Linear Fractional Transformations of Circular Domains in Operator Spaces

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0. Introduction

Our object is to study domains which are the region of negative definiteness of an operator-valued Hermitian form defined on a space of operators and to investigate the biholomorphic linear fractional transformations between them. This is a unified setting in which to consider operator balls, operator half-planes, strictly J-contractive operators, strictly J-dissipative operators, etc., and the biholomorphic images of these domains under linear fractional transformations. Our approach is close in spirit to that of Potapov [28], Krein and Smuljan [27] and Smuljan [33]. At the same time, because we consider subspaces of operators, our circular domains include the matrix balls which E. Cartan [6] obtained as the classical bounded symmetric domains and they include the Siegel domains of genus 2 and 3 which Pyatetskii-Shapiro [29] associates with these domains as well as the infinite dimensional analogues of both types of domains given in [18] and [19]. Thus we are able to use the ideas of functional analysis and operator theory to study a highly general class of domains in a setting which allows explicit constructions and computations.

The extensive applications of linear fractional transformations of matrices and operators to problems in functional analysis is well documented in a series of recent volumes on operator theory edited by I. Gohberg and others. In particular, their importance is established in the study of indefinite inner product spaces [15, 2], the spectral theory of operators [14, 12], the study of the Riccati equation and control theory [37, 10, 16, 3], and the mathematical analysis of electrical networks [11, 21].

A treatment of circular domains for the complex plane is given by Schwerdtfeger [32] and the several variable case is discussed by Hua [24, 25]. Hua considers projective spaces and thus does not discuss questions of invertibility which are important for determining holomorphic equivalence of domains and for applications to operator theory. The problem of holomorphic equivalence is solved for a different kind of circular domains in [4]. A discussion of linear fractional transformations in a more general context than ours is given by N. Young [36]. See [30] and [31] for more properties of linear fractional transformations. See Upmeier [35] for an exposition of the infinite dimensional theory of bounded symmetric domains and extensions.

Section 1 discusses basic definitions and facts about linear fractional transformations of operators and their coefficient matrices. It is shown that a linear fractional transformation which is defined at some point may be decomposed as a product of elementary transformations. To avoid degeneracy, the coefficient matrix is usually assumed to be invertible. In that case, the domain and range of a linear fractional transformation and expressions for its inverse are given. Finally, necessary and sufficient conditions are obtained for a linear fractional transformation to map a domain in one space of operators onto a subset with non-empty interior in another space of operators. These conditions are algebraic in nature and are important for statements of later results.

Our definition of circular domains is given in Section 2. Various examples are given to show that these cover a wide range of domains of interest in several complex variables and functional analysis. If E is the coefficient of the quadratic term in the definition of a circular domain $\mathcal{D}$, then $\mathcal{D}$ is convex when $E \geq 0$. According to our basic lemma, to show that a linear fractional transformation T maps a given circular domain $\mathcal{D}_1$ onto the corresponding circular domain $\mathcal{D}_2$ computed from the coefficient matrix of T, it suffices to show that T is defined on $\mathcal{D}_1$ and that T^{-1} is defined on $\mathcal{D}_2$. Our main result is that if both circular domains $\mathcal{D}_1$ and $\mathcal{D}_2$ satisfy the convexity condition mentioned, then T is a biholomorphic mapping of $\mathcal{D}_1$ onto $\mathcal{D}_2$. We also show that under a compactness condition, if $\mathcal{D}_1$ satisfies the convexity condition and if T maps $\mathcal{D}_1$ onto a domain in another operator space, then that domain is $\mathcal{D}_2$. The last result of Section 2 obtains canonical models for domains which are holomorphically equivalent to the open unit ball of a space of operators under a linear fractional transformation. These model domains include the operator Siegel domains of [19].

Section 3 discusses a dual notion of circular domains. Every dual circular domain is the set of adjoints of the operators in some circular domain and hence facts about circular domains can be carried over to analogous facts about dual circular domains. Our main result in this section is an equivalence of operator inequalities corresponding to the equality of a circular domain and a dual circular domain. We present three applications. One of them contains the result of Ginzburg that the adjoint of a strictly J-contractive operator is strictly J-contractive when $J = I - 2\,E$ and E is a projection with finite rank.

1. Prerequisites and Identities

Let H and K be Hilbert spaces. We consider

$$M = \begin{bmatrix} A & B \\ C & D \end{bmatrix}, \quad \begin{array}{ll} A \in \mathcal{L}(K), & B \in \mathcal{L}(H,K) \\ C \in \mathcal{L}(K,H), & D \in \mathcal{L}(H). \end{array} \tag{1}$$

Clearly $M \in \mathcal{L}(K \times H)$ and, conversely, any $M \in \mathcal{L}(K \times H)$ can be written in the form (1). The *linear fractional transformation* T *with coefficient matrix* M is the function with values in $\mathcal{L}(H,K)$ defined by

$$T(Z) = (AZ + B)(CZ + D)^{-1} \tag{2}$$

for $Z \in \mathcal{L}(H,K)$ where $(CZ+D)^{-1}$ exists. For example, if $n = \dim\ H$ and $m = \dim\ K$ are finite, then $\mathcal{L}(H,K)$ can be identified with the space of all $m \times n$ matrices and

$$T(Z) = \frac{[\alpha_{ij}(Z)]}{\det(CZ+D)},$$

where $\alpha_{ij}(Z)$ is the determinant of $CZ + D$ with row j replaced by row i of $AZ + B$. One can prove this by taking transposes in the equation $W(CZ+D) = AZ + B$ and solving for W with Cramer's rule.

Proposition 1 *Let M be as in (1) and suppose* $\dim\ H < \infty$*. If M^{-1} exists, then there is a $Z_0 \in \mathcal{L}(H,K)$ such that $(CZ_0 + D)^{-1}$ exists.*

The assumption dim $H < \infty$ cannot be omitted. Indeed, given H with dim $H = \infty$, let $K = H$ and let $A \in \mathcal{L}(H)$ be an operator with left inverse D but no right inverse. (For example, A could be the shift operator on $H = \ell^2$ and D would be its adjoint.) Take $M = \begin{bmatrix} A & I - AD \\ 0 & D \end{bmatrix}$. Then $M^{-1} = \begin{bmatrix} D & 0 \\ I - AD & A \end{bmatrix}$ but $CZ + D = D$ is not invertible. Thus the linear fractional transformation with coefficient matrix M is undefined for all operators in $\mathcal{L}(H, K)$ although M is invertible.

Proof. We first show that in the case where $H = K$, if $(CZ + D)^{-1}$ does not exist for any $Z \in \mathcal{L}(H)$ then there is a non-zero $x \in H$ satisfying both $x^t C = 0$ and $x^t D = 0$. Put $n = \dim H$. It suffices to show that all determinants formed from n rows of $\begin{bmatrix} C^t \\ D^t \end{bmatrix}$ are 0 since then the null space of $\begin{bmatrix} C^t \\ D^t \end{bmatrix}$ is nontrivial [5, p. 373]. By hypothesis, $\det(CZ + D) = 0$ for all $n \times n$ matrices Z. Clearly, $CZ + D = \begin{bmatrix} C^t \\ D^t \end{bmatrix}^t \begin{bmatrix} Z \\ I \end{bmatrix}$ and hence by the Cauchy-Binet formula [23, p. 14], $\det(CZ + D)$ is the sum of the $\binom{2\,n}{n}$ products of pairs of the n th order determinants which can be formed by choosing n rows from $\begin{bmatrix} C^t \\ D^t \end{bmatrix}$ and the same n rows from $\begin{bmatrix} Z \\ I \end{bmatrix}$.

Now choose any n rows of $\begin{bmatrix} C^t \\ D^t \end{bmatrix}$. This determines a set α of row numbers of C^t and a set β of row numbers of D^t. Choose Z so that the rows of Z whose number is not in α are 0 rows and so that the other rows of Z are the rows of I with row numbers not in β. Then the determinant formed from the rows of $\begin{bmatrix} Z \\ I \end{bmatrix}$ corresponding to the n rows chosen from $\begin{bmatrix} C^t \\ D^t \end{bmatrix}$ is the determinant of a permutation matrix and so it is ± 1. Also the determinant formed from any other n rows of $\begin{bmatrix} Z \\ I \end{bmatrix}$ is 0. Thus the determinant formed from any chosen n rows of $\begin{bmatrix} C^t \\ D^t \end{bmatrix}$ is $\pm \det(CZ + D) = 0$. This establishes what we intended to show first.

To deduce the proposition, let $M^{-1} = \begin{bmatrix} P & Q \\ R & L \end{bmatrix}$ and note that $CQ + DL = I$ since $MM^{-1} = I$. If $(CQX + D)^{-1}$ does not exist for any $X \in \mathcal{L}(H)$, then by what we have established there exists a non-zero $x \in H$ with $x^t CQ = 0$ and $x^t D = 0$. But then $x^t = x^t(CQ + DL) = 0$, a contradiction. Thus $(CZ_0 + D)^{-1}$ exists for some $Z_0 \in \mathcal{L}(H, K)$.

Proposition 2 *Let M be as in (1) and suppose $(CZ_0 + D)^{-1}$ exists for some $Z_0 \in \mathcal{L}(H, K)$. Put $X_0 = (CZ_0 + D)^{-1}C$ and $W_0 = T(Z_0)$, where T is as in (2). Then*

$$M = \begin{bmatrix} I & W_0 \\ 0 & I \end{bmatrix} \begin{bmatrix} A - W_0 C & 0 \\ 0 & CZ_0 + D \end{bmatrix} \begin{bmatrix} I & 0 \\ X_0 & I \end{bmatrix} \begin{bmatrix} I & -Z_0 \\ 0 & I \end{bmatrix}. \quad (3)$$

Also, M^{-1} exists iff. $(A - W_0 C)^{-1}$ exists. If $(A - W_0 C)^{-1}$ exists, then

$$M^{-1} = \begin{bmatrix} P & Q \\ R & L \end{bmatrix}, \text{ where}$$

$$R = -X_0(A - W_0C)^{-1}, \qquad L = (CZ_0 + D)^{-1} - RW_0,$$

$$P = (A - W_0C)^{-1} + Z_0R, \quad Q = -PW_0 + Z_0(CZ_0 + D)^{-1}.$$

Note that the usual formulae [9, p. 3-4] for the inverse of block matrices are consequences of Proposition 2 in the cases where $Z_0 = 0$ and $W_0 = 0$. The decomposition of linear fractional transformations corresponding to (3) is given by the following:

Proposition 3 *If T is as in (2) and if $(CZ_0 + D)^{-1}$ exists, then $T = T_4 \circ T_3 \circ T_2 \circ T_1$, where*

$$T_1(Z) = Z - Z_0, \qquad T_3(Z) = (A - W_0C)Z(CZ_0 + D)^{-1},$$

$$T_2(Z) = Z(I + X_0Z)^{-1}, \quad T_4(Z) = Z + W_0.$$

Proofs. Formula (3) of Proposition 2 follows by block multiplication of operator matrices. It is easy to show that if D_1^{-1} exists, then $\begin{bmatrix} A_1 & 0 \\ C_1 & D_1 \end{bmatrix}^{-1}$ exists if and only if A_1^{-1} exists and in that case

$$\begin{bmatrix} A_1 & 0 \\ C_1 & D_1 \end{bmatrix}^{-1} = \begin{bmatrix} A_1^{-1} & 0 \\ -D_1^{-1}C_1A_1^{-1} & D_1^{-1} \end{bmatrix}.$$

A similar result holds for $\begin{bmatrix} A_1 & B_1 \\ 0 & D_1 \end{bmatrix}^{-1}$. In particular, the second factor on the right-hand side of (3) is invertible if and only if $A - W_0C$ is invertible and the other factors are always invertible. The remainder of Proposition 2 follows from this and the fact that the inverse of a product is the product of the inverses in reverse order.

Proposition 3 can be verified directly by a straightforward algebraic computation.

Proposition 4 *Suppose* M^{-1} *exists where* M *is as in (1) and write* $M^{-1} = \begin{bmatrix} P & Q \\ R & L \end{bmatrix}$. *If* $W \in \mathcal{L}(H, K)$, *then* $(A - WC)^{-1}$ *exists if and only if* $(RW + L)^{-1}$ *exists and*

$$(A - WC)^{-1}(WD - B) = (PW + Q)(RW + L)^{-1}. \tag{4}$$

If T *is the linear fractional transformation with coefficient matrix* M, *then* T *is a biholomorphic mapping of* $\mathcal{D}_1 = \{Z \in \mathcal{L}(H, K) : (CZ + D)^{-1}$ *exists*$\}$ *onto* $\mathcal{D}_2 = \{W \in \mathcal{L}(H, K) : (A - WC)^{-1}$ *exists*$\}$ *and* T^{-1} *is the linear fractional transformation above with coefficient matrix* M^{-1}.

Proof. Let S be the linear fractional transformation with coefficient matrix M^{-1}. By multiplying out

$$\begin{bmatrix} I & -Z \\ 0 & I \end{bmatrix} M^{-1} M \begin{bmatrix} Z \\ I \end{bmatrix} = \begin{bmatrix} 0 \\ I \end{bmatrix},$$

we see that

$$R(AZ + B) + L(CZ + D) \quad = \quad I$$

$$(P - ZR)(AZ + B) + (Q - ZL)(CZ + D) \;=\; 0$$

so $RW + L = (CZ + D)^{-1}$ and $PW + Q = Z(RW + L)$, where $W = T(Z)$. Hence $S(T(Z)) = Z$ for all $Z \in \mathcal{D}_1$. Interchanging the roles of S and T, we have that $T(S(W)) = W$ for all $W \in \hat{\mathcal{D}}_2$, where $\hat{\mathcal{D}}_2 = \{W \in \mathcal{L}(H, K) : (RW + L)^{-1} \text{ exists}\}$. Hence $S = T^{-1}$ and T is a biholomorphic mapping of $\mathcal{D}_1$ onto $\hat{\mathcal{D}}_2$.

Now we show that $\hat{\mathcal{D}}_2 = \mathcal{D}_2$. If $W \in \hat{\mathcal{D}}_2$, then $W = T(Z)$ for some $Z \in \mathcal{D}_1$ so $W \in \mathcal{D}_2$ by Proposition 2. Let $W \in \mathcal{D}_2$ and define $Z = (A - WC)^{-1}(WD - B)$. Then $AZ + B = W(CZ + D)$ so

$$\begin{bmatrix} I & -W \\ 0 & I \end{bmatrix} M \begin{bmatrix} I & Z \\ 0 & I \end{bmatrix} = \begin{bmatrix} A - WC & 0 \\ C & CZ + D \end{bmatrix}$$

and this is invertible. Hence $Z \in \mathcal{D}_1$. Then $W = T(Z)$ so $W \in \hat{\mathcal{D}}_2$.

Equality (4) follows from cross multiplication and the identity $MM^{-1} = I$.

Call a subset Γ of a normed linear space X *thick* if for any open connected set Ω in X which contains Γ, every holomorphic function $f : \Omega \to \mathbb{C}$ which vanishes on Γ vanishes on all of Ω. By the identity theorem [22, Th. 3.16.4], a set is thick whenever it has non-empty interior. The following gives necessary and sufficient conditions for a linear fractional transformation to be a biholomorphic mapping between domains in spaces of operators.

Proposition 5 *Let $\mathcal{A}$ and $\mathcal{B}$ be closed complex subspaces of $\mathcal{L}(H,K)$ and let T be a linear fractional transformation which is defined at some point $Z_0 \in \mathcal{A}$ and which has an invertible coefficient matrix M given by (1). Put $W_0 = T(Z_0)$ and $X_0 = (CZ_0 + D)^{-1}C$. If each of the conditions*

(i) $ZX_0Z \in \mathcal{A}$ *for all* $Z \in \mathcal{A}$

(ii) $\mathcal{B} = (A - W_0C)\mathcal{A}(CZ_0 + D)^{-1}$

(iii) $W_0 \in \mathcal{B}$

holds, then T is a biholomorphic mapping of $\mathcal{E}_1$ onto $\mathcal{E}_2$, where

$$\begin{aligned}\mathcal{E}_1 &= \mathrm{Comp}_{Z_0}(\{Z \in \mathcal{A} : \ (CZ + D)^{-1} \ \textit{exists}\}),\\ \mathcal{E}_2 &= \mathrm{Comp}_{W_0}(\{W \in \mathcal{B} : \ (A - WC)^{-1} \ \textit{exists}\}).\end{aligned}$$

Moreover, there is an invertible affine linear fractional transformation of $\mathcal{E}_1$ onto $\mathcal{E}_2$ which takes Z_0 to W_0. Conversely, if T is defined on a connected domain $\mathcal{D} \subseteq \mathcal{A}$ containing Z_0 and if $T(\mathcal{D})$ is a thick subset of $\mathcal{B}$, then (i), (ii) and (iii) hold.

Here Comp_P denotes the connected component containing P. Note that by Proposition 5, if (i), (ii) and (iii) hold for some $Z_0 \in \mathcal{A}$, then they hold for any other value of Z_0 in $\mathcal{E}_1$. An important special case of Proposition 5 occurs when

$$T(Z) = (I - AB^*)^{-\frac{1}{2}}(Z + A)(I + B^*Z)^{-1}(I - B^*A)^{\frac{1}{2}}, \tag{5}$$

where $\sigma(B^*A)$ does not intersect $[1, \infty)$ and the square roots are defined by the holomorphic functional calculus. (See [19, p. 146].) It follows from Proposition 5 with $Z_0 = 0$ and part (c) of Lemma 6 below that if $A \in \mathcal{A}$ and if $ZB^*Z \in \mathcal{A}$ for all $Z \in \mathcal{A}$, then T is a biholomorphic mapping of $\mathcal{E}_1$ onto $\mathcal{E}_2$ and

$$\mathcal{E}_1 = \Omega_1, \quad \mathcal{E}_2 = A - (I - AB^*)^{\frac{1}{2}}\Omega_1(I - B^*A)^{\frac{1}{2}},$$

where Ω_1 is given by (6) below with $X_0 = B^*$.

Lemma 6 *Suppose $ZX_0Z \in \mathcal{A}$ whenever $Z \in \mathcal{A}$. Then*

a) $Zp(X_0Z) \in \mathcal{A}$ *b)* $p(ZX_0)W + Wp(X_0Z) \in \mathcal{A}$

c) $p(ZX_0)Wp(X_0Z) \in \mathcal{A}$ *d)* $WX_0p(ZX_0)W \in \mathcal{A}$

for all $Z, W \in \mathcal{A}$ and all polynomials p.

Proof. Let $Z, W \in \mathcal{A}$ and put $W_n = Z(X_0Z)^n$ for $n \geq 0$. By hypothesis,

$$ZX_0W + WX_0Z = (Z + W)X_0(Z + W) - ZX_0Z - WX_0W \in \mathcal{A}.$$

Taking $W = W_n$, we see that $W_{n+1} \in \mathcal{A}$ if $W_n \in \mathcal{A}$. Hence (a) follows by induction. Also (b) follows from the relation

$$(ZX_0)^{n+1}W + W(X_0Z)^{n+1} = W_nX_0W + WX_0W_n \in \mathcal{A},$$

which holds by what we have shown. Let $L(p, W)$ be the left-hand side of (b). Then by (b),

$$2\,p(ZX_0)Wp(X_0Z) = L(p, L(p, W)) - L(p^2, W) \in \mathcal{A},$$

which proves (c). Finally, applying (c) in the case where $p(z) \equiv z$, we have that $WX_0(ZX_0)^{n+1}W = WX_0W_nX_0W \in \mathcal{A}$. Hence (d) follows.

Proof of Proposition 5. Note that $(A - W_0C)^{-1}$ exists by Proposition 2. By (ii) and (iii), the transformations T_1, T_3 and T_4 of Proposition 3 are everywhere defined biholomorphic maps. Thus to show that T is a biholomorphic map of $\mathcal{E}_1$ onto $\mathcal{E}_2$, it suffices to show that T_2 is a biholomorphic map of $\Omega_1 = T_1(\mathcal{E}_1)$ onto $\Omega_2 = T_3^{-1} \circ T_4^{-1}(\mathcal{E}_2)$. It follows from the identities

$$\begin{aligned} I + X_0T_1(Z) &= (CZ_0 + D)^{-1}(CZ + D) \\ A - WC &= (A - W_0C)(I - ZX_0), \end{aligned}$$

where $W = T_4 \circ T_3(Z)$, that

$$\Omega_1 = \mathrm{Comp}_0(\{Z \in \mathcal{A} : (I + X_0Z)^{-1} \text{ exists}\}) \qquad (6)$$

$$\Omega_2 \quad = \quad \mathrm{Comp}_0(\{Z \in \mathcal{A} : (I - X_0 Z)^{-1} \text{ exists}\}). \tag{7}$$

Clearly $T_4 \circ T_3 \circ (-T_1)$ is affine and it maps $\mathcal{E}_1$ onto $\mathcal{E}_2$ since $\Omega_2 = -\Omega_1$.

Since $(I + X_0 Z)^{-1}$ is a limit in the operator norm of polynomials in $X_0 Z$ when $||X_0 Z|| < 1$, it follows from (i) and part (a) of Lemma 6 that $T_2(Z) \in \mathcal{A}$ whenever $Z \in \mathcal{A}$ and $||Z|| < 1/||X_0||$. Since T_2 is holomorphic in Ω_1, this implies that $T_2(\Omega_1) \subseteq \mathcal{A}$ by the Hahn-Banach and identity theorems. In fact, $T_2(\Omega_1) \subseteq \Omega_2$ since if $Z \in \Omega_1$ and $W = T_2(Z)$ then $I - X_0 W = (I + X_0 Z)^{-1}$. Define $T_2^{-1}(W) = W(I - X_0 W)^{-1}$ and note that $T_2^{-1} \circ T_2(Z) = Z$ for $Z \in \Omega_1$. A similar argument applied to T_2^{-1} shows that $T_2^{-1}(\Omega_2) \subseteq \Omega_1$ and $T_2 \circ T_2^{-1}(W) = W$ for all $W \in \Omega_2$. Hence T_2 is a biholomorphic mapping of Ω_1 onto Ω_2, as required.

To prove the converse assertion, suppose $T(\mathcal{D})$ is a thick subset of $\mathcal{B}$. Clearly (iii) holds. Put $R = T_3 \circ T_2$ and $\mathcal{D}_1 = T_1(\mathcal{D})$. Note that $T_3 = DR(0)$ maps $\mathcal{A}$ into $\mathcal{B}$ since $R(\mathcal{D}_1) \subseteq \mathcal{B}$. Define $R^{-1} = T_2^{-1} \circ T_3^{-1}$, where T_2^{-1} is as given above and $T_3^{-1}(W) = (A - W_0 C)^{-1} W (C Z_0 + D)$. Then $\Gamma = R(\mathcal{D}_1)$ is a thick subset of $\mathcal{B}$ and $R^{-1}(W) \in \mathcal{A}$ for all $W \in \Gamma$ since $R^{-1}(R(Z)) = Z$ for all $Z \in \mathcal{D}_1$. Hence if $\Omega = \mathrm{Comp}_0(\{W \in \mathcal{B} : R^{-1}(W) \text{ exists}\})$, then $R^{-1}(\Omega) \subseteq \mathcal{A}$ since Ω is a connected neighborhood of 0 containing Γ and R^{-1}

is holomorphic on Ω. Consequently, $T_3^{-1} = DR^{-1}(0)$ maps $\mathcal{B}$ into $\mathcal{A}$, which proves (ii). Also $T_2(\mathcal{D}_1) \subseteq \mathcal{A}$ since $T_2 = T_3^{-1} \circ R$. Given $Z \in \mathcal{A}$, it follows that

$$Z(I + tX_0Z)^{-1} = \frac{1}{t}T_2(tZ) \in \mathcal{A}$$

for all small enough $t \neq 0$. Differentiating with respect to t at $t = 0$, we see that $ZX_0Z \in \mathcal{A}$. This proves (i).

2. Circular Domains

If $Z \in \mathcal{L}(H)$, we write

$$\text{Re } Z = \frac{Z + Z^*}{2}, \quad \text{Im } Z = \frac{Z - Z^*}{2i}.$$

Let $\mathcal{A}$ be a closed complex subspace of $\mathcal{L}(H, K)$. A *circular domain* in $\mathcal{A}$ is a set of the form

$$\mathcal{D}(J, \mathcal{A}) = \{Z \in \mathcal{A} : \begin{bmatrix} Z \\ I \end{bmatrix}^* J \begin{bmatrix} Z \\ I \end{bmatrix} < 0\},$$

where $J \in \mathcal{L}(K \times H)$ is self-adjoint. Hence a circular domain is an open set but it is not necessarily connected. We write $\mathcal{D}(J)$ for $\mathcal{D}(J, \mathcal{A})$ when $\mathcal{A}$ is understood. Since there is a decomposition

$$J = \begin{bmatrix} E & F \\ F^* & G \end{bmatrix} \tag{8}$$

as in (1) where $E^* = E$ and $G^* = G$, we have

$$\mathcal{D}(J) = \{Z \in \mathcal{A} : \ Z^*EZ + 2\,\mathrm{Re}\,F^*Z + G < 0\}. \tag{9}$$

For example, the circular domains in $\mathbb{C}$ are any open disc, the exterior of any open disc, any half-plane, any punctured plane, the entire plane and the empty set.

Example 1 Take $J = J_0 = \begin{bmatrix} I & 0 \\ 0 & -I \end{bmatrix}$. Then

$$\mathcal{D}(J) = \{Z \in \mathcal{A} : \ ||Z|| < 1\},$$

where $||\ \ ||$ is the operator norm. Thus $\mathcal{D}(J)$ is the open unit ball of $\mathcal{A}$, which we denote by $\mathcal{A}_0$. If $\mathcal{A}$ is a J^*-algebra, i.e., $ZZ^*Z \in \mathcal{A}$ whenever $Z \in \mathcal{A}$, then $\mathcal{D}(J)$ is a bounded symmetric homogeneous domain. In fact, it is shown in [18, Th. 2] that

$$T_B(Z) = (I - BB^*)^{-\frac{1}{2}}(Z + B)(I + B^*Z)^{-1}(I - B^*B)^{\frac{1}{2}}$$

is a biholomorphic mapping of $\mathcal{A}_0$ onto itself for each $B \in \mathcal{A}_0$. This also follows from the remarks after (5) and Theorem 8 below. For more on J^*-algebras, see [20] and [7, §9.1].

Example 2 Let V be a partial isometry in $\mathcal{A}$ and take

$$J = \begin{bmatrix} I - VV^* & -iV \\ iV^* & -(I - V^*V) \end{bmatrix}.$$

Then

$$\mathcal{D}(J) = \{Z \in \mathcal{A} : \; 2\,\mathrm{Im}\, V^*Z - Z^*(I - VV^*)Z + I - V^*V > 0\}.$$

If $(\mathcal{A}, V)$ is a P^*-algebra, i.e. $VZ^*Z + ZZ^*V \in \mathcal{A}$ whenever $Z \in \mathcal{A}$ and $(I - VV^*)\mathcal{A}(I - V^*V)$ is a J^*-algebra, then it is shown in [19] that $\mathcal{D}(J)$ is a homogeneous operator Siegel domain of genus 3. When $V = 0$ this reduces to the result mentioned in Example 1. When $V = I$, $\mathcal{D}(J)$ is the set of strictly dissipative operators in a JC^*-algebra [35].

Example 3   A given Hilbert space H can be identified with $\mathcal{A} = \mathcal{L}(\mathbb{C}, H)$. Hence any circular domain $\mathcal{D}$ in H is of the form

$$\mathcal{D} = \{x \in H : \; (Ex, x) + 2\,\mathrm{Re}\,(x, f) + g < 0\},$$

where $E \in \mathcal{L}(H)$ is self adjoint, $f \in H$ and $g \in \mathbb{R}$.

Example 4   Given a Hilbert space H, the space $H \times \mathbb{C}$ is a Hilbert space. Hence we may take $E = \begin{bmatrix} I & 0 \\ 0 & -1 \end{bmatrix}$, $f = 0$ and $g = 0$ in Example 3 to obtain

$$\mathcal{D} = \{(x, z) \in H \times \mathbb{C} : \; ||x|| < |z|\}.$$

This circular domain is homogeneous. In fact, the group of linear automorphisms of $\mathcal{D}$ operates transitively on $\mathcal{D}$. To see this, let $(v, \lambda) \in \mathcal{D}$ and let

$L \in \mathcal{L}(H \times \mathbb{C})$ be given by

$$L = \begin{bmatrix} \overline{\lambda} A & v \\ v^* & \lambda \end{bmatrix}, \quad A = E_v + \sqrt{1 - \frac{||v||^2}{|\lambda|^2}}(I - E_v),$$

where the elements of $H \times \mathbb{C}$ are viewed as column matrices and where E_v is the projection of H onto the space spanned by v. Then

$$L^* E L = (|\lambda|^2 - ||v||^2)E, \quad L^{-1} = \frac{1}{|\lambda|^2 - ||v||^2} \begin{bmatrix} \lambda A & -v \\ -v^* & \overline{\lambda} \end{bmatrix}.$$

Hence L is an automorphism of $\mathcal{D}$ and $L(0,1) = (v, \lambda)$.

Next we list some elementary geometrical properties of circular domains. Suppose that $\mathcal{D}$ is a non-empty circular domain in $\mathcal{A}$ given by (9). If $E \geq 0$, then $\mathcal{D}$ is convex. In fact, if $E > 0$ and $E^{-1}F \in \mathcal{A}$, then $\mathcal{D}$ is affinely equivalent to the open unit ball of $\mathcal{B} = E^{1/2}\mathcal{A}Y^{-1/2}$, where $Y = F^*E^{-1}F - G > 0$. (Compare [33, Lemma 2.1].) If we know only that E^{-1} exists and $E^{-1}F \in \mathcal{A}$, then $\mathcal{D}$ is circled with respect to $Z_0 = -E^{-1}F$ i.e., $\lambda(\mathcal{D} - Z_0) \subseteq \mathcal{D} - Z_0$ whenever $|\lambda| = 1$, and if, in addition, $Y \geq 0$ or $Y \leq 0$, then $\lambda(\mathcal{D} - Z_0) \subseteq \mathcal{D} - Z_0$ whenever $0 < |\lambda| \leq 1$ or $|\lambda| \geq 1$, respectively. The first assertion follows immediately from the fact that if Z_1 and Z_2 are in $\mathcal{D} = \mathcal{D}(J)$ and if $Z = tZ_1 + (1-t)Z_2$, where $0 \leq t \leq 1$, then

$$\begin{aligned} \begin{bmatrix} Z \\ I \end{bmatrix}^* J \begin{bmatrix} Z \\ I \end{bmatrix} &= t \begin{bmatrix} Z_1 \\ I \end{bmatrix}^* J \begin{bmatrix} Z_1 \\ I \end{bmatrix} + (1-t) \begin{bmatrix} Z_2 \\ I \end{bmatrix}^* J \begin{bmatrix} Z_2 \\ I \end{bmatrix} \\ &\quad - t(1-t)(Z_2 - Z_1)^* E (Z_2 - Z_1). \end{aligned}$$

To prove the second assertion, observe that $T_1(Z) = Z - Z_0$ is a biholomorphic mapping of $\mathcal{D}$ onto $\mathcal{D}_1 = \{Z \in \mathcal{A} : \ Z^*EZ < Y\}$ by Lemma 7 (below), and $Y > 0$ since $\mathcal{D}_1 \neq \emptyset$. Also $T_2(Z) = E^{1/2}ZY^{-1/2}$ is a biholomorphic mapping of $\mathcal{D}_1$ onto $\mathcal{D}_2 = \{W \in \mathcal{B} : \ ||W|| < 1\}$ by Lemma 7. The third assertion follows from Lemma 7 and the identity

$$|\lambda|^2 (M^{-1})^* J M^{-1} = J + (1 - |\lambda|^2) \begin{bmatrix} 0 & 0 \\ 0 & Y \end{bmatrix},$$

where M is the coefficient matrix for $T_\lambda(Z) = \lambda(Z - Z_0) + Z_0$.

In general, one cannot expect linear fractional transformations to map circular domains biholomorphically onto circular domains since the range of a linear fractional transformation defined on a circular domain may not completely cover a circular domain. The simplest example of this is the transformation $T(z) = z^{-1}$ which maps the circular domain $\mathcal{D}_1 = \{z \in \mathbb{C} : \ |z| > 1\}$ biholomorphically onto the domain $\mathcal{D}_2 = \{z \in \mathbb{C} : \ 0 < |z| < 1\}$. Even when a linear fractional transformation with coefficient matrix M maps a circular domain $\mathcal{D}_1 = \mathcal{D}(J_1)$ onto a circular domain $\mathcal{D}_2$, it may not be true that $\mathcal{D}_2 = \mathcal{D}(J_2)$, where $J_2 = (M^{-1})^* J_1 M^{-1}$. For example, let $T(z) = 1/z$ and $\mathcal{D}_1 = \{z \in \mathbb{C} : \ z \neq 0\}$. Then $J_1 = \begin{bmatrix} -1 & 0 \\ 0 & 0 \end{bmatrix}$ so $\mathcal{D}(J_2) = \mathbb{C}$ but $\mathcal{D}_2 = \mathcal{D}_1$. We shall obtain conditions which imply that a linear fractional transformation is defined on a convex circular domain and that its image is a circular domain given by the coefficient matrix.

Lemma 7 *Suppose the hypotheses of Proposition 5 are satisfied. Given a self-adjoint* $J_1 \in \mathcal{L}(K \times H)$*, put* $J_2 = (M^{-1})^* J_1 M^{-1}$*. Then* T *is a biholomorphic mapping of* $\mathcal{D}_1 = \mathcal{D}(J_1, \mathcal{A}) \cap \mathcal{E}_1$ *onto* $\mathcal{D}_2 = \mathcal{D}(J_2, \mathcal{B}) \cap \mathcal{E}_2$*.*

Proof. Let $W = T(Z)$. Then

$$\begin{bmatrix} W \\ I \end{bmatrix} = M \begin{bmatrix} Z \\ I \end{bmatrix} (CZ + D)^{-1}$$

so

$$\begin{bmatrix} W \\ I \end{bmatrix}^* J_2 \begin{bmatrix} W \\ I \end{bmatrix} = [(CZ + D)^{-1}]^* \begin{bmatrix} Z \\ I \end{bmatrix}^* J_1 \begin{bmatrix} Z \\ I \end{bmatrix} (CZ + D)^{-1}. \quad (10)$$

Hence $T(\mathcal{D}_1) \subseteq \mathcal{D}_2$ by Proposition 5. By Proposition 4, the same argument applies to T^{-1} to show that $T^{-1}(\mathcal{D}_2) \subseteq \mathcal{D}_1$.

By Lemma 7, the open operator balls with invertible left and right radii as defined in [33] are circular domains since they are just the images of the open unit ball of $\mathcal{L}(H, K)$ under affine linear fractional transformations with invertible coefficient matrices.

In the theorem below and throughout, we assume that a self-adjoint operator $J \in \mathcal{L}(K \times H)$ is written as in (8) with corresponding subscripts when J is subscripted.

Theorem 8 *Suppose the hypotheses of Proposition 5 are satisfied. Given a self-adjoint* $J_1 \in \mathcal{L}(K \times H)$, *put* $J_2 = (M^{-1})^* J_1 M^{-1}$ *and suppose* $Z_0 \in \mathcal{D}(J_1, \mathcal{A})$. *If* $E_1 \geq 0$ *and* $E_2 \geq 0$, *then* T *is a biholomorphic mapping of* $\mathcal{D}_1 = \mathcal{D}(J_1, \mathcal{A})$ *onto* $\mathcal{D}_2 = \mathcal{D}(J_2, \mathcal{B})$.

For example, if J_1 is as in Example 1, then $\mathcal{D}(J_1, \mathcal{A})$ is the open unit ball of $\mathcal{A}$ and the hypotheses on J_1 and J_2 are satisfied when D^{-1} exists and $||D^{-1}C|| \leq 1$. (Take $Z_0 = 0$ in (11) below.) A converse to Theorem 8 holds for this choice of J_1 when $J_2 = J_1$ and $\mathcal{A} = \mathcal{L}(H, K)$. See Krein and Smuljan [27, Th. 3.2].

The assumption that T is defined at some point $Z_0 \in \mathcal{D}(J_1, \mathcal{A})$ cannot be omitted from Theorem 8. For example, let $\mathcal{A} = \mathcal{B} = \mathcal{L}(H)$, $T(Z) = Z^{-1}$, $E_1 \geq 0$, $G_1 \geq 0$ and suppose F_1 is not invertible. Then $E_2 = G_1 \geq 0$. However, T is not defined anywhere in $\mathcal{D}_1$ since if $Z \in \mathcal{D}_1$, then Re $F_1^* Z < 0$ and this implies that $F_1^* Z$ is invertible because the spectrum of an operator is contained in the closure of its numerical range. (Note that if $\mathcal{D}_1$ is not empty, then F_1 must have a left inverse.)

As an example of Theorem 8, we state an extension to circular domains of a part of the fundamental theorem of [27]. (Note that J-expansive rather than

J-contractive coefficient matrices are considered in [27] since the associated linear fractional transformations are defined differently there.)

Corollary 9 *Given a self-adjoint $J \in \mathcal{L}(K \times H)$ with $E \geq 0$, suppose $M \in \mathcal{L}(K \times H)$ is invertible and satisfies $M^*JM \leq J$. Let $\mathcal{A} = \mathcal{L}(H, K)$. If the linear fractional transformation T with coefficient matrix M is defined at some point of $\mathcal{D}(J, \mathcal{A})$, then T maps $\mathcal{D}(J, \mathcal{A})$ biholomorphically onto a convex circular subdomain of itself.*

The hypothesis $E_2 \geq 0$ in Theorem 8 is equivalent to

$$X_0^* \begin{bmatrix} Z_0 \\ I \end{bmatrix}^* J_1 \begin{bmatrix} Z_0 \\ I \end{bmatrix} X_0 - 2\,\mathrm{Re}\,(E_1 Z_0 + F_1) X_0 + E_1 \geq 0 \tag{11}$$

since $E_2 = Y^* J_1 Y$, where $Y = \begin{bmatrix} I - Z_0 X_0 \\ -X_0 \end{bmatrix} (A - W_0 C)^{-1}$, and this follows from Proposition 2 since $Y = \begin{bmatrix} P \\ R \end{bmatrix}$. The next result shows that under a compactness restriction the hypothesis $E_2 \geq 0$ may be replaced by the assumption that T is defined on $\mathcal{D}_1$.

Proposition 10 *Let $\mathcal{A}$ and $\mathcal{B}$ be closed complex subspaces of $\mathcal{L}(H, K)$ and let T be a linear fractional transformation with an invertible coefficient matrix M given by (1). Suppose C is compact or $\mathcal{A}$ contains only compact operators. Let J_1 and J_2 be as in Theorem 8 and suppose $\mathcal{D}(J_2, \mathcal{B})$ is connected. If $E_1 \geq 0$ and if T maps $\mathcal{D}_1 = \mathcal{D}(J_1, \mathcal{A})$ onto a domain $\mathcal{D}_2$ in $\mathcal{B}$, then $\mathcal{D}_2 = \mathcal{D}(J_2, \mathcal{B})$.*

The domain $\mathcal{D}_2$ need not be convex, as Example 5 (below) shows. However, it is easy to show that if $\mathcal{A}$ is a Hilbert space (in the sense of Example 3) then $E_2 \geq 0$ and, in particular, $\mathcal{D}_2$ is convex.

Proofs. We deduce Theorem 8 from Lemma 7. Let T_1, T_2, T_3 and T_4 be defined as in Proposition 3. By Lemma 7, T_1 maps $\mathcal{D}_1$ biholomorphically onto $\mathcal{D}(J_a, \mathcal{A})$ and $T_3^{-1} \circ T_4^{-1}$ maps $\mathcal{D}_2$ biholomorphically onto $\mathcal{D}(J_b, \mathcal{A})$, where

$$E_a = E_1, \quad E_b = (A - W_0 C)^* E_2 (A - W_0 C).$$

It suffices to show that T_2 is a biholomorphic mapping of $\mathcal{D}(J_a, \mathcal{A})$ onto $\mathcal{D}(J_b, \mathcal{A})$. To do this, we first show that $\mathcal{D}(J_a, \mathcal{A}) \subseteq \Omega_1$, where Ω_1 is given by (6), and this is equivalent to the invertibility of $I + X_0 Z$ for each $Z \in \mathcal{D}(J_a, \mathcal{A})$ since $\mathcal{D}(J_a, \mathcal{A})$ is a convex set containing 0.

By Proposition 3, $J_a = M_2^* J_b M_2$, where M_2 is the coefficient matrix of T_2, and hence

$$\begin{bmatrix} Z \\ I \end{bmatrix}^* J_a \begin{bmatrix} Z \\ I \end{bmatrix} = Z^* E_b Z + \operatorname{Re} Y^* (I + X_0 Z), \tag{12}$$

where $Y = 2\, F_b^* Z + G_b (I + X_0 Z)$. Also, $F_b = F_a - X_0^* G_a$ and $G_b = G_a$, so $Y = 2\, F_a^* Z + G_a (I - X_0 Z)$. Clearly $G_a < 0$ since $0 \in \mathcal{D}(J_a, \mathcal{A})$ so $P = (-G_a)^{\frac{1}{2}}$ is an invertible positive operator. Suppose $Z \in \mathcal{D}(J_a, \mathcal{A})$ and put $U = 2\,(F_a^* Z + G_a)$. Then by (9), $\operatorname{Re} U < 0$ since $E_a \geq 0$ so U^{-1} exists and by (12), Re $Y^*(I + X_0 Z) < 0$ since $E_b \geq 0$. Let $W = P(I + X_0 Z)$ and put $Q = WU^{-1}P$. Then $Y = P(P^{-1}U + W)$ so

$$(U^{-1}P)^* Y^* (I + X_0 Z) U^{-1} P = (Q + I)^* Q.$$

Hence Re $(Q + I)^* Q < 0$ so $(2\,Q + I)^*(2\,Q + I) < I$ and consequently $||2\,Q + I|| < 1$. Therefore $-2\,Q = I - (2\,Q + I)$ is invertible and it follows that $I + X_0 Z$ is invertible. A similar argument with the roles of J_a and J_b interchanged shows that $\mathcal{D}(J_b, \mathcal{A}) \subseteq \Omega_2$, where Ω_2 is given by (7). This proves Theorem 8.

We prove Proposition 10 by a variant of the above argument. It suffices to show that $(I - X_0 Z)^{-1}$ exists whenever $Z \in \mathcal{D}(J_b, \mathcal{A})$. By an argument similar to the one establishing (12), we have

$$\begin{bmatrix} Z \\ I \end{bmatrix}^* J_b \begin{bmatrix} Z \\ I \end{bmatrix} = Z^* E_a Z + \text{Re } Y^*(I - X_0 Z),$$

where $Y = 2F_a^* Z + G_a(I - X_0 Z)$. Let $Z \in \mathcal{D}(J_b, \mathcal{A})$. Then Re $Y^*(I - X_0 Z) < 0$ since $E_1 \geq 0$, so $I - X_0 Z$ has a left inverse. By hypothesis, $X_0 Z$ is compact so $I - X_0 Z$ is invertible by [17, prob. 140] or Lemma 18 below.

Corollary 9 follows easily from Theorem 8 with $J_1 = J$. Indeed, $J_1 \leq J_2$ by hypothesis so $\mathcal{D}(J_2, \mathcal{A}) \subseteq \mathcal{D}(J_1, \mathcal{A})$ and $0 \leq E_1 \leq E_2$.

Remark The proofs of Theorem 8 and Proposition 10 are much easier when both H and K are finite dimensional since left invertibility of operators on these spaces implies invertibility. Indeed, it follows from $J_1 = M^* J_2 M$ that

$$\begin{bmatrix} Z \\ I \end{bmatrix}^* J_1 \begin{bmatrix} Z \\ I \end{bmatrix} = (AZ+B)^* E_2 (AZ+B) + \mathrm{Re}\, Y^*(CZ+D), \qquad (13)$$

where $Y = 2\,F_2^*(AZ+B) + G_2(CZ+D)$. Hence if $E_2 \geq 0$ then $\mathrm{Re}\, Y^*(CZ+D) < 0$ for all $Z \in \mathcal{D}(J_1, \mathcal{A})$, so $T(Z)$ is defined in $\mathcal{D}(J_1, \mathcal{A})$. A similar argument assuming $E_1 \geq 0$ shows that $T^{-1}(Z)$ is defined in $\mathcal{D}(J_2, \mathcal{B})$.

The following theorem obtains up to affine equivalence all domains which are holomorphically equivalent to the open unit ball of a space of operators under a linear fractional transformation satisfying a mild restriction. These domains are extensions of the operator Siegel domains of Example 2.

Theorem 11

a) Let $\mathcal{A}$ be a closed complex subspace of $\mathcal{L}(H, K)$ and suppose $F \in \mathcal{A}$ satisfies $ZF^*Z \in \mathcal{A}$ for all $Z \in \mathcal{A}$ and $||F|| \leq 1$. Then $B = F[I + (I - F^*F)^{\frac{1}{2}}]^{-1}$ satisfies the same hypotheses as F and

$$S_B(Z) = i\,(I + BB^*)^{-\frac{1}{2}}(Z + B)(I - B^*Z)^{-1}(I + B^*B)^{\frac{1}{2}}$$

is a biholomorphic mapping of the open unit ball $\mathcal{A}_0$ of $\mathcal{A}$ onto the convex circular domain

$$\mathcal{H}_F = \{Z \in \mathcal{A}: \ 2\, Im\, F^*Z - Z^*(I - FF^*)^{\frac{1}{2}}Z + (I - F^*F)^{\frac{1}{2}} > 0\}.$$

b) Let $\mathcal{A}$ and $\mathcal{B}$ be closed complex subspaces of $\mathcal{L}(H,K)$ and suppose that T is a linear fractional transformation which maps the open unit ball $\mathcal{A}_0$ of $\mathcal{A}$ biholomorphically onto a domain $\mathcal{D}$ in $\mathcal{B}$. Let the coefficient matrix of T be given by (1), put $X_0 = D^{-1}C$ and suppose $X_0^* \in \mathcal{A}$. Then $\mathcal{D}$ is a convex circular domain and there is an invertible affine linear fractional transformation of $\mathcal{D}$ onto a domain $\mathcal{H}_F$ as in part (a). Also $\mathcal{D}$ is bounded if and only if $||X_0|| < 1$, and in that case there is an invertible affine linear fractional transformation of $\mathcal{D}$ onto $\mathcal{A}_0$.

The following example shows that the condition $X_0^* \in \mathcal{A}$ cannot be omitted from part (b).

Example 5 Let $\mathcal{A} = \{ \begin{bmatrix} z_1 & z_2 \\ 0 & z_3 \end{bmatrix} : \ z_1, z_2, z_3 \in \mathbb{C}\}$, $X_0 = \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix}$ and $T(Z) = Z(I + X_0Z)^{-1}$. Then T is a biholomorphic mapping of $\mathcal{A}_0$ onto a domain $\mathcal{D}$ in $\mathcal{A}$ which is not convex and hence is not affinely equivalent to any domain $\mathcal{H}_F$. To see this, note that

$$\mathcal{A}_0 = \{ \begin{bmatrix} z_1 & z_2 \\ 0 & z_3 \end{bmatrix} : \ |z_2|^2 < (1 - |z_1|^2)(1 - |z_3|^2), \ |z_3| < 1\}$$

$$T^{-1}(\begin{bmatrix} z_1 & z_2 \\ 0 & z_3 \end{bmatrix}) = \frac{1}{1-z_1}\begin{bmatrix} z_1 & z_2 + z_1 z_3 \\ 0 & z_3(1-z_1) \end{bmatrix}$$

so

$$\mathcal{D} = \{ \begin{bmatrix} z_1 & z_2 \\ 0 & z_3 \end{bmatrix} : \ |z_2 + z_1 z_3|^2 < (1 - 2\,\mathrm{Re}\, z_1)(1 - |z_3|^2), \ |z_3| < 1\}.$$

Clearly $0 \in \mathcal{D}$ and given $0 < r < 1$, if $z_1 = r/2 + i\,s$, $z_3 = r$ and $z_2 = -z_1 z_3$, then $Z_s = \begin{bmatrix} z_1 & z_2 \\ 0 & z_3 \end{bmatrix} \in \mathcal{D}$ for all real s. However, if $0 < t < 1$, it is easy to verify that $tZ_s \notin \mathcal{D}$ for all large enough s. Hence $\mathcal{D}$ is not starlike with respect to 0 and, in particular, not convex.

Proof. To prove (a), note that $||B|| \leq 1$ since $B^*B \leq I$. By part (a) of Lemma 6, $Fp(F^*F) \in \mathcal{A}$ for all polynomials p so $B \in \mathcal{A}$. Also, by part (d) of Lemma 6, if $Z \in \mathcal{A}$ then $Z[Fp(F^*F)]^*Z \in \mathcal{A}$ for all polynomials p so $ZB^*Z \in \mathcal{A}$. By our remarks following (5) and Theorem 8, we conclude that S_B is a biholomorphic mapping of $\mathcal{A}_0$ onto $\mathcal{D}(J_2, \mathcal{A})$, where $J_2 = (M^{-1})^* J_1 M^{-1}$ and J_1 is as in Example 1. Since

$$M^{-1} = \begin{bmatrix} -i\,(I + BB^*)^{-\frac{1}{2}} & -(I + BB^*)^{-\frac{1}{2}} B \\ -i\,(I + B^*B)^{-\frac{1}{2}} B^* & (I + B^*B)^{-\frac{1}{2}} \end{bmatrix}$$

and

$$\begin{aligned} F &= 2\,B(I + B^*B)^{-1} \\ (I - FF^*)^{\frac{1}{2}} &= (I - BB^*)(I + BB^*)^{-1} \end{aligned}$$

$$(I - F^*F)^{\frac{1}{2}} \;\; = \;\; (I - B^*B)(I + B^*B)^{-1},$$

we have

$$J_2 = \begin{bmatrix} (I - FF^*)^{\frac{1}{2}} & -i\,F \\ i\,F^* & -(I - F^*F)^{\frac{1}{2}} \end{bmatrix}. \tag{14}$$

This proves part (a).

To prove part (b), we first show that $||X_0^*|| \leq 1$. If $|\lambda| > ||X_0^*||$, then $I + X_0(-\lambda^{-1}X_0^*)$ is invertible since T is defined in $\mathcal{A}_0$ and $X_0^* \in \mathcal{A}$. Hence the spectral radius r of $X_0 X_0^*$ satisfies $r \leq ||X_0^*||$, so $||X_0^*|| \leq 1$ since $r = ||X_0^*||^2$. Let $B = -X_0^*$ and observe that $B \in \mathcal{A}$ and $||B|| \leq 1$. It is an easy exercise to modify the proof of Proposition 5 (so that invertibility of M is not assumed) to show that $ZB^*Z \in \mathcal{A}$ whenever $Z \in \mathcal{A}$. Hence the proof of part (a) shows that S_B is a biholomorphic mapping of $\mathcal{A}_0$ onto $\mathcal{H}_F$, where $F = 2\,B(I + B^*B)^{-1}$. By applying Proposition 3 with $Z_0 = 0$, we may write $T = T_a \circ T_2$ and $S_B = S_a \circ S_2$, where both T_a and S_a are invertible affine linear fractional transformations and where $S_2 = T_2$ by our choice of B. Hence $S_B = S_a \circ T_a^{-1} \circ T$ so $S_a \circ T_a^{-1}$ maps $\mathcal{D}$ onto $\mathcal{H}_F$.

If $||X_0|| < 1$, take $B = X_0^*$ and recall that the transformation T_B of Example 1 is a biholomorphic mapping of $\mathcal{A}_0$ onto itself. Hence the above argument holds with S_B replaced by T_B to prove the the existence of the required affine mapping of $\mathcal{D}$ onto $\mathcal{A}_0$. The existence of this mapping can also be deduced from Lemma 6 and the comments following Example 4.

If $\mathcal{D}$ is bounded, then T_2 is bounded on $\mathcal{A}_0$ by some number M. To show that $||X_0|| < 1$, suppose $||X_0|| = 1$ and take $Z = -tX_0^*$, where $0 < t < 1$. Since $Z^*Z \leq M^2(I + X_0Z)^*(I + X_0Z)$, by the spectral mapping theorem, $st^2 \leq M^2(1 - st)^2$ for all $s \in \sigma(X_0X_0^*)$. We may take $s = t = 1$ in this inequality to obtain the desired contradiction.

3. Dual Circular Domains

Because of the general non-commutativity of operators, there is another obvious choice for the definition of linear fractional transformations of operators. Let $N \in \mathcal{L}(K \times H)$ and write

$$N = \begin{bmatrix} P & Q \\ R & L \end{bmatrix} \tag{15}$$

as in (1). Define the *linear fractional transformation* S *with alternate coefficient matrix* N by

$$S(Z) = (ZR + P)^{-1}(ZL + Q) \tag{16}$$

for $Z \in \mathcal{L}(H, K)$ where $(ZR + P)^{-1}$ exists. By Proposition 4, these transformations agree with the linear fractional transformations previously defined in (2). Specifically, let J_0 be as in Example 1 and put $M = J_0N^{-1}J_0$. If

T is the linear fractional transformation with coefficient matrix M, then S and T both have the same domain of definition and are equal there. (For example, $N = M$ for the transformations (5) by [19, §3].) It is easy to see that if $W = S(Z)$ then

$$[I\ W] = (ZR + P)^{-1}[I\ Z]N$$

so

$$[I\ W]J_2[I\ W]^* = (ZR + P)^{-1}[I\ Z]NJ_2N^*[I\ Z]^*[(ZR + P)^{-1}]^*, \tag{17}$$

where $J_2 \in \mathcal{L}(K \times H)$. In view of Lemma 7, this suggests the following definition, which leads directly to an analogous result.

Let $\mathcal{A}$ be a closed complex subspace of $\mathcal{L}(H, K)$. A *dual circular domain* in $\mathcal{A}$ is a set of the form

$$\tilde{\mathcal{D}}(J, \mathcal{A}) = \{Z \in \mathcal{A} :\ [I\ Z]J[I\ Z]^* > 0\},$$

where $J \in \mathcal{L}(K \times H)$ is self-adjoint. The decomposition (8) gives

$$\tilde{\mathcal{D}}(J, \mathcal{A}) = \{Z \in \mathcal{A} :\ E + 2\operatorname{Re} ZF^* + ZGZ^* > 0\}.$$

It is not difficult to reformulate and prove our results about circular domains for dual circular domains. Basically, $\mathcal{D}(J, \mathcal{A})$ is replaced by $\tilde{\mathcal{D}}(J, \mathcal{A})$,

$J_2 = (M^{-1})^* J_1 M^{-1}$ is replaced by $J_2 = N^{-1} J_1 (N^{-1})^*$, and $E \geq 0$ is replaced by $G \leq 0$. Analogous arguments or the following lemma can be used to carry over results for circular domains to dual circular domains.

Define $\mathcal{D}^* = \{Z^* : Z \in \mathcal{D}\}$ for sets $\mathcal{D} \subseteq \mathcal{A}$ and define $T^*(Z) = T(Z^*)^*$ for functions $T : \mathcal{D}_1 \to \mathcal{D}_2$. Also, define $\hat{J} = -U^* J U$ for $J \in \mathcal{L}(K \times H)$, where $U \in \mathcal{L}(H \times K, K \times H)$ is given by $U = \begin{bmatrix} 0 & I \\ I & 0 \end{bmatrix}$. (Thus $\mathcal{A}^*$ is not the dual space of $\mathcal{A}$ and $U^* \neq U$ unless $H = K$.)

Lemma 12

a) $\mathcal{D}$ is a dual circular domain in $\mathcal{A}$ iff. $\mathcal{D}^*$ is a circular domain in $\mathcal{A}^*$. Moreover, $\tilde{\mathcal{D}}(J, \mathcal{A})^* = \mathcal{D}(\hat{J}, \mathcal{A}^*)$.

b) T is a linear fractional transformation which maps a domain $\mathcal{D}_1$ in $\mathcal{A}$ biholomorphically onto a domain $\mathcal{D}_2$ in $\mathcal{B}$ iff. T^* is a linear fractional transformation which maps $\mathcal{D}_1^*$ biholomorphically onto $\mathcal{D}_2^*$. Moreover, if N is an alternate coefficient matrix for T, then $M = \hat{N}^*$ is a coefficient matrix for T^*.

c) If $M = \hat{N}^*$, then $J_2 = N^{-1} J_1 (N^{-1})^*$ iff. $\hat{J}_2 = (M^{-1})^* \hat{J}_1 M^{-1}$.

The proof of the above lemma is straightforward and will be omitted. Note that by Proposition 2, the value of X_0 for T is $R(Z_0 R + P)^{-1}$ so the corresponding value for T^* is X_0^*. Also, observe that if $J = \begin{bmatrix} E & F \\ F^* & G \end{bmatrix}$ then $\hat{J} = \begin{bmatrix} -G & -F^* \\ -F & -E \end{bmatrix}$. Thus, for example, $\tilde{\mathcal{D}}(J, \mathcal{A})$ is convex when $G \leq 0$. As another example, we state an analogue of Theorem 8 for dual circular domains.

Theorem 13 *Suppose the hypotheses of Proposition 5 are satisfied. Given a self-adjoint $J_1 \in \mathcal{L}(K \times H)$, put $J_2 = N^{-1} J_1 (N^{-1})^*$, where $N = J_0 M^{-1} J_0$, and suppose $Z_0 \in \tilde{\mathcal{D}}(J_1, \mathcal{A})$. If $G_1 \leq 0$ and $G_2 \leq 0$, then T is a biholomorphic mapping of $\tilde{\mathcal{D}}(J_1, \mathcal{A})$ onto $\tilde{\mathcal{D}}(J_2, \mathcal{B})$.*

The hypothesis $G_2 \leq 0$ is equivalent to

$$X_0[I\ Z_0] J_1 [I\ Z_0]^* X_0^* - 2\,\mathrm{Re}\, X_0(F_1 + Z_0 G_1) + G_1 \leq 0.$$

Note that if N is as in (15), it follows from $J_1 = N J_2 N^*$ that

$$[I\ Z] J_1 [I\ Z]^* = \mathrm{Re}\,(ZR + P) Y^* + (ZL + Q) G_2 (ZL + Q)^*, \tag{18}$$

where $Y = 2(ZL + Q) F_2^* + (ZR + P) E_2$. Thus a remark for dual circular domains analogous to the one before Theorem 11 is true.

Many circular domains are also dual circular domains. Specifically, given a $J \in \mathcal{L}(K \times H)$, we wish to find a $J' \in \mathcal{L}(K \times H)$ such that $\mathcal{D}(J, \mathcal{A}) =$

$\tilde{\mathcal{D}}(J', \mathcal{A})$. For example, if $J = J_0$ then $\mathcal{D}(J, \mathcal{A}) = \{Z \in \mathcal{A} : \ ||Z|| < 1\}$ and $J' = J_0$ since $Z^*Z - I < 0$ iff. $I - ZZ^* > 0$. Also, if $J = \begin{bmatrix} 0 & -i\,Y \\ i\,Y^* & 0 \end{bmatrix}$, where Y is an invertible operator in $\mathcal{L}(H, K)$, then

$$\mathcal{D}(J, \mathcal{A}) = \{Z \in \mathcal{A} : \ \mathrm{Im}\, Y^* Z > 0\} \tag{19}$$

and $J' = \begin{bmatrix} 0 & i(Y^{-1})^* \\ -i\,Y^{-1} & 0 \end{bmatrix}$ since $(Y^{-1})^*(\mathrm{Im}\, Y^*Z)Y^{-1} = \mathrm{Im}\, ZY^{-1}$.

The following theorem shows how to compute the representation of a circular domain as a dual circular domain when the matrix giving the domain is Hermitian congruent to the matrix of a domain where the representation as a dual circular domain is known. (Note that this does not require the domains to be holomorphically equivalent.) We state our result in operator-theoretic terms. Let $\mathcal{N}$ be the set of normal operators on H and let $\mathcal{K}$ be the set of compact operators on H.

Theorem 14 *Let $J_1, J_1' \in \mathcal{L}(K \times H)$ be self-adjoint with $E_1 \geq 0$ and $G_1' \leq 0$, and suppose*

$$\begin{bmatrix} Z \\ I \end{bmatrix}^* J_1 \begin{bmatrix} Z \\ I \end{bmatrix} < 0 \ \textit{ iff. } \ [I\ Z]J_1'[I\ Z]^* > 0$$

for all $Z \in \mathcal{L}(H, K)$. Let M be an invertible operator as in (1) and put

$$J_2 = M^* J_1 M, \quad J_2' = N J_1' N^*,$$

where $N = J_0 M^{-1} J_0$. *Suppose* $(CZ_0 + D)^{-1}$ *exists for some* $Z_0 \in \mathcal{L}(H, K)$. *If* $Z \in \mathcal{L}(H, K)$ *and if one of the conditions*

a) $(CZ + D)^{-1}$ *exists,*
b) $E_2 \geq 0, G'_2 \leq 0$ *and* Z_0 *satisfies both inequalities of* (20),
c) $X_0(Z - Z_0) \in \mathcal{N} + \mathcal{K}$,
d) $(Z - Z_0)X_0 \in \mathcal{N} + \mathcal{K}$,

holds, then

$$\begin{bmatrix} Z \\ I \end{bmatrix}^* J_2 \begin{bmatrix} Z \\ I \end{bmatrix} < 0 \quad \textit{iff.} \quad [I \ Z] J'_2 [I \ Z]^* > 0. \tag{20}$$

Clearly (c) and (d) are satisfied if C is compact. Also, note that if $J'_1 = J_0 J_1^{-1} J_0$, then $J'_2 = J_0 J_2^{-1} J_0$. The following contains a result of Ginzburg [13] on uniformly J-expansive operators. (See also [26, p. 77].)

Corollary 15 *Let* Y *be an invertible self-adjoint operator on a Hilbert space* H *and let* $Z \in \mathcal{L}(H)$. *If one of the operators* $(I-Y)(I-Z)$ *or* $(I-Z)(I-Y)$ *is in* $\mathcal{N} + \mathcal{K}$ *or if* $(I - Z)^{-1}$ *exists, then* $Z^*YZ < Y$ *iff.* $ZY^{-1}Z^* < Y^{-1}$.

Clearly the first assumption holds when $I - Y$ or $I - Z$ is compact. The conclusion of Corollary 15 is false without some restriction on Y or Z. For example, suppose $H = H_1 \times H_2$, where dim $H_2 = \infty$, and let V be a non-unitary isometry on H_2. Take $Y = \begin{bmatrix} I & 0 \\ 0 & -I \end{bmatrix}$ and $Z = \begin{bmatrix} 0 & 0 \\ 0 & rV \end{bmatrix}$, where $r > 1$. Then $Z^*YZ < Y$ but $ZYZ^* < Y$ is false.

Corollary 16 *Let* $J \in \mathcal{L}(K \times H)$ *be given by (8) and suppose* $E > 0$ *and* $Y = F^*E^{-1}F - G > 0$. *Then* $\mathcal{D}(J, \mathcal{A}) = \tilde{\mathcal{D}}(J', \mathcal{A})$, *where* $J' = J_0 J^{-1} J_0$. *Specifically,*

$$J' = \begin{bmatrix} E^{-1} - E^{-1}FY^{-1}F^*E^{-1} & -E^{-1}FY^{-1} \\ -(E^{-1}FY^{-1})^* & -Y^{-1} \end{bmatrix}.$$

For example, Corollary 16 in the case $E = Q^{-1}$, $F = 0$ and $G = -P$ implies [1].

Corollary 17 *Suppose* $\mathcal{A}$ *and* F *satisfy the hypotheses of part (a) of Theorem 11. Then*

$$\mathcal{H}_F = \{Z \in \mathcal{A} : \; 2\, Im\, ZF^* - Z(I - F^*F)^{\frac{1}{2}} Z^* + (I - FF^*)^{\frac{1}{2}} > 0\}.$$

Lemma 18 *If* $F \in \mathcal{N} + \mathcal{K}$ *and if* F *has a left or right inverse, then* F *is invertible.*

Proof (See [8, Ch. 5].) Suppose F has a left inverse. Write $F = N + K$, where $N \in \mathcal{N}$ and $K \in \mathcal{K}$, and let π be the natural homomorphism of $\mathcal{L}(H)$ onto the Calkin algebra $\mathcal{C}$. Then $\pi(F)$ is a normal element of $\mathcal{C}$ with a left inverse so F is Fredholm and $\operatorname{ind} F = \operatorname{ind} N = 0$. By hypothesis, $\ker F = \{0\}$ so $(\operatorname{ran} F)^{\perp} = \{0\}$. Hence $\operatorname{ran} F = H$ so F is invertible. If F has a right inverse then the above argument applies to F^*.

Proof of Theorem 14. We first consider the case where condition (a) holds. Let T be given by (2), let N be given by (15) and let S be given by (16). By our previous comments, $T(Z)$ and $S(Z)$ are both defined and equal to some operator W. By our hypothesis, (10) holds with J_1 and J_2 interchanged and (17) holds with J_2 replaced by J_1'. This implies (20). More generally, (20) holds for any Z such that $CZ + D$ is invertible when either one of the operator inequalities in (20) holds. We will show that this is the case when one of the conditions (b), (c) or (d) holds. Let $\mathcal{A} = \mathcal{L}(H, K)$.

If (b) holds, then $T(Z)$ is defined in $\mathcal{D}(J_2)$ by Theorem 8 and $T(Z)$ is defined on $\tilde{\mathcal{D}}(J_2')$ by Theorem 13 so (20) holds.

Suppose (c) holds and let $Z \in \mathcal{D}(J_2)$. By our hypotheses, $E_1 \geq 0$ and (13) holds with J_1 and J_2 interchanged so $CZ + D$ has a left inverse. Then $F_l = I + X_0(Z - Z_0)$ has a left inverse since $CZ + D = (CZ_0 + D)F_l$. Hence by Lemma 18, F_l is invertible so $CZ + D$ is invertible.

Now let $Z \in \tilde{D}(J_2')$. By our hypotheses, $G_1' \leq 0$ and (18) holds with J_1 replaced by J_2' and J_2 replaced by J_1', so $ZR + P$ has a right inverse. Then $F_r = I + (Z - Z_0)X_0$ has a right inverse Y since $ZR + P = F_r(Z_0R + P)$. Hence F_l has a right inverse given by $I - X_0Y(Z - Z_0)$ so F_l is invertible by Lemma 18. Therefore $CZ + D$ is invertible. A similar argument also leads to this conclusion when (d) is assumed.

Proofs of Corollaries. To prove Corollary 15, we first consider the case where one of the mentioned operators is in $\mathcal{N}+\mathcal{K}$. Take $J_1 = J_1' = J_0$ and $M = \frac{1}{2}\begin{bmatrix} I+Y & I-Y \\ I-Y & I+Y \end{bmatrix}$ in Theorem 14. Then $J_2 = \begin{bmatrix} Y & 0 \\ 0 & -Y \end{bmatrix}$ and $J_2' = \begin{bmatrix} Y^{-1} & 0 \\ 0 & -Y^{-1} \end{bmatrix}$ since $J_2' = J_0 J_2^{-1} J_0$. If $Z_0 = I$, then $X_0 = (I-Y)/2$. By hypothesis, (c) or (d) holds so the required equivalence follows from from (20).

Now suppose $(I-Z)^{-1}$ exists and let J_1 and J_1' be those given for (19). Take $M = \frac{1}{\sqrt{2}}\begin{bmatrix} i\,I & i\,I \\ -I & I \end{bmatrix}$. Then J_2 and J_2' are as in the previous case. By hypothesis, (a) holds so the required equivalence follows from (20).

Corollary 16 follows immediately from Theorem 14 with $J_1 = J_1' = J_0$, $J_2 = J$ and $M = \begin{bmatrix} E^{\frac{1}{2}} & E^{-\frac{1}{2}}F \\ 0 & Y^{\frac{1}{2}} \end{bmatrix}$ since (a) holds.

To prove Corollary 17, take $J_1' = J_1 = J_0$ and let M be the inverse of the coefficient matrix of S_B. Then J_2 is given by (14) and $J_2' = J_0 J_2^{-1} J_0 = J_0 J_2 J_0$ since $J_2^2 = I$. Now (b) holds with $Z_0 = iB$ so $\mathcal{H}_F = \tilde{\mathcal{D}}(J_2', \mathcal{A})$ by (20).

References

[1] Anon, *Positive definite matrices*, Problem 6095, Amer. Math. Monthly **85**(1978), 59.

[2] T. Ya. Azizov and I.S. Iokhvidov, *Linear Operators in Spaces with an Indefinite Metric*, Wiley, Chichester, 1989.

[3] J. A. Ball, I. Gohberg and L. Rodman, *Interpolation of Rational Matrix Functions*, Operator Theory: Advances and Applications, Vol 45, Birkhäuser Verlag, Basel, 1990.

[4] E. Bedford and J. Dadok, *Generalized Reinhardt domains*, Journal of Geometric Analysis **1**(1991), 1-17.

[5] R. Bellman, *Introduction to Matrix Analysis*, McGraw-Hill, New York 1970.

[6] E. Cartan, *Sur les domaines bornés homogènes de l'espace de n variables l'espace de n variables complexes*, Abh. Math. Sem. Univ. Hamburg **11**(1935), 116-162.

[7] S. Dineen, *The Schwarz Lemma*, Oxford Mathematical Monographs, Oxford, 1989.

[8] R. G. Douglas, *Banach Algebra Techniques in Operator Theory*, Academic Press, New York, 1972.

[9] H. Dym, *J Contractive Matrix Functions, Reproducing Kernel Hilbert Spaces and Interpolation*, Regional Conference Series in Mathematics, No. 71, AMS, Providence, 1989.

[10] H. Dym and A. Iacob, *Positive definite extensions, canonical equations and inverse problems*, in Topics in Operator Theory Systems and Networks (Ed. H. Dym and I. Gohberg), Operator Theory: Advances and Applications, Vol 12, p. 141-240, Birkhäuser Verlag, Basel, 1984.

[11] A. V. Efimov and V. P. Potapov, *J-expanding matrix functions and their role in the analytical theory of electrical circuits*, Russian Math. Surveys **28**(1973) No. 1, 69-140.

[12] C. Foias and A. E. Frazho, *The Commutant Lifting Approach to Interpolation Problems*, Operator Theory: Advances and Applications, Vol 44, Birkhäuser Verlag, Basel, 1990.

[13] Ju. P. Ginzburg, *On J-nonexpansive operators in Hilbert space*, Naučn. Zap. Fiz.-Mat. Fak. Odessa Gos. Ped. Inst. **22**(1958), 13-20 (Russian).

[14] I. Gohberg, Editor, *I. Schur Methods in Operator Theory and Signal Processing*, Operator Theory: Advances and Applications, Vol 18, Birkhäuser Verlag, Basel, 1986.

[15] I. Gohberg, P. Lancaster and L. Rodman, *Matrices and Indefinite Scalar Products*, Operator Theory: Advances and Applications, Vol 8, Birkhäuser Verlag, Basel, 1983.

[16] I. Gohberg and S. Rubinstein, *Proper contractions and their unitary minimal completions*, in Topics in Interpolation Theory of Rational Matrix-valued Functions (Ed. I. Gohberg), Operator Theory: Advances and Applications, Vol 33, p.223-247, Birkhäuser Verlag, Basel, 1986.

[17] P. R. Halmos, *A Hilbert Space Problem Book*, Van Nostrand, Princeton, 1967.

[18] L. A. Harris, *Bounded symmetric homogeneous domains in infinite dimensional spaces*, in Infinite Dimensional Holomorphy (Eds. T. L. Hayden and T. J. Suffridge), Lecture Notes in Mathematics, Vol 364, p. 13-40, Springer Verlag, Berlin, 1973.

[19] L. A. Harris, *Operator Siegel domains*, Proc. Royal Soc. Edinburgh **79A** (1977), 137-156.

[20] L. A. Harris, *A generalization of C*-algebras*, Proc. London Math. Soc. **42**(1981), 331-361.

[21] J. W. Helton, *Operator Theory, Analytic Functions, Matrices and Electrical Engineering*, Regional Conference Series in Mathematics, No. 68, AMS, Providence, 1987.

[22] E. Hille and R. S. Phillips, *Functional Analysis and Semi-Groups*, Amer. Math. Soc. Colloq. Publ., Vol. 31, AMS, Providence, 1957.

[23] A. S. Householder, *The Theory of Matrices in Numerical Analysis*, Blaisdell, New York, 1964; Dover, New York, 1975.

[24] Loo-Keng Hua, *On the theory of automorphic functions of a matrix variable I - Geometrical Basis*, Amer. J. Math. **66**(1944), 470-488.

[25] Loo-Keng Hua, *On the theory of automorphic functions of a matrix variable II - The classification of hypercircles under the symplectic group*, Amer. J. Math. **66**(1944), 531-563.

[26] I. S. Iohvidov, M. G. Krein and H. Langer, *Introduction to the Spectral Theory of Operators in Spaces with an Indefinite Metric*, Akademie-Verlag, Berlin, 1982.

[27] M. G. Krein and Ju. L. Smuljan, *On linear-fractional transformations with operator coefficients*, Mat. Issled. **2**(1967), 64-96; English transl. in Amer. Math. Soc. Transl. **103**(1974), 125-152.

[28] V. P. Potapov, *The multiplicative structure of J-contractive matrix functions*, Trudy Moskov. Mat. Obšč. **4**(1955), 125-236; English transl. in Amer. Math. Soc. Transl. **15**(1960), 131-243.

[29] I. I. Pyatetskii-Shapiro, *The Geometry of the Classical Domains and the theory of Automorphic Functions*, Fizmatgiz, Moscow, 1961; French transl. Donod, Paris, 1966; English transl. Gordon and Breach, New York, 1969.

[30] R. M. Redheffer, *On a certain linear fractional transformation*, J. Math. and Physics **39**(1960), 269-286.

[31] B. Schwarz and A. Zaks, *Matrix möbius transformations*, Comm. in Algebra **19**(1981), 1913-1968.

[32] H. Schwerdtfeger, *Geometry of Complex Numbers*, University of Toronto Press, Toronto, 1962; Dover, New York, 1979.

[33] Ju. L. Smuljan, *On operator balls and linear-fractional transformations with operator coefficients*, Math. USSR Sbornik **6**(1968), 309-326.

[34] Ju. L. Smuljan, *General linear-fractional transformations of operator balls*, Siberian Math. J. **19**(1978), 293-298.

[35] H. Upmeier, *Symmetric Banach Manifolds and Jordan C*-Algebras*, North-Holland, Amsterdam, Math. Studies, Vol. 104, 1985.

[36] N. J. Young, *Linear fractional transformations in rings and modules*, Linear Alg. Appl. **56**(1984), 251-290.

[37] M. Kh. Zakhar-Itkin, *The matrix Riccati differential equation and the semi-group of linear fractional transformations*, Russian Math. Surveys **28**(1973) No. 3, 89-131.

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