Factorizations of Operator Matrices

Lawrence A. Harris
Mathematics Department
University of Kentucky
Lexington, Kentucky 40506

ABSTRACT
This note gives explicit factorizations of a 2×2 operator matrix as a product of an upper triangular operator matrix and an involutory, unitary or J-unitary operator matrix. A pattern is given for construction of factorizations of this kind.

Let H and K be Hilbert spaces and let $\mathcal{L}(H, K)$ denote the space of all bounded linear operators from H to K. Put $\mathcal{L}(H) = \mathcal{L}(H, H)$. Throughout, $M = \begin{bmatrix} A & B \\ C & D \end{bmatrix}$ denotes any given operator in $\mathcal{L}(K \times H)$. To describe factorizations of M, define

$$T(Z) = (AZ + B)(CZ + D)^{-1}, \tag{1}$$
$$S(W) = (A - WC)^{-1}(WD - B)$$

for $Z, W \in \mathcal{L}(H, K)$. By [4, Prop. 4], if M^{-1} exists then $[A - T(Z)C]^{-1}$ exists whenever $Z \in \mathcal{L}(H, K)$ and $(CZ + D)^{-1}$ exists, and $[CS(W) + D]^{-1}$ exists whenever $W \in \mathcal{L}(H, K)$ and $(A - WC)^{-1}$ exists; moreover, $S = T^{-1}$.

Theorem 1 *Put*

$$R = \begin{bmatrix} -(A - W_0C) & AZ_0 + W_0D \\ 0 & CZ_0 + D \end{bmatrix},$$
$$U = \begin{bmatrix} -(I - YX) & 2Y - YXY \\ X & I - XY \end{bmatrix}.$$

a) Suppose $(CZ_0 + D)^{-1}$ *exists for some* $Z_0 \in \mathcal{L}(H, K)$*. Take* $X = (CZ_0 + D)^{-1}C$*,* $Y = Z_0$ *and* $W_0 = T(Z_0)$*. Then* $M = RU$*.*

b) Suppose $(A - W_0C)^{-1}$ *exists for some* $W_0 \in \mathcal{L}(H, K)$*. Take* $X = -C(A - W_0C)^{-1}$*,* $Y = W_0$ *and* $Z_0 = S(W_0)$*. Then* $M = UR$*.*

In both cases, $U^2 = I$*.*

Let $J \in \mathcal{L}(K \times H)$ be given by $J = \begin{bmatrix} I & 0 \\ 0 & -I \end{bmatrix}$. Recall [1] that an $M \in \mathcal{L}(K \times H)$ is called J-self-adjoint if $M^* = JMJ$ and J-unitary if $M^*JM = MJM^* = J$.

Theorem 2 *Put*

$$U = \begin{bmatrix} (I + X^*X)^{-1/2} & -X^*(I + XX^*)^{-1/2} \\ (I + XX^*)^{-1/2}X & (I + XX^*)^{-1/2} \end{bmatrix}.$$

a) Suppose D^{-1} *exists and take* $X = D^{-1}C$*. Then* $M = RU$*, where*

$$R = \begin{bmatrix} (A - BX)(I + X^*X)^{-1/2} & (B + AX^*)(I + XX^*)^{-1/2} \\ 0 & D(I + XX^*)^{1/2} \end{bmatrix}.$$

b) Suppose A^{-1} *exists and take* $X = CA^{-1}$*. Then* $M = UR$*, where*

$$R = \begin{bmatrix} (I + X^*X)^{1/2}A & (I + X^*X)^{-1/2}(B + X^*D) \\ 0 & (I + XX^*)^{-1/2}(D - XB) \end{bmatrix}.$$

In both cases, U *is unitary and* J*-self-adjoint.*

Theorem 3 *Put*

$$U = \begin{bmatrix} (I - X^*X)^{-1/2} & X^*(I - XX^*)^{-1/2} \\ (I - XX^*)^{-1/2}X & (I - XX^*)^{-1/2} \end{bmatrix}.$$

a) Suppose D^{-1} *exists and take* $X = D^{-1}C$*. If* $||X|| < 1$*, then* $M = RU$*, where*

$$R = \begin{bmatrix} (A - BX)(I - X^*X)^{-1/2} & (B - AX^*)(I - XX^*)^{-1/2} \\ 0 & D(I - XX^*)^{1/2} \end{bmatrix}.$$

b) Suppose A^{-1} *exists and take* $X = CA^{-1}$*. If* $||X|| < 1$*, then* $M = UR$*, where*

$$R = \begin{bmatrix} (I - X^*X)^{1/2}A & (I - X^*X)^{-1/2}(B - X^*D) \\ 0 & (I - XX^*)^{-1/2}(D - XB) \end{bmatrix}.$$

In both cases, U *is* J*-unitary and self-adjoint.*

Note that the operators U in Theorems 1–3 admit the factorizations

$$
\begin{aligned}
U &= \begin{bmatrix} I & Y \\ 0 & I \end{bmatrix} \begin{bmatrix} -I & 0 \\ X & I \end{bmatrix} \begin{bmatrix} I & -Y \\ 0 & I \end{bmatrix}, \\
U &= \begin{bmatrix} (I+X^*X)^{-1/2} & 0 \\ 0 & (I+XX^*)^{-1/2} \end{bmatrix} \begin{bmatrix} I & -X^* \\ X & I \end{bmatrix}, \\
U &= \begin{bmatrix} (I-X^*X)^{-1/2} & 0 \\ 0 & (I-XX^*)^{-1/2} \end{bmatrix} \begin{bmatrix} I & X^* \\ X & I \end{bmatrix},
\end{aligned}
$$

respectively, and that the operator matrices in each of the last two factorizations commute. Parts (a) of Theorems 1–3 may be viewed as the operator matrix versions of factorizations for linear fractional transformations given in [4] (in the sentence below formula (7) and in Theorem 11b.) The above theorems can be verified directly by multiplication of operator matrices or they can be deduced from the following general result.

Proposition 4 *Let* $U = \begin{bmatrix} U_1 & U_2 \\ U_3 & U_4 \end{bmatrix}$ *be an invertible operator in* $\mathcal{L}(K \times H)$.

a) Suppose $U_3Z_0+U_4$ *is invertible for some* $Z_0 \in \mathcal{L}(H,K)$. *Then* $M = RU$ *for some* $R \in \mathcal{L}(K \times H)$ *of the form* $R = \begin{bmatrix} R_1 & R_2 \\ 0 & R_3 \end{bmatrix}$ *if and only if both* $C = R_3U_3$ *and* $D = R_3U_4$, *and in that case*

$$
\begin{aligned}
R_3 &= (CZ_0 + D)(U_3Z_0 + U_4)^{-1}, && (2) \\
R_1 &= [A - P(Z_0)U_3][U_1 - U(Z_0)U_3]^{-1}, && (3) \\
R_2 &= P(Z_0) - R_1U(Z_0), && (4)
\end{aligned}
$$

where

$$
\begin{aligned}
& P(Z) = (AZ + B)(U_3Z + U_4)^{-1}, \\
& U(Z) = (U_1Z + U_2)(U_3Z + U_4)^{-1}. && (5)
\end{aligned}
$$

b) Suppose $U_1 - W_0U_3$ *is invertible for some* $W_0 \in \mathcal{L}(H,K)$. *Then* $M = UR$ *for some* $R \in \mathcal{L}(K \times H)$ *of the form* $R = \begin{bmatrix} R_1 & R_2 \\ 0 & R_3 \end{bmatrix}$ *if and only if both* $A = U_1R_1$ *and* $C = U_3R_1$, *and in that case*

$$
\begin{aligned}
R_1 &= (U_1 - W_0U_3)^{-1}(A - W_0C), && (6) \\
R_3 &= [U_4 + U_3V(W_0)]^{-1}[D + U_3Q(W_0)], && (7) \\
R_2 &= V(W_0)R_3 - Q(W_0), && (8)
\end{aligned}
$$

where

$$\begin{aligned} Q(W) &= (U_1 - WU_3)^{-1}(WD - B), \\ V(W) &= (U_1 - WU_3)^{-1}(WU_4 - U_2). \end{aligned}$$

Corollary 5 *Let T and U be linear fractional transformations with coefficients given by (1) and (5), respectively. Suppose that the coefficient matrix of U is invertible and that $U_3Z_0 + U_4$ is invertible for some $Z_0 \in \mathcal{L}(H,K)$. If both $C = R_3U_3$ and $D = R_3U_4$ for some invertible $R_3 \in \mathcal{L}(H)$, then $T = \phi \circ U$, where ϕ is the affine linear fractional transformation given by*

$$\phi(Z) = R_1[Z - U(Z_0)]R_3^{-1} + T(Z_0)$$

and R_1 is given by (3).

Proof. To prove part (a), suppose $C = R_3U_3$ and $D = R_3U_4$. Clearly (2) holds. Let R_1 and R_2 be defined by (3) and (4). Then

$$R_1U_1 + R_2U_3 = R_1[U_1 - U(Z_0)U_3] + P(Z_0)U_3 = A.$$

It follows from this and (4) that

$$R_1U_2 + R_2U_4 = R_1(U_1Z_0 + U_2) + R_2(U_3Z_0 + U_4) - AZ_0 = B.$$

Hence $M = RU$. Conversely, any R with the given triangular form which satisfies $M = RU$ clearly satisfies $C = R_3U_3$ and $D = R_3U_4$ and thus is given by formulas (2)–(4) since there can be at most one such R by the invertibility of U.

One can prove part (b) similarly by observing that by (7) and (8),

$$U_3R_2 + U_4R_3 = [U_4 + U_3V(W_0)]R_3 - U_3Q(W_0) = D$$

and that by this and (8),

$$U_1R_2 + U_2R_3 = (U_1 - W_0U_3)R_2 + (U_2 - W_0U_4)R_3 + W_0D = B. \qquad \square$$

All the results given above hold with no change in the more general case where M and R are in $\mathcal{L}(K_1 \times H_1, K_2 \times H_2)$ and U is in $\mathcal{L}(K_1 \times H_1)$ in case (a) and U is in $\mathcal{L}(K_2 \times H_2)$ in case (b). Other factorization theorems are given for block matrices in [2, Section II] and [3]. The author has given another factorization theorem for operator matrices in [4, Prop. 2]. See [5] for a survey of some operator factorization theorems.

References

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