Multivariate Markov Polynomial Inequalities and Chebyshev Nodes

Lawrence A. Harris
Department of Mathematics, University of Kentucky
Lexington, Kentucky 40506-0027

Abstract

This article considers the extension of V. A. Markov's theorem for polynomial derivatives to polynomials with unit bound on the closed unit ball of any real normed linear space. We show that this extension is equivalent to an inequality for certain directional derivatives of polynomials in two variables that have unit bound on the Chebyshev nodes. We obtain a sharpening of the Markov inequality for polynomials whose values at specific points have absolute value less than one. We also obtain an interpolation formula for polynomials in two variables where the interpolation points are Chebyshev nodes.

1. Introduction.

Throughout X and Y denote arbitrary real normed linear spaces and m and k are any integers with $m \geq 1$ and $0 \leq k \leq m$. If $f : \Omega \to Y$ is a function defined on a domain $\Omega \subseteq X$ and if $x \in \Omega$, the notation $\hat{D}^k f(x)$ denotes the homogeneous polynomial associated with the kth order Fréchet derivative at x, i.e.,

$$\hat{D}^k f(x)y = D^k f(x)(y, \ldots, y)$$

for all $y \in X$. (Recall [3, p. 179] that the kth order Fréchet derivative of f at a point is a continuous symmetric k-linear mapping from $X \times \cdots \times X$ to Y.) When $\hat{D}^k f(x)$ exists, it is always given by a kth order directional derivative at x, i.e.,

$$\hat{D}^k f(x)y = \frac{d^k}{dt^k}\, f(x + ty)\Big|_{t=0} \tag{1}$$

for all $y \in X$. By the usual definition,

$$\|\hat{D}^k f(x)\| = \sup\{\|\hat{D}^k f(x)y\| : \ \|y\| \leq 1\}.$$

Let $T_m(t) = \cos(m \arccos(t))$ denote the Chebyshev polynomial of degree m. Recently, V. I. Skalyga [13, Theorem 2] solved the longstanding problem of extending V. A. Markov's inequality for polynomial derivatives to real normed linear spaces (see [4] and [5]). An important case of his highly general results is the following.

Theorem 1. *Let $P : X \to Y$ be any polynomial of degree at most m satisfying $\|P(x)\| \leq 1$ for all $x \in X$ with $\|x\| \leq 1$. Then*

$$\|\hat{D}^k P(x)\| \leq T_m^{(k)}(1) \text{ for all } x \in X \text{ with } \|x\| \leq 1 \text{ and} \tag{2}$$

$$\|\hat{D}^k P(x)\| \leq T_m^{(k)}(\|x\|) \text{ for all } x \in X \text{ with } \|x\| \geq 1. \tag{3}$$

When $X = Y = \mathbb{R}$, inequality (2) reduces to the classical inequality of V. A. Markov, which is a rather deep theorem (see [11]), and inequality (3) reduces to a theorem of Schur, which is an elementary consequence of the Lagrange interpolation formula (see [9]). Thus it is interesting that (2) follows easily from the case $X = \ell_1(\mathbb{R}^2)$, $x = (1, 0)$ and $Y = \mathbb{R}$ of (3). (See Section 3.)

For the case $k = 1$, an elementary proof of (2) was given by Sarantopoulos in [10] and an easy extension of his argument proves (3). (See [5, p. 321] and note that $|p(r)| \leq T'_m(r)$ for $r \geq 1$ by the standard argument with Lagrange interpolation [6, p. 41].)

Let

$$\alpha_k(r) = \frac{r+1}{2(k+1)m^2} T_m^{(k+1)}(r), \qquad \beta_k(r) = \frac{r-1}{2(k+1)m^2} T_m^{(k+1)}(r), \tag{4}$$

$$\gamma_k(r) = \frac{1}{m^2 r} \left[\frac{(r^2-1) T_m^{(k+1)}(r)}{k+1} + k T_m^{(k-1)}(r) \right]. \tag{5}$$

Here we take $T_m^{(k-1)} = 0$ when $k = 0$. Note that $\alpha_k(r)$, $\beta_k(r)$ and $\gamma_k(r)$ are nonnegative for $r \geq 1$ since $T_m^{(k)}(r) \geq 0$ for $r \geq 1$ by (3) or [9, p. 8]. A main result is the following, which we deduce as a consequence of Theorem 1.

Theorem 2. *Let $P : X \to \mathbb{R}$ be a polynomial of degree at most m satisfying $|P(x)| \leq 1$ for all $x \in X$ with $\|x\| \leq 1$ and let k be even. Then*

$$\|\hat{D}^k P(x)\| \leq T_m^{(k)}(1) - \frac{T_m^{(k+1)}(1)}{(k+1)m^2}(1 - |P(x)|) \tag{6}$$

for all $x \in X$ with $\|x\| \leq 1$ and

$$\|\hat{D}^k P(x)\| \leq T_m^{(k)}(\|x\|) - \alpha_k(\|x\|)(1 - |P(u)|) - \beta_k(\|x\|)(1 - |P(-u)|) \tag{7}$$

for all $x \in X$ *with* $\|x\| \geq 1$*, where* $u = \dfrac{x}{\|x\|}$ *. If, in addition,* m *is even, then we may subtract the terms*

$$\frac{k}{m^2} T_m^{(k-1)}(1)(1 - |P(0)|) \quad \text{and} \quad \gamma_k(\|x\|)(1 - |P(0)|)$$

from the right-hand sides of inequalities (6) and (7), respectively.

2. Chebyshev nodes.

Recall that the Chebyshev points are defined by

$$h_n = \cos\left(\frac{n\pi}{m}\right), \quad n = 0, \ldots, m,$$

and satisfy $T_m(h_n) = (-1)^n$ for these n. Combining results of Duffin and Schaeffer [2] and of Rogosinksi [9], we have the following.

Theorem 3. *Let* $p(t)$ *be any polynomial of degree at most* m *satisfying* $|p(h_n)| \leq 1$ *for all* $n = 0, \ldots, m$*. Then*

$$\begin{aligned} |p^{(k)}(t)| &\leq T_m^{(k)}(1) \text{ whenever } -1 \leq t \leq 1 \text{ and} \\ |p^{(k)}(t)| &\leq T_m^{(k)}(|t|) \text{ whenever } |t| \geq 1. \end{aligned}$$

To discuss extensions of Theorem 3 to two dimensions, we define nodes for polynomial interpolation in $\mathbb{R}^2$ which we call Chebyshev nodes. For each m, we divide the ordered pairs of Chebyshev points into two disjoint sets of Chebyshev nodes. Specifically, let Q be the set of all ordered pairs (n, q) of integers with $0 \leq n, q \leq m$ and let

$$Q_k = \{(n, q) \in Q : \ n - q = k \bmod 2\}$$

for any nonnegative integer k. Define $\mathcal{N}_k$ be the set of nodes in $\mathbb{R}^2$ given by

$$\mathcal{N}_k = \{(h_n, h_q) : \ (n, q) \in Q_k\}.$$

We say that the nodes of $\mathcal{N}_0$ are the *even Chebyshev nodes* and the nodes of $\mathcal{N}_1$ are the *odd Chebyshev nodes*. Thus the even Chebyshev nodes are the ordered pairs of Chebyshev points with indices that are both even or both odd and the odd Chebyshev nodes are the ordered pairs of Chebyshev points with one of the indices even and the other odd.

Clearly $\mathcal{N}_0$ and $\mathcal{N}_1$ are disjoint sets and every ordered pair of Chebyshev points lies in $\mathcal{N}_0$ or $\mathcal{N}_1$. Also $\mathcal{N}_k$ is the set of even (resp., odd) Chebyshev nodes when

k is an even (resp., odd) integer. Letting $n(S)$ denote the number of elements of a set S, we have

$$\begin{aligned} n(\mathcal{N}_1) = n(\mathcal{N}_0) &= \frac{(m+1)^2}{2} \text{ for } m \text{ odd}, \\ n(\mathcal{N}_1) = n(\mathcal{N}_0) - 1 &= \frac{m(m+2)}{2} \text{ for } m \text{ even}. \end{aligned} \tag{8}$$

The Chebyshev nodes have appeared previously in [7] and [14] in connection with a minimal cubature formula for the product Chebyshev weight function. The following is a consequence of the generalization given in [1, Corollary 2.2].

Theorem 4. *Let* $w(t) = \dfrac{1}{\pi\sqrt{1-t^2}}$ *and let* $p(s,t)$ *be a polynomial in two variables with degree at most* $2m-1$*. Let* $k=0$ *or* $k=1$*. Then*

$$\int_{-1}^{1}\int_{-1}^{1} p(s,t)w(s)w(t)\,ds\,dt = \frac{2}{m^2}\sum_{(n,q)\in Q_k} c_n c_q p(h_n,h_q),$$

where c_j *is 1 except that it is* $1/2$ *when* $j=0$ *or* $j=m$*.*

Put $I = \{(i,j):\ 0 \le j \le i \le 2m-1\}$. Since the polynomials $T_{i-j}(s)T_j(t)$, $(i,j)\in I$, are an orthogonal basis for the space of polynomials in two variables of degree at most $2m-1$, Theorem 4 is easily seen to be equivalent to the identity

$$\frac{2}{m^2}\sum_{(n,q)\in Q_k} c_n c_q T_{i-j}(h_n)T_j(h_q) = \begin{cases} 1 & \text{if } i=0 \\ 0 & \text{if } i>0 \end{cases},$$

for all $(i,j)\in I$.

Let $(n,q)\in Q$. Clearly $(n,q)\in Q_k$, where $k=0$ or $k=1$. We say that a polynomial $P_{n,q}$ of degree at most m is a *Lagrange polynomial for* $\mathcal{N}_k$ if $P_{n,q}(h_n,h_q)=1$ and $P_{n,q}(x)=0$ whenever $x\in\mathcal{N}_k$ and $x\neq(h_n,h_q)$. Yuan Xu in [14] has obtained an explicit expression for such Lagrange polynomials, namely,

$$P_{n,q}(s,t) = \frac{2}{m^2}c_n c_q G(s,t,h_n,h_q), \tag{9}$$

where

$$\begin{aligned} G(s,t,u,v) &= 4\sum_{i=0}^{m}{}'' \sum_{j=0}^{i}{}'' T_{i-j}(s)T_j(t)T_{i-j}(u)T_j(v) \\ &\quad -\frac{1}{2}\left[T_m(s)T_m(u) + T_m(t)T_m(v)\right]. \end{aligned}$$

Here the $''$ in a sum indicates that the first and last terms of the sum should be divided by 2.

Given k, define

$$V_j(s,t) = T_{m-j}(s)T_j(t) - (-1)^k T_j(s)T_{m-j}(t), \quad j = 0, \ldots, m, \tag{10}$$

and note that the identity $T_{m-j}(h_n) = (-1)^n T_j(h_n)$ implies that $V_j(x) = 0$ whenever $x \in \mathcal{N}_k$ and $j = 0, \ldots, m$. Thus Lagrange polynomials are far from unique since any linear combination of the $V_j{}'s$ can be added to any of these polynomials.

Let $\mathcal{P}_m(\mathbb{R}^2)$ denote the space of all real-valued polynomials of degree at most m in two variables. Below is an interpolation formula for the Chebyshev nodes. (Compare [1].)

Theorem 5. *Let $k = 0$ or $k = 1$. If $p \in \mathcal{P}_m(\mathbb{R}^2)$ then*

$$p = \sum_{(n,q)\in Q_k} p(h_n, h_q)P_{n,q} + \overline{p}_k,$$

where $\overline{p}_k$ is a linear combination of the polynomials (10).

Proof. By subtracting the sum in the theorem from p, we may suppose that $p(h_n, h_q) = 0$ for all $(n,q) \in Q_k$. Let c be the number of elements in the set Q_k and define a linear transformation $L : \mathcal{P}_m(\mathbb{R}^2) \to \mathbb{R}^c$ by

$$L(p) = (p(h_n, h_q) : \ (n,q) \in Q_k)\,.$$

The vectors $L(P_{n,q})$, $(n,q) \in Q_k$, are distinct standard basis vectors and so are linearly independent. Thus $\mathrm{rank}(L) = c$ so $\mathrm{nullity}(L) = d - c$, where d is the dimension of $\mathcal{P}_m(\mathbb{R}^2)$, i.e., $d = (m+1)(m+2)/2$. However, it is easy to verify using (8) that $d - c$ is the dimension of the space spanned by V_j, $j = 0, \ldots, m$. Since this space is a subspace of $\mathrm{null}(L)$, it is the entire space $\mathrm{null}(L)$.

The author was lead to consider the Chebyshev nodes because they appear in linear inequalities for polynomials in two variables when the Chebyshev polynomials in one variable are extremal. To state this more precisely, define a norm on $\mathcal{P}_m(\mathbb{R}^2)$ by

$$\|p\| = \max\{|p(s,t)| : \ -1 \le s,t \le 1\}.$$

Lemma 6. *Let ℓ be a linear functional on $\mathcal{P}_m(\mathbb{R}^2)$ and suppose*

$$\ell(P) = \ell(Q) = \|\ell\|,$$

where $P(s,t) = T_m(s)$, $Q(s,t) = \epsilon T_m(t)$ and ϵ is a constant that is either 1 or -1. Then

$$|\ell(p)| \le \|\ell\| \max\{|p(x)| : \ x \in \mathcal{N}_k\}$$

for all $p \in \mathcal{P}_m(\mathbb{R}^2)$, where $k = 0$ when $\epsilon = 1$ and $k = 1$ when $\epsilon = -1$.

Proof. Suppose $\ell \not\equiv 0$. By a theorem of Rivlin and Shapiro (see [8, p. 98–99]), there exist nonzero real numbers $\alpha_1, \ldots, \alpha_n$ and points $x_1, \ldots, x_n$ in $[-1,1] \times [-1,1]$ such that

$$\|\ell\| = \sum_{j=1}^{n} |\alpha_j| \quad \text{and} \quad \ell(p) = \sum_{j=1}^{n} \alpha_j p(x_j) \tag{11}$$

for all $p \in \mathcal{P}_m(\mathbb{R}^2)$. Moreover, if $\|p\| = 1$ and $\ell(p) = \|\ell\|$, then $p(x_j) = \operatorname{sign} \alpha_j$ for each $j = 1, \ldots, n$. Thus by hypothesis, for each $j = 1, \ldots, n$, we have $P(x_j) = Q(x_j) = \pm 1$. Writing $x_j = (s_j, t_j)$, we see that $T_m(s_j) = \epsilon T_m(t_j) = \pm 1$ so $x_j \in \mathcal{N}_0$ when $\epsilon = 1$ and $x_j \in \mathcal{N}_1$ when $\epsilon = -1$. Therefore, the lemma follows from (11).

3. Equivalent inequalities.

In this section we list a number of equivalent inequalities for polynomials p in two variables from which Theorem 1 can be easily deduced. Our starting point is the following simple equivalent statement.

A. *If $p \in \mathcal{P}_m(\mathbb{R}^2)$ and if $|p(u,v)| \le 1$ whenever $|u| + |v| \le 1$, then*

$$\left| \frac{d^k}{dt^k}\, p(r,t)\Big|_{t=0} \right| \le T_m^{(k)}(r) \quad \text{for } r \ge 1.$$

Clearly (A) follows from the second inequality of Theorem 1 for the case $X = \mathbb{R}^2$ with $\|(u,v)\| = |u| + |v|$ and $x = (r, 0)$. Thus Theorem 1 $\Rightarrow$ (A). To show that (A) $\Rightarrow$ Theorem 1, let P be as in Theorem 1 and note that we may assume that $Y = \mathbb{R}$ by composing P with a linear functional and applying the Hahn-Banach theorem. Given $x, y \in X$ with $\|x\| \le 1$ and $\|y\| \le 1$, define $p(u,v) = P(ux + vy)$ for $u, v \in \mathbb{R}$. Then $|p(u,v)| \le 1$ whenever $|u| + |v| \le 1$ by the triangle inequality and

$$\frac{d^k}{dt^k}\, p(r,t)\Big|_{t=0} = \hat{D}^k P(rx) y$$

by (1). Thus the first inequality of Theorem 1 follows from (A) with $r = 1$. The second inequality of Theorem 1 follows with x replaced by z if one chooses $r = \|z\|$ and takes $x = z/r$. (Compare [5, Lemma 9].)

In order to apply Theorem 5 and Lemma 6, we use the change of variables $u = (s+t)/2$, $v = (s-t)/2$ and the identity $\max\{|s|, |t|\} = |u| + |v|$ to reformulate statement (A). Define $\phi(u,v) = (u+v, u-v)$. If $p \in \mathcal{P}_m(\mathbb{R}^2)$, then $p \circ \phi \in \mathcal{P}_m(\mathbb{R}^2)$,

$$\max_{-1 \le s,t \le 1} |p(s,t)| = \max_{|u|+|v| \le 1} |p \circ \phi(u,v)|,$$

and

$$\frac{d^k}{dt^k}\, p(r+t, r-t) = \frac{d^k}{dt^k}\, p \circ \phi(r,t).$$

Therefore we need only consider the linear functionals

$$\ell_k(p) = \left.\frac{d^k}{dt^k} p(r+t, r-t)\right|_{t=0}, \tag{12}$$

where r is a fixed real number. The alternate expressions

$$\ell_k(p) = \hat{D}^k p(r,r)(1,-1) = \left.\left(\frac{\partial}{\partial s} - \frac{\partial}{\partial t}\right)^k p(s,t)\right|_{s=t=r}$$

are easy to deduce from (1) and the chain rule.

Thus statement (A) is equivalent to the following.

B. *If* $p \in \mathcal{P}_m(\mathbb{R}^2)$ *and if* $|p(s,t)| \le 1$ *whenever* $-1 \le s, t \le 1$*, then*

$$|\hat{D}^k p(r,r)(1,-1)| \le T_m^{(k)}(r) \quad \text{for } r \ge 1.$$

Before continuing with our list of equivalent inequalities, we establish some elementary properties of the functionals ℓ_k. We first observe that if $V(s,t)$ is a polynomial satisfying $V(t,s) = -(-1)^k V(s,t)$ for all $s, t \in \mathbb{R}$ (such as the polynomials in (10)), then

$$\hat{D}^k V(r,r)(1,-1) = 0 \tag{13}$$

for all real r. To see this it suffices to show that $f(t) = V(r+t, r-t)$ satisfies $f^{(k)}(0) = 0$. By hypothesis, we have $f(-t) = -(-1)^k f(t)$ and we obtain the required equality by taking the kth derivative of both sides at $t = 0$.

In particular, $\ell_k(\overline{p}_k) = 0$ where $\overline{p}_k$ is as in Theorem 5 and hence

$$\ell_k(p) = \sum_{(n,q)\in Q_k} \ell_k(P_{n,q}) p(h_n, h_q) \tag{14}$$

for all $p \in \mathcal{P}_m(\mathbb{R}^2)$. Since $P_{n,q}(t,s) = P_{q,n}(s,t)$ for all $s, t \in \mathbb{R}$, an argument similar to the above shows that $\ell_k(P_{n,q}) = (-1)^k \ell_k(P_{q,n})$ for all $(n,q) \in Q$.

Although we make no use of it, there is an explicit formula when $k = m$ given by

$$\ell_m(p) = (m-1)!2^{m-1} \sum_{n=0}^{m}{}'' \, (-1)^n p(h_n, h_{m-n})$$

for all $p \in \mathcal{P}_m(\mathbb{R}^2)$. One can prove this directly by computing the highest coefficient in the classical Lagrange interpolation formula for $p(t,-t)$ at the Chebyshev points and observing that this coefficient is $\ell_m(p)/m!$. (See [8, p. 41].)

The following statement, which clearly contains (B), can be viewed as an extension of Theorem 3 to two variables.

C. *If $p \in \mathcal{P}_m(\mathbb{R}^2)$ and if $|p(x)| \leq 1$ whenever $x \in \mathcal{N}_k$, then*

$$|\hat{D}^k p(r,r)(1,-1)| \leq T_m^{(k)}(r) \quad \text{for } r \geq 1.$$

Thus (C) $\Rightarrow$ (B). Conversely, (B) $\Rightarrow$ (C) by Lemma 6 with $\ell = \ell_k$ and $\epsilon = (-1)^k$. When $k = m$, the second inequality of Theorem 3 for $p(t,-t)$ shows that statement (C) holds when the set of nodes $\mathcal{N}_k$ is replaced by the subset $\{(h_n, h_{m-n}) : \ n = 0, ..., m\}$ since $h_{m-n} = -h_n$.

Statement (C) cannot be extended to derivatives in directions other than $(1,-1)$ and $(1,1)$. Indeed, for fixed $r \geq 1$, k and $(a,b) \in \mathbb{R}^2$, if there exists a constant M such that $|\hat{D}^k p(r,r)(a,b)| \leq M$ for all p as in (C), then $b = -a$ when k is odd and $b = \pm a$ when k is even. (Consider $p = cV_m$ for large numbers c.)

Note that (C) can be formulated as a linear program. Before [12] appeared, the author verified the statement numerically up to $m = 48$ when $r = 1$ using AMPL and solver MINOS. An equivalent linear program is the following.

D. *If $p \in \mathcal{P}_m(\mathbb{R}^2)$ and if $(-1)^n p(h_n, h_q) \leq 0$ whenever $(n,q) \in Q_k$, then*

$$\hat{D}^k p(r,r)(1,-1) \leq 0 \quad \text{for } r \geq 1.$$

To show that (D) $\Rightarrow$ (C), let p be as in (C) and define $\overline{p}(s,t) = \epsilon p(s,t) - T_m(s)$, where $\epsilon = \pm 1$. Then

$$(-1)^n \overline{p}(h_n, h_q) = (-1)^n \epsilon p(h_n, h_q) - 1 \leq 0$$

for all $(n,q) \in Q_k$ so $\ell_k(\overline{p}) \leq 0$ by (D), where ℓ_k is as in (12). Hence (C) follows since $\ell_k(\overline{p}) = \epsilon \ell_k(p) - T_m^{(k)}(r)$.

Choosing $p = -(-1)^n P_{n,q}$ in (D), we obtain the following.

E. *If $r \geq 1$ then $(-1)^n \hat{D}^k P_{n,q}(r,r)(1,-1) \geq 0$ whenever $(n,q) \in Q_k$.*

To show that (E) $\Rightarrow$ (D), suppose p is as in (D). Then $p(h_n, h_q)\ell_k(P_{n,q}) \leq 0$ for all $(n,q) \in Q_k$ and hence $\ell_k(p) \leq 0$ by (14).

To complete our arguments for equivalence, we show that (C) $\Rightarrow$ (E). Let $(n,q) \in Q_k$ and define $p(s,t) = T_m(s) - (-1)^n P_{n,q}(s,t)$. Clearly $|p(x)| \leq 1$

whenever $x \in \mathcal{N}_k$ since $p(h_n, h_q) = 0$. Hence $\ell_k(p) \leq T_m^{(k)}(r)$ by (C) and $\ell_k(p) = T_m^{(k)}(r) - (-1)^n \ell_k(P_{n,q})$ so $(-1)^n \ell_k(P_{n,q}) \geq 0$.

It is easy to verify that (E) together with (14) is equivalent to the assertion that the linear program dual to (D) has a solution.

4. Extensions.

In this section we extend Statement (C) to the case where the hypotheses hold except on a subset of the Chebyshev nodes. The values of a polynomial at nodes of this subset will add or subtract from the upper bound according as they are outside or inside the interval $[-1, 1]$.

Theorem 7. *Let S be a subset of Q_k and suppose $p \in \mathcal{P}_m(\mathbb{R}^2)$ satisfies $|p(h_n, h_q)| \leq 1$ for all $(n, q) \in Q_k$ with $(n, q) \notin S$. Let $r \geq 1$. Then, with the notation of (12),*

$$|\ell_k(p)| \leq T_m^{(k)}(r) + \sum_{(n,q)\in S} (-1)^n \ell_k(P_{n,q})(|p(h_n, h_q)| - 1), \tag{15}$$

where the sum has the value zero when S is empty. Further, for any real values $(a_{n,q} : (n, q) \in S)$ there exists a $p \in \mathcal{P}_m(\mathbb{R}^2)$ for which equality holds in (15) such that $|p(h_n, h_q)| = |a_{n,q}|$ for all $(n, q) \in S$ and $|p(h_n, h_q)| \leq 1$ for all $(n, q) \in Q_k$ with $(n, q) \notin S$.

Proof. Subtraction of (14) with $p(s, t) = T_m(s)$ from (14) gives

$$\ell_k(p) - T_m^{(k)}(r) = \sum_{(n,q)\in Q_k} (-1)^n \ell_k(P_{n,q})[(-1)^n p(h_n, h_q) - 1].$$

Now by hypothesis and (E),

$$(-1)^n \ell_k(P_{n,q})(|p(h_n, h_q)| - 1) \leq 0$$

for all $(n, q) \in Q_k$ with $(n, q) \notin S$. Thus,

$$\ell_k(p) - T_m^{(k)}(r) \leq \sum_{(n,q)\in S} (-1)^n \ell_k(P_{n,q})(|p(h_n, h_q)| - 1).$$

Hence (15) holds since p can be replaced by $-p$ in the above.

To show the second part of the Theorem, let

$$p(s, t) = T_m(s) + \sum_{(n,q)\in S} (-1)^n (|a_{n,q}| - 1) P_{n,q}(s, t).$$

Then $p(h_n, h_q) = (-1)^n |a_{n,q}|$ for $(n, q) \in S$ and $p(h_n, h_q) = (-1)^n$ for $(n, q) \in Q_k$ with $(n, q) \notin S$. Clearly equality holds in (15).

Note that Theorem 7 shows that Statement (C) still holds when nodes (h_n, h_q) with $\ell_k(P_{n,q}) = 0$ are removed from $\mathcal{N}_k$. For example, when k is even and $r = 1$, a formula below shows that $\ell_k(P_{m,m}) = (-1)^m \beta_k(1) = 0$. Thus the following holds.

Corollary 8. *Let $p \in \mathcal{P}_m(\mathbb{R}^2)$. If k is even and if $|p(x)| \le 1$ for all $x \in \mathcal{N}_0$ with $x \ne (-1,-1)$, then*

$$|\hat{D}^k p(1,1)(1,-1)| \le T_m^{(k)}(1).$$

One can verify directly that each of the polynomials

$$\begin{aligned} P_{0,0}(s,t) &= \frac{1}{4m^2}(s+t+2)\frac{T_m(s)-T_m(t)}{s-t}, \\ P_{m,m}(s,t) &= P_{0,0}(-s,-t), \\ P_{n,n}(s,t) &= \frac{(-1)^n}{m^2}(s^2+t^2-2)\frac{T_m(s)-T_m(t)}{s^2-t^2} \quad \text{when } m=2n, \end{aligned}$$

satisfies the definition of a Lagrange polynomial for the even Chebyshev nodes, where m is even in the third equation. Alternately, these polynomials can be obtained by summing the series in (9). To obtain derivatives, first observe that if $p(t)$ is a polynomial with $p(0)=0$ and if $q(t)=p(t)/t$, then $p^{(j)}(0) = jq^{(j-1)}(0)$ for positive integers j. By routine differentiation, if k is even, we obtain

$$\ell_k(P_{0,0}) = \alpha_k(r), \qquad (-1)^m \ell_k(P_{m,m}) = \beta_k(r),$$

$$(-1)^n \ell_k(P_{n,n}) = \gamma_k(r) \quad \text{when } m=2n,$$

where $\alpha_k(r)$, $\beta_k(r)$ and $\gamma_k(r)$ are given by (4) and (5). As expected from our discussion after (14), if k is odd, then $\ell_k(P_{0,0})=0$, $\ell_k(P_{m,m})=0$, and $\ell_k(P_{n,n})=0$, where in the last equation m is even and $m=2n$. Substituting the previous identities into Theorem 7, we obtain the following.

Theorem 9. *Let $p \in \mathcal{P}_m(\mathbb{R}^2)$ and suppose $|p(x)| \le 1$ for all $x \in \mathcal{N}_0$ different from $(1,1)$, $(-1,-1)$ and $(0,0)$. Let k be even and $r \ge 1$. If m is even, then*

$$\begin{aligned} |\ell_k(p)| \le\ & T_m^{(k)}(r) + \alpha_k(r)(|p(1,1)|-1) + \beta_k(r)(|p(-1,-1)|-1) \\ & + \gamma_k(r)(|p(0,0)|-1). \end{aligned} \tag{16}$$

If m is odd, then (16) holds with the last term removed. Both inequalities are sharp.

Proof of Theorem 2. Let P be as in Theorem 2 and let $x, y \in X$ with $\|x\| \le 1$ and $\|y\| \le 1$. Define

$$p(s,t) = P\left(\frac{s+t}{2}x + \frac{s-t}{2}y\right)$$

and note that $\ell_k(p) = \hat{D}^k P(rx)y$ by (1). Then (6) follows from Theorem 9 with $r=1$ and (7) follows with x replaced by z if one chooses $r = \|z\|$ and takes $x = z/r$.

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