - th)^{-1}(\lambda x) = \lambda x + th\left((I - th)^{-1}(\lambda x)\right),$$

and it follows that

$$\frac{\partial f(\lambda,0)}{\partial t} = \ell\left(h(\lambda x)\right).$$

Hence, it follows from (4) with $\lambda = s$ that

$$\operatorname{Re}\ell(h_s(x)) \le (1-s^2)\operatorname{Re}\ell(h(0)).$$

Therefore,

$$\sup \operatorname{Re} V(h_s) \le (1-s^2)p(h(0)),$$

and thus (a) holds. □

Lemma 4. *Let Δ be the open unit disc of the complex plane and suppose $f : \Delta \times [0,1) \to \Delta$ is a function which is holomorphic in the first variable and right differentiable in the second at $t = 0$. If $f(\lambda, 0) = \lambda$ for all $\lambda \in \Delta$,* then

$$\operatorname{Re}\overline{\lambda}\,\frac{\partial f(\lambda,0)}{\partial t} \le (1-|\lambda|^2)\operatorname{Re}\overline{\lambda}\,\frac{\partial f(0,0)}{\partial t}, \qquad \lambda \in \Delta. \tag{4}$$

Proof. Define

$$g(\lambda,t) = \frac{f(\lambda,t)-f(0,t)}{1-f(\lambda,t)\overline{f(0,t)}}.$$

Then

$$|g(\lambda,t)| \le |\lambda|$$

for $\lambda \in \Delta$ and $0 \le t < 1$ by the Schwarz Lemma. Hence,

$$\operatorname{Re}\overline{\lambda}\,\frac{\partial g(\lambda,0)}{\partial t} \le 0 \tag{5}$$

since

$$\operatorname{Re}\overline{\lambda}[g(\lambda,t)-g(\lambda,0)] \le 0.$$

By the quotient rule,

$$\frac{\partial g(\lambda,0)}{\partial t} = \frac{\partial f(\lambda,0)}{\partial t} - \frac{\partial f(0,0)}{\partial t} + \lambda^2\overline{\frac{\partial f(0,0)}{\partial t}}. \tag{6}$$

The required inequality (4) follows from (5) and (6). □

Note that our proof that (a) implies (b) in Theorem 3 used only the hypothesis that

$$\lim_{s\to 1^-} \sup \operatorname{Re} W(h_s) \leq 0.$$

Since V can be replaced by W in Corollary 9, the same holds for Theorem 1. It follows that if $h : B \to X$ is holomorphic, then

$$L(h) = \lim_{s\to 1^-} \sup \operatorname{Re} W(h_s).$$

Indeed, suppose M is the right-hand side of this equality and M is finite. Then $h - MI$ satisfies the mentioned hypothesis, and hence $L(h - MI) \leq 0$ by Theorem 1. Therefore, $L(h) \leq M$. On the other hand, by definition, $W(h_s) \subseteq V(h_s)$, so $M \leq L(h)$.

3. Applications to Bloch radii

Definition 3. We say that positive numbers r and P are *Bloch radii* for h if $h(B_r) \supseteq B_P$ and $h^{-1} : B_P \to B_r$ is a well-defined holomorphic function.

For example, if $h(0) = 0$ and $Dh(0) = I$, then the inverse function theorem shows that Bloch radii exist for h. We find Bloch radii that hold for all members of a class of holomorphic functions with restricted numerical range. See [8] for Bloch radii under various norm restrictions.

Our approach to this problem is based on the following refinement of Corollary 2:

Lemma 5. *Let $h : B \to X$ be a holomorphic function and suppose $N(s)$ is an lower semi-continuous function on (0,1) satisfying*

$$\sup \operatorname{Re} V\left((I - h)_s\right) \leq N(s), \quad 0 < s < 1.$$

Suppose also that $\Phi(s) = s - N(s)$ assumes its maximum value P at r and that $P > 0$. *Then r and P are Bloch radii for h.*

Proof. Put

$$f := PI - h_r.$$

If $0 < s < 1$, $\|x\| = 1$ and $\ell \in J(x)$, then

$$\begin{aligned}\operatorname{Re}\ell(f_s(x)) &= \operatorname{Re}\ell(Psx) + \operatorname{Re}\ell\left((I-h)_{rs}(x)\right) - \operatorname{Re}\ell(rsx) \\ &\leq Ps + N(rs) - rs,\end{aligned}$$

so

$$\lim_{s\to 1^-} \sup \operatorname{Re} V(f_s) \leq P - \Phi(r) = 0.$$

Hence, f is dissipative. Thus Lemma 5 follows from part (b) of Theorem 1 with $t = 1/P$, since then $I - tf = th_r$. □

Theorem 6. *Let $h : B \to X$ be a holomorphic function with $h(0) = 0$ and $Dh(0) = I$. If $L(h) \leq M$, then $M \geq 1$ and Bloch radii for h are given by*

$$r = 1 - \sqrt{1 - \frac{1}{2M-1}}, \quad P = \left(\sqrt{2M-1} - \sqrt{2\,(M-1)}\right)^2.$$

Proof. Without loss of generality, we may assume that $L(h) < M$. Then

$$\sup \operatorname{Re} V(h_s) < sM, \quad 1 - \delta < s < 1 \tag{7}$$

for some $\delta > 0$. Let $\|x\| = 1$ and $\ell \in J(x)$. The function

$$f(\lambda) = \frac{1}{s\lambda}\ell\left(h(s\lambda x)\right)$$

is holomorphic on Δ with $f(0) = 1$ and

$$\operatorname{Re} f(\lambda) < M$$

whenever $|\lambda| = 1$ by (1) and (7). The same inequality holds for all $\lambda \in \Delta$ by the maximum principle. In particular, $M > 1$.

Putting

$$g(\lambda) = \frac{M - f\,(\lambda)}{M-1},$$

we have that g is a function of the class of Carathéodory, i.e.,

$$\operatorname{Re} g\,(\lambda) \geq 0 \quad \text{and} \quad g\,(0) = 1. \tag{8}$$

Therefore, as is well known [12, p. 169],

$$\operatorname{Re} g\,(\lambda) \le \frac{1+\mid\lambda\mid}{1-\mid\lambda\mid},$$

so

$$\operatorname{Re} f\,(\lambda) \ge M - (M-1)\,\frac{1+\mid\lambda\mid}{1-\mid\lambda\mid}. \tag{9}$$

Let $\lambda = t$ in (9), where $0 < t < 1$. Then

$$\begin{aligned} \operatorname{Re} \ell\left(tx - \frac{1}{s}h(stx)\right) &= t - \operatorname{Re} t f(t) \\ &\le \frac{2(M-1)}{1-t}t^2 := N(t). \end{aligned}$$

Letting $s \to 1^-$, we obtain

$$\sup \operatorname{Re} V((I-h)_t) \le N(t).$$

A calculation shows that $\Phi(t) = t - N(t)$ assumes its maximum at r and that $\Phi(r) = P$. Hence, by Lemma 5, the function h has Bloch radii r and P. □

Recall that the numerical radius $|V(h)|$ is defined in [7] by

$$|V(h)| = \lim_{s \to 1^-} |V(h_s)|.$$

One can take $M = |V(h)|$ or $M = \|h\|$, provided M is finite, since

$$L(h) \le |V(h)| \le \|h\|.$$

It was shown in [8, Theorem 3] that if $M = \|h\|$ and if h is normalized so that $h(0) = 0$ and $Dh(0) = I$, then

$$r_1 = \frac{1}{\sqrt{4M^2+1}}, \quad P_1 = \frac{1}{2M + \sqrt{4M^2+1}}$$

are Bloch radii for h. This was deduced from the inequality

$$\|(I-h)(x)\| \le 2M\left(1 - \sqrt{1 - \|x\|^2}\right), \quad x \in B, \tag{10}$$

by using a version of the inverse function theorem that depends on the Earle-Hamilton fixed point theorem. Alternately, starting from (10), one arrives at these same values by applying Lemma 5 with

$$N(s) = 2M(1 - \sqrt{1 - s^2}).$$

It is easy to verify that when $M = \|h\|$,

$$\begin{aligned} P_1 \geq P \quad &\text{for} \quad M \geq 1 + \frac{1}{\sqrt{3}}, \\ P > P_1 \quad &\text{for} \quad M < 1 + \frac{1}{\sqrt{3}}. \end{aligned}$$

Thus the value of P obtained from Theorem 6 improves the value given previously in [8, Theorem 3] when $\|h\| < 1 + 1/\sqrt{3}$, but not otherwise.

Theorem 7. *Let $h : B \to X$ be a holomorphic function with $h(0) = 0$ and $Dh(0) = I$. Suppose $M' = L(I - h)$ is finite. Then the values r and P below are Bloch radii:*
If $M' \leq 2/3$, then

$$r = 1, \qquad P = 1 - M'.$$

If $M' > 2/3$, then

$$r = \sqrt{1 + \frac{1}{2M' - 1}} - 1, \quad P = \left(\sqrt{2M'} - \sqrt{2M' - 1}\right)^2.$$

Proof. We proceed as in the proof of Theorem 6 but with h replaced by $I - h$ and M replaced by M'. Then we arrive at a function $f : \Delta \to \mathbb{C}$ satisfying $f(0) = 0$ and $\operatorname{Re} f(\lambda) < M'$ for all λ in Δ. Hence, by [12, p. 173],

$$\operatorname{Re} f(\lambda) \leq \frac{2M'|\lambda|}{1 + |\lambda|}, \quad \lambda \in \Delta.$$

Let $\lambda = t$, where $0 < t < 1$. Then

$$\operatorname{Re} \ell\left((I - h)(stx)\right) \leq \frac{2st^2}{1 + t} M'.$$

Letting $s \to 1^-$, we obtain

$$\sup \operatorname{Re} V\left((I-h)_t\right) \leq \frac{2t^2}{1+t} M' := N(t).$$

Now Theorem 7 follows from Lemma 5 and a calculation. □

4. Distortion form of Schwarz's Lemma

In this section we obtain a distortion form of the Schwarz Lemma [4] for holomorphic functions with restricted numerical range. Another version is given in [6] for holomorphic functions mapping the open unit ball into the closed unit ball. Our proof depends on an inequality given in [7] that uses the numerical radius to estimate the norm. Specifically, if $P_m : X \to X$ is a continuous homogeneous polynomial of degree m, then

$$\|P_m\| \leq k_m |W(P_m)|, \quad m \geq 1, \tag{11}$$

where $k_m = m^{m/(m-1)}$ when $m > 1$ and $k_1 = e$.

Theorem 8. *Let $h : B \to X$ be a holomorphic function with $h(0) = 0$ and $Dh(0) = I$. If $L(h)$ is finite, then*

$$\|h(x) - x\| \leq \frac{8\|x\|^2}{(1 - \|x\|)^2} \left(L(h) - 1\right)$$

for all $x \in B$. In particular, $h = I$ when $L(h) \leq 1$.

Corollary 9. *If $h : B \to X$ is a holomorphic function and $L(h)$ is finite, then h_s is bounded on B for each s with $0 < s < 1$.*

Proof. Corollary 9 follows from Theorem 8 with h replaced by

$$h - h(0) - Dh(0) + I.$$

To prove Theorem 8, we begin with the observation that since h is holomorphic at 0, there exists an $r > 0$ such that the power series expansion

$$h(x) = \sum_{m=1}^{\infty} P_m(x) \tag{12}$$

holds for all $x \in X$ with $\|x\| < r$.

Proceeding as in the proof of Theorem 6, we arrive at (8). By [12, p. 170], each of the coefficients of g has modulus at most 2 and hence

$$|\ell(P_m(sx))| \le 2s(M-1), \quad m > 1,$$

for all s satisfying $1-\delta \le s < 1$. Therefore,

$$|V(P_m)| \le 2(M-1),$$

and hence

$$\|P_m\| \le 4(M-1)m, \quad m > 1,$$

by (11). (Observe that $k_m \le 2\,m$ for $m > 1$.)

Thus the right-hand side of (12) converges uniformly on each ball B_s, where $0 < s < 1$, and hence (12) holds for all $x \in B$ by the identity theorem. Therefore,

$$\begin{aligned}\|h(x)-x\| &\le \sum_{m=2}^{\infty} \|P_m\|\,\|x\|^m \\ &\le 4(M-1)\sum_{m=2}^{\infty} m\|x\|^m \\ &\le 4(M-1)\frac{2\|x\|^2}{(1-\|x\|)^2}\end{aligned}$$

for $x \in B$. The required inequality follows since $M > L(h)$ was arbitrary. □

Here is another proof of Theorem 8 for the case $L(h) \le 1$: Put $H = h - I$ and note that $L(H) \le 0$ by hypothesis. Then $f := (I-H)^{-1}$ is a holomorphic function of B into itself by Theorem 1. Also, $f(0) = 0$ and $Df(0) = I$. Hence $f = I$ by the Schwarz Lemma of Harris and it follows that $h = I$.

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