Computation of Functions of Certain Operator Matrices

Lawrence A. Harris
Mathematics Department
University of Kentucky
Lexington, Kentucky 40506

ABSTRACT
This note gives a simple method to compute the entries of holomorphic functions of a 2×2 block or operator matrix which can be written as a product. To illustrate this method, the entries are given for the exponential, fractional powers and inverse of such operator matrices.

If H and K are Hilbert spaces, we denote the space of bounded linear operators from H to K by $\mathcal{L}(H,K)$. Our main result is the following:

Theorem 1 *Let $B_1, C_1 \in \mathcal{L}(H,K_1)$ and $B_2, C_2 \in \mathcal{L}(H,K_2)$. Suppose f is a function which is holomorphic in an open set D containing the spectrum $\sigma(C_1^* B_1 + C_2^* B_2)$ and 0. Then*

$$f\left(\begin{bmatrix} B_1C_1^* & B_1C_2^* \\ B_2C_1^* & B_2C_2^* \end{bmatrix}\right) = \begin{bmatrix} f(0)I + B_1RC_1^* & B_1RC_2^* \\ B_2RC_1^* & f(0)I + B_2RC_2^* \end{bmatrix},$$

where $R = g(C_1^* B_1 + C_2^* B_2)$ and g is the holomorphic extension to D of $g(z) = [f(z) - f(0)]/z$.

Note that when $H = \mathbb{C}$ the operators B_1, B_2, C_1 and C_2 may be identified with vectors in K_1 or K_2. The proof of Theorem 1 follows immediately from the lemma below with $B = \begin{bmatrix} B_1 \\ B_2 \end{bmatrix}$, $C = \begin{bmatrix} C_1 \\ C_2 \end{bmatrix}$ and $K = K_1 \times K_2$.

Lemma 2 *Let $B, C \in \mathcal{L}(H, K)$. Suppose f is a function which is holomorphic in an open set D containing $\sigma(C^*B)$ and 0. Then*

$$f(BC^*) = f(0)I + Bg(C^*B)C^*.$$

In particular, the above lemma contains the well-known formula (eg. [6, p. 264])

$$(\lambda I - BC^*)^{-1} = \frac{1}{\lambda}\,[I + B(\lambda I - C^*B)^{-1}C^*], \tag{1}$$

which holds when $\lambda \notin \sigma(C^*B)$ and $\lambda \neq 0$.

Proof. Let $S = \sigma(C^*B) \cup \{0\}$. Then $\sigma(BC^*) \subseteq S$ by (1). Clearly $f(z) = f(0) + zg(z)$ for $z \in D$ so $f(BC^*) = f(0)I + BC^*g(BC^*)$ by the holomorphic functional calculus [4, §5.2]. It suffices to show that $C^*g(BC^*) = g(C^*B)C^*$. Since $(\lambda I - C^*B)C^* = C^*(\lambda I - BC^*)$, it follows that $C^*(\lambda I - BC^*)^{-1} = (\lambda I - C^*B)^{-1}C^*$ for all $\lambda \notin S$. Let Γ be an oriented envelope of S with respect to $g(z)$. Then by Theorems 3.3.2 and 5.2.4 of [4],

$$\begin{aligned} C^*g(BC^*) &= \frac{1}{2\pi i}\int_\Gamma g(\lambda)C^*(\lambda I - BC^*)^{-1}d\lambda \\ &= \frac{1}{2\pi i}\int_\Gamma g(\lambda)(\lambda I - C^*B)^{-1}C^*d\lambda \\ &= g(C^*B)C^*, \end{aligned}$$

as required.

Corollary 3 $$\exp(\begin{bmatrix} B_1C_1^* & B_1C_2^* \\ B_2C_1^* & B_2C_2^* \end{bmatrix}) = \begin{bmatrix} I + B_1RC_1^* & B_1RC_2^* \\ B_2RC_1^* & I + B_2RC_2^* \end{bmatrix},$$

where

$$R = \sum_{n=0}^{\infty} \frac{(C_1^*B_1 + C_2^*B_2)^n}{(n+1)!}\ .$$

Let S be a compact set in $\mathbb{C}$ containing 1 and suppose that 0 is in the unbounded component of the complement of S. It is not difficult to show that there is a simply connected domain D containing S and an n *th* root function $z \longrightarrow z^{1/n}$ which is holomorphic on D with $1^{1/n} = 1$. By the functional calculus, any such function defines an n *th* root for any bounded linear operator A on H satisfying $\sigma(A) \subseteq S$. In particular, when A is a positive operator, we may define $A^{1/n}$ to be the unique positive n *th* root of A. (Existence and uniqueness can be proved by a simple modification of the argument for the square root given in [2, Prop. 4.33] or as in [1, 5].)

Corollary 4 *Suppose that m and n are positive integers and that 1 is in the unbounded component of the complement of $\sigma(C_1^*B_1 + C_2^*B_2)$. Then*

$$\begin{bmatrix} I - B_1C_1^* & -B_1C_2^* \\ -B_2C_1^* & I - B_2C_2^* \end{bmatrix}^{m/n} = \begin{bmatrix} I - B_1RC_1^* & -B_1RC_2^* \\ -B_2RC_1^* & I - B_2RC_2^* \end{bmatrix},$$

where $R = \phi(I - C_1^*B_1 - C_2^*B_2)$ and

$$\phi(z) = \frac{1 + z + \cdots + z^{m-1}}{1 + z^{m/n} + \cdots + z^{(n-1)m/n}} \ .$$

In particular, when the exponent is $1/2$, $R = [I + (I - C_1^*B_1 - C_2^*B_2)^{1/2}]^{-1}$.

Corollary 5 *Under the conditions of Corollary 4,*

$$\begin{bmatrix} I - B_1C_1^* & -B_1C_2^* \\ -B_2C_1^* & I - B_2C_2^* \end{bmatrix}^{-m/n} = \begin{bmatrix} I + B_1RC_1^* & B_1RC_2^* \\ B_2RC_1^* & I + B_2RC_2^* \end{bmatrix},$$

where $R = (I - C_1^*B_1 - C_2^*B_2)^{-m/n}\phi(I - C_1^*B_1 - C_2^*B_2)$. In particular, when the exponent is $-1/2$, $R = (I - C_1^*B_1 - C_2^*B_2)^{-1/2}[I + (I - C_1^*B_1 - C_2^*B_2)^{1/2}]^{-1}$.

The following formula is useful for computing inverses. In particular, it allows one to deduce Corollary 5 from Corollary 4 (by incorporating R into B_1 and B_2). Compare [3, Prop. 2].

Corollary 6 *Suppose λ is a non-zero complex number outside of $\sigma(C_1^*B_1 + C_2^*B_2)$. Then*

$$\begin{bmatrix} \lambda I - B_1C_1^* & -B_1C_2^* \\ -B_2C_1^* & \lambda I - B_2C_2^* \end{bmatrix}^{-1} = \begin{bmatrix} \lambda^{-1}I + B_1RC_1^* & B_1RC_2^* \\ B_2RC_1^* & \lambda^{-1}I + B_2RC_2^* \end{bmatrix},$$

where $R = \lambda^{-1}(\lambda I - C_1^*B_1 - C_2^*B_2)^{-1}$.

Proofs of Corollaries. Corollary 3 follows immediately from Theorem 1 with $f(z) = \exp(z)$. To deduce Corollary 4, put $S = \sigma(I - C^*B) \cup \{1\}$ and let $z \to z^{1/n}$ be an n *th* root function as described in the comments preceding Corollary 4. Take $f(z) = (1 - z)^{m/n}$ and note that R is as asserted since $g(z) = -\phi(1 - z)$. The proof of Corollary 5 is similar but with $f(z) = (1 - z)^{-m/n}$ and $g(z) = (1 - z)^{-m/n}\phi(1 - z)$. Corollary 6 follows immediately from Theorem 1 with $f(z) = (\lambda - z)^{-1}$. It also follows immediately from (1).

References

[1] D. R. Brown and M. J. O'Malley, *On n th roots of positive operators*, Amer. Math. Monthly **87**(1980), 380-382.

[2] R. G. Douglas, *Banach Algebra Techniques in Operator Theory*, Academic Press, New York, 1972.

[3] L. A. Harris, *Linear fractional transformations of circular domains in operator spaces*, Indiana Univ. Math. J. **41**(1992), 125-147.

[4] E. Hille and R. S. Phillips, *Functional Analysis and Semi-Groups*, Amer. Math. Soc. Colloq. Publ., Vol. 31, AMS, Providence, 1957.

[5] N. M. Rice, *On n th roots of positive operators*, Amer. Math. Monthly **89**(1982), 313-314.

[6] A. E. Taylor, *Introduction to Functional Analysis*, Wiley, New York, 1958.