OPTIMAL OSCILLATION POINTS FOR POLYNOMIALS OF RESTRICTED GROWTH ON THE REAL LINE

LAWRENCE A. HARRIS
Department of Mathematics, University of Kentucky
Lexington, Kentucky 40506-0027

1. Introduction

Let ϕ be a continuous even function on the real line with positive values. Given a positive integer m, we consider the class of all real polynomials p satisfying

$$|p(x)| \leq \phi(x)^m, \quad -\infty < x < \infty.$$

Computational methods are given in [3] to determine the maximum possible absolute value of each of the coefficients of the polynomials in this class when ϕ satisfies some restrictions. It is also shown that there exist polynomials P and Q in the class such that each of the nonzero coefficients of P and Q have the largest possible absolute value of any polynomial in the class. The extremal polynomials P and Q are of degrees $m-1$ and m, respectively, with one of the polynomials even and the other odd. Together their coefficients give the best estimates on the absolute values of all of the coefficients of polynomials in the class.

Each of the polynomials P and Q is characterized by a decreasing sequence of points $x_1, \ldots, x_m$ and $y_1, \ldots, y_{m-1}$, respectively, where equality holds in the inequalities

$$|P(x)| \leq \phi(x)^m, \qquad |Q(x)| \leq \phi(x)^m,$$

with alternating signs at the corresponding sequence of points (and also at ±∞ for Q). Such sequences are referred to as an optimal sequence of ϕ-oscillation points (for P) and an optimal sequence of (ϕ, ∞)-oscillation points (for Q). The proof of the existence of these sequences given in [3] uses the Tchebycheff alternation theorem. It requires some restrictions on ϕ, especially in the case of Q with m even where ϕ is required to be self-inversive.

In this paper we remove this restriction on ϕ and establish the existence of optimal sequences of oscillation points using elementary arguments. These depend on apparently new interpolation identities (Lemmas 10 and 11). Moreover, we show that the sequences can be chosen so that their points are symmetric with respect to the origin. Thus arguments of Rogosinski [10] can be extended to show that if a polynomial p satisfies the inequality $|p(x)| \leq \phi(x)^m$ at one of the optimal sequences (and at ∞ for Q), then the nonzero coefficients of the associated extremal polynomial dominate the corresponding coefficients of p.

To make our exposition self contained, we include the proofs of some basic facts about optimal sequences of oscillation points given in [3]. We have applied our results to obtain the optimal coefficients in Markov's inequality for homogeneous polynomials on real Banach spaces. See [2] and [3] for this and other applications.

2. The extremal polynomials

Throughout, unless otherwise mentioned, ϕ denotes a positive continuous function on $(-\infty, \infty)$ satisfying $\phi(-x) = \phi(x)$ for all real x. We may also require several of the restrictions on ϕ given below:

i) $\phi(x) \geq \phi(0)$,

ii) $\lim_{x\to\infty} \dfrac{\phi(x)}{x} = A, \quad A \neq 0$,

iii) $\phi(x) \geq A|x|$.

For example, the function $\phi(x) = (1 + |x|^p)^{1/p}$ satisfies all of the conditions (i)-(iii) with $A = 1$ when $0 < p < \infty$.

Let m be a positive integer. Given a decreasing sequence X of numbers $x_1, \ldots, x_m$, set

$$\omega_j(x, X) = \prod_{i \neq j}(x - x_i), \quad \omega(x, X) = \prod_{i=1}^{m}(x - x_i),$$

where $j = 1, \ldots, m$. When $m = 1$, take $\omega_1(x, X) \equiv 1$. Define P to be the polynomial of degree at most $m - 1$ satisfying

$$P(x_j) = (-1)^{j-1}\phi(x_j)^m, \quad j = 1, \ldots, m. \tag{1}$$

By the Lagrange interpolation formula,

$$P(x) = \sum_{j=1}^{m}(-1)^{j-1}\phi(x_j)^m \frac{\omega_j(x, X)}{\omega_j(x_j, X)} , \tag{2}$$

and the coefficient $f(x_1, \ldots, x_m)$ of x^{m-1} in $P(x)$ is given by

$$f(x_1, \ldots, x_m) = \sum_{j=1}^{m} \frac{\phi(x_j)^m}{|\omega_j(x_j, X)|} , \tag{3}$$

since the sign of $\omega_j(x_j, X)$ is $(-1)^{j-1}$.

Let Y be a decreasing sequence of numbers $y_1, \ldots, y_{m-1}$, where $m > 1$, and suppose ϕ satisfies (ii). Define Q to be the polynomial having no term of degree $m - 1$ and satisfying both

$$Q(y_j) = (-1)^j\phi(y_j)^m, \quad j = 1, \ldots, m - 1, \quad \text{and} \tag{4}$$

$$\lim_{x\to\infty} \frac{Q(x)}{\phi(x)^m} = 1. \tag{5}$$

Put $s_1 = \sum_1^{m-1} y_j$, $s_2 = \sum_1^{m-1} y_j^2$ and $s_3 = \sum_{i<j} y_i y_j$. Then

$$(x + s_1)\omega(x, Y) = x^m + (s_3 - s_1^2)x^{m-2} + \cdots. \tag{6}$$

Also, Q has degree m and leading coefficient A^m by (ii) and (5). Hence the Lagrange interpolation formula applies to $Q(x) - A^m(x + s_1)\omega(x, Y)$ to show that

$$Q(x) = A^m(x + s_1)\omega(x, Y) - \sum_{j=1}^{m-1}(-1)^{j-1}\phi(y_j)^m \frac{\omega_j(x, Y)}{\omega_j(y_j, Y)} . \tag{7}$$

If $g(y_1, \ldots, y_{m-1})$ is the negative of the coefficient of x^{m-1} in $Q(x)$, then

$$g(y_1, \ldots, y_{m-1}) = \frac{A^m}{2} \, (s_1^2 + s_2) + \sum_{j=1}^{m-1} \frac{\phi(y_j)^m}{|\omega_j(y_j, Y)|} \, , \tag{8}$$

since $s_3 - s_1^2 = -(s_1^2 + s_2)/2$ and this follows from the identity $s_1^2 = s_2 + 2s_3$. Note that both f and g are symmetric functions. The arguments above show that for a given ϕ, P and Q are uniquely determined by the sequences X and Y, respectively.

To see the relation between P and Q, add two additional points to the sequence $y_1, \ldots, y_{m-1}$ by defining $y_0 = t$ and $y_m = -t$, where $t > y_1$ and $-t < y_{m-1}$. Let Y_t be this new sequence and let P_t be the polynomial of degree at most m satisfying $P_t(y_j) = (-1)^j \phi(y_j)^m$ for $j = 0, \ldots, m$.

Proposition 1.

$$\lim_{t \to \infty} P_t(x) = Q(x), \qquad -\infty < x < \infty.$$

Proof. Let $R_t(x)$ be the sum of the first and last terms of the Lagrange interpolation formula for P_t at the points of Y_t and let $S_t(x)$ be the sum of the rest of the terms. Then $P_t(x) = R_t(x) + S_t(x)$, where

$$\begin{aligned} R_t(x) &= \frac{\phi(t)^m \omega(x, Y)(x+t)}{\omega(t, Y)(2t)} + (-1)^m \frac{\phi(t)^m \omega(x, Y)(x-t)}{\omega(-t, Y)(-2t)} \, , \\ S_t(x) &= \sum_{j=1}^{m-1} (-1)^j \phi(y_j)^m \frac{\omega_j(x, Y)(x-t)(x+t)}{\omega_j(y_j, Y)(y_j - t)(y_j + t)} \, . \end{aligned}$$

It suffices to show

$$\lim_{t \to \infty} R_t(x) = A^m (x + s_1) \omega(x, Y). \tag{9}$$

To do this, define

$$\gamma(t, Y) = \prod_{j=1}^{m-1} \left(1 - \frac{y_j}{t}\right)^{-1}$$

and note that

$$R_t(x) = \frac{\phi(t)^m \omega(x, Y)}{2t^m} \, \left[(x+t)\gamma(t, Y) + (x-t)\gamma(-t, Y)\right].$$

Then (9) follows since

$$\gamma(t,Y) = 1 + \frac{s_1}{t} + \frac{\sigma(t)}{t^2},$$

where $\sigma(t)$ is bounded as $t \to \pm\infty$. In fact, $\sigma(t) \to (s_1^2 + s_2)/2$. $\square$

3. Optimal oscillation points

Definition. Let m be a positive integer. We call a decreasing sequence of numbers $x_1, \ldots, x_m$ an *optimal sequence of ϕ-oscillation points* if the polynomial P defined by (1) satisfies $|P(x)| \le \phi(x)^m$ for all real x. We call a decreasing sequence of numbers $y_1, \ldots, y_{m-1}$ an *optimal sequence of (ϕ, ∞)-oscillation points* if the polynomial Q defined by (4) and (5) satisfies $|Q(x)| \le \phi(x)^m$ for all real x.

We refer to the polynomials P and Q in the preceding definition as the oscillating polynomials determined by the corresponding sequences. We shall see later (Proposition 4 below) that for a given m, these oscillating polynomials are independent of the particular sequence of oscillation points used to define them. Thus, for example, if the equation $|P(x)| = \phi(x)^m$ has only m solutions, no other optimal sequence of ϕ-oscillation points with m points exists. The analogous result holds also for (ϕ, ∞)-oscillation points.

The next proposition, which is based on an argument of Rogosinski [10], shows that the optimal sequences of oscillation points give optimal estimates on the highest three coefficients and that P and Q are extremal polynomials. Optimal estimates on the remaining coefficients are obtained later (in Theorems 6 and 7) since these require more careful analysis.

Proposition 2. *Let $p(x) = a_0 + a_1 x + \cdots + a_m x^m$ be a polynomial satisfying $|p(x)| \le \phi(x)^m$ for all real x, and let f and g be given by (3) and (8). Then*

$$|a_{m-1}| \le f(x_1, \ldots, x_m) \tag{10}$$

whenever $x_1, \ldots, x_m$ are distinct real numbers. If $m > 1$ and (ii) holds, then

$$|a_{m-2}| \le g(y_1, \ldots, y_{m-1}) \tag{11}$$

whenever $y_1, \ldots, y_{m-1}$ *are distinct real numbers. Equality holds in (10) if the points* $x_1, \ldots, x_m$ *when arranged in decreasing order are an optimal sequence of* ϕ*-oscillation points and* $p = P$*. Equality holds in (11) if the points* $y_1, \ldots, y_{m-1}$ *when arranged in decreasing order are an optimal sequence of* (ϕ, ∞)*-oscillation points and* $p = Q$*.*

Proof. Since f and g are symmetric functions, we may reorder the x_j's and y_j's into decreasing sequences X and Y. To prove (10), note that by replacing $p(x)$ by $[p(x) - (-1)^m p(-x)]/2$, we may suppose that the degree of p is at most $m - 1$. Then by the Lagrange interpolation formula,

$$p(x) = \sum_{j=1}^{m} p(x_j) \frac{\omega_j(x, X)}{\omega_j(x_j, X)} \ .$$

Equating the coefficients of x^{m-1} on each side, we have that

$$a_{m-1} = \sum_{j=1}^{m} \frac{p(x_j)}{\omega_j(x_j, X)} \ , \tag{12}$$

and hence $|a_{m-1}| \le f(x_1, \ldots, x_m)$.

To prove (11), note that by replacing $p(x)$ by $[p(x) + (-1)^m p(-x)]/2$, we may suppose that $a_{m-1} = 0$. Let

$$r(x) = p(x) - a_m(x + s_1)\omega(x, Y).$$

Then r is a polynomial of degree at most $m - 2$ by (6). Since r satisfies $r(y_j) = p(y_j)$ for $j = 1, \ldots, m - 1$, by the Lagrange interpolation formula for r,

$$p(x) = a_m(x + s_1)\omega(x, Y) + \sum_{j=1}^{m-1} p(y_j) \frac{\omega_j(x, Y)}{\omega_j(y_j, Y)} \ .$$

Collecting the coefficients of x^{m-2} on both sides and applying $s_1^2 = s_2 + 2s_3$, we obtain

$$a_{m-2} = -\frac{a_m}{2} \left(s_1^2 + s_2\right) + \sum_{j=1}^{m-1} \frac{p(y_j)}{\omega_j(y_j, Y)} \ . \tag{13}$$

Therefore, $|a_{m-2}| \le g(y_1, \ldots, y_{m-1})$ since $|a_m| \le A^m$ by (ii).

To prove the assertions about equality, let X and Y be the given optimal sequences and let P and Q be the corresponding oscillating polynomials. Let A_{m-1} be the coefficient of x^{m-1} in $P(x)$ and let B_{m-2} be the coefficient of x^{m-2} in $Q(x)$. Since the degree of P is at most $m-1$ and the sign of $\omega_j(x_j, X)$ is $(-1)^{j-1}$, it follows from (12) with $p = P$ that $A_{m-1} = f(x_1, \ldots, x_m)$. In the same way, it follows from (13) with $p = Q$ that $B_{m-2} = -g(y_1, \ldots, y_{m-1})$. □

The following shows that one can compute optimal sequences of ϕ- or (ϕ, ∞)-oscillation points when they exist by finding a point where the corresponding function is minimum.

Proposition 3. *If there exists an optimal sequence of ϕ-oscillation points with m points, then f assumes its minimum at a point $(x_1, \ldots, x_m)$ if and only if the coordinates of this point arranged in decreasing order are an optimal sequence of ϕ-oscillation points. If there exists an optimal sequence of (ϕ, ∞)-oscillation points with $m-1$ points, then g assumes its minimum at a point $(y_1, \ldots, y_{m-1})$ if and only if the coordinates of this point arranged in decreasing order are an optimal sequence of (ϕ, ∞)-oscillation points.*

Proposition 4. *All optimal sequences of ϕ-oscillation points with the same number of points define the same oscillating polynomial P. All optimal sequences of (ϕ, ∞)-oscillation points with the same number of points define the same oscillating polynomial Q.*

Proof of Propositions 3 and 4. By Proposition 2, the function f attains its minimum at each optimal sequence of ϕ-oscillation points with m points. Likewise, the function g attains its minimum at each optimal sequence of (ϕ, ∞)-oscillation points with $m-1$ points.

To prove the converse, let P be the oscillating polynomial determined by a given optimal sequence of ϕ-oscillation points with m points and let A_{m-1} be the coefficient of x^{m-1} in $P(x)$. If f assumes its minimum at a point $(x_1, \ldots, x_m)$, then $A_{m-1} = f(x_1, \ldots, x_m)$ by Proposition 2. Reorder the x_j's into a decreasing sequence X. If $(-1)^{j-1}P(x_j) < \phi(x_j)^m$ for some j, then

$$\sum_{j=1}^{m} \frac{P(x_j)}{\omega_j(x_j, X)} = \sum_{j=1}^{m} \frac{(-1)^{j-1}P(x_j)}{|\omega_j(x_j, X)|} < \sum_{j=1}^{m} \frac{\phi(x_j)^m}{|\omega_j(x_j, X)|} ,$$

so $A_{m-1} < f(x_1, \dots, x_m)$ by (12), a contradiction. Hence $(-1)^{j-1}P(x_j) = \phi(x_j)^m$ for $j = 1, \dots, m$. Thus X is an optimal sequence of ϕ-oscillation points and P is also the oscillating polynomial determined by X. This proves the first part of Proposition 3.

To prove the first part of Proposition 4, suppose now that X is another optimal sequence of ϕ-oscillation points $x_1, \dots, x_m$. By our preliminary comments, the function f assumes its minimum at $(x_1, \dots, x_m)$. Hence by what we have shown, P is also the oscillating polynomial determined by X, as required.

A similar argument based on (13) establishes the case of (ϕ, ∞)-oscillation points. □

4. Main results

The parity of a number refers to whether it is even or odd.

Theorem 5. *(Existence.) Suppose ϕ satisfies (ii) and let m be a positive integer.*

1) There exists an optimal sequence of ϕ-oscillation points with m points. If ϕ also satisfies (iii), then there exists an optimal sequence of (ϕ, ∞)-oscillation points with $m-1$ points when $m > 1$.

2) If ϕ also satisfies (i) when the number of points is odd, then optimal sequences of ϕ- and (ϕ, ∞)-oscillation points can be chosen to be symmetric with respect to the origin when they exist.

When optimal sequences of oscillation points exist, one can obtain these and the corresponding extremal polynomials P and Q by minimizing f and g and applying Propositions 2 and 3. If the points of an optimal sequence are symmetric with respect to the origin, an argument of Rogosinski [10] applies to show that Q and P alternate as extremal polynomials for estimates on the coefficients in descending order.

Theorem 6. *(Opposite parity.) Suppose there exists an optimal sequence of ϕ-oscillation points $x_1, \dots, x_m$ which are symmetric with respect to the origin and let P be the corresponding oscillating polynomial. If p is a polynomial of degree at most m which satisfies*

$$|p(x_j)| \le \phi(x_j)^m, \quad j = 1, \dots, m,$$

then

$$|p^{(k)}(0)| \leq |P^{(k)}(0)| \tag{14}$$

for all k *with parity opposite to* m *and* $0 \leq k < m$*. Moreover, if equality holds in (14) for some* k *with* $0 < k < m$*, then*

$$p(x) + (-1)^{m-1}p(-x) = 2\epsilon P(x)$$

for all real x*, where* $\epsilon = \pm 1$*.*

Theorem 7. *(Same parity.) Let* $m > 1$*. Suppose there exists an optimal sequence of* (ϕ, ∞)*-oscillation points* $y_1, \ldots, y_{m-1}$ *which are symmetric with respect to the origin and let* Q *be the corresponding oscillating polynomial. If* p *is a polynomial of degree at most* m *which satisfies*

$$|p(y_j)| \leq \phi(y_j)^m, \quad j = 1, \ldots, m-1$$

and

$$\limsup_{y \to \infty} \frac{|p(y)|}{\phi(y)^m} \leq 1, \tag{15}$$

then

$$|p^{(k)}(0)| \leq |Q^{(k)}(0)| \tag{16}$$

for all k *with the same parity as* m *and* $0 \leq k \leq m$*. Moreover, if equality holds in (16) for some* k *with* $0 < k < m$*, then*

$$p(x) + (-1)^m p(-x) = 2\epsilon Q(x)$$

for all real x*, where* $\epsilon = \pm 1$*.*

Corollary 8. *Suppose* ϕ *satisfies (i), (ii) and (iii). If* p *is a polynomial of degree at most* m *satisfying* $|p(x)| \leq \phi(x)^m$ *whenever* $-\infty < x < \infty$*, then*

$$|p^{(k)}(0)| \leq |P^{(k)}(0)| \quad \text{or} \quad |p^{(k)}(0)| \leq |Q^{(k)}(0)|$$

according as k *has parity opposite to* m *or the same as* m*, where* $1 \leq k \leq m$*.*

It follows from part (2) of Theorem 5 and Proposition 3 that one can compute optimal sequences of oscillation points and the corresponding oscillating polynomials by minimizing f and g over points whose coordinates are symmetric with respect to the origin. This computation can be done using the identities below and the multivariate minimization Fortran subroutine LBFGS [6], which is available online from Netlib.

Proposition 9.
1) Let $x_1, \ldots, x_m$ *be an optimal sequence of* ϕ*-oscillation points which are symmetric with respect to the origin and let* U *be the sequence* $x_1^2, \ldots, x_n^2$*, where* $n = [(m+1)/2]$*. (Note that* $x_n = 0$ *when* m *is odd.) Then*

$$\begin{aligned} f(x_1, \ldots, x_m) &= \sum_{j=1}^{n} \frac{\phi(x_j)^m}{|\omega_j(x_j^2, U)|} && (m \text{ odd}) \\ &= \sum_{j=1}^{n} \frac{\phi(x_j)^m}{x_j |\omega_j(x_j^2, U)|} && (m \text{ even}). \end{aligned}$$

2) Let $y_1, \ldots, y_{m-1}$ *be an optimal sequence of* (ϕ, ∞)*-oscillation points which are symmetric with respect to the origin and let* U *be the sequence* $y_1^2, \ldots, y_n^2$*, where* $n = [m/2]$*. (Note that* $y_n = 0$ *when* m *is even.) Then*

$$\begin{aligned} g(y_1, \ldots, y_{m-1}) &= A^m \sum_{j=1}^{n} y_j^2 + \sum_{j=1}^{n} \frac{\phi(y_j)^m}{|\omega_j(y_j^2, U)|} && (m \text{ even}) \\ &= A^m \sum_{j=1}^{n} y_j^2 + \sum_{j=1}^{n} \frac{\phi(y_j)^m}{y_j |\omega_j(y_j^2, U)|} && (m \text{ odd}) \end{aligned}$$

5. Proof of existence theorem

Our proof of Theorem 5 is based on the following lemma.

Lemma 10. *Let* $\psi : (-\infty, \infty) \to [0, \infty)$ *be a continuous function, let* X *be a decreasing sequence of numbers* $x_1, \ldots, x_m$*, and let* P *and* f *be defined by (2) and (3), respectively, where* $\phi(x)^m$ *is replaced by* $\psi(x)$*. Then*

$$\begin{aligned} &\psi(x) - (-1)^{k-1} P(x) = \\ &\qquad [f(x_1, \ldots, x_{k-1}, x, x_{k+1}, \ldots, x_m) - f(x_1, \ldots, x_m)]\, |\omega_k(x, X)| \qquad (17) \end{aligned}$$

for all x satisfying $x_{k+1} < x < x_{k-1}$, where $1 \leq k \leq m$. Here it is understood that x_{k-1} does not appear when $k = 1$ and x_{k+1} does not appear when $k = m$.

Proof. Let X_k be the sequence obtained from X by replacing the point x_k by an arbitrary number x, where $1 \leq k \leq m$. Then

$$\omega_j(x_j, X_k) = \frac{\omega_j(x_j, X)(x_j - x)}{x_j - x_k}\ , \quad j \neq k, \tag{18}$$

$$\omega_k(x, X_k) = \omega_k(x, X) = \frac{\omega_j(x, X)(x - x_j)}{x - x_k} \tag{19}$$

Let R be the right-hand side of (17). Then subtracting corresponding terms of R and factoring, we have

$$R = \left(\sum_{j \neq k} \frac{\psi(x_j)}{|\omega_j(x_j, X)|}\ R_j \right) + \psi(x) - \frac{\psi(x_k)|\omega_k(x, X)|}{|\omega_k(x_k, X)|}\ ,$$

where

$$R_j = \left[\frac{|\omega_j(x_j, X)|}{|\omega_j(x_j, X_k)|}\ - 1 \right] |\omega_k(x, X)|.$$

Applying (18) and (19), we have

$$\begin{aligned} R_j &= \frac{|x_j - x_k| - |x_j - x|}{|x_j - x|}\ |\omega_k(x, X)| \\ &= [|x_j - x_k| - |x_j - x|] \left| \frac{\omega_j(x, X)}{x - x_k}\ \right|. \end{aligned}$$

Now suppose $x_{k+1} < x < x_{k-1}$ and $x \neq x_k$, with the convention mentioned for out of range subscripts. Then the sign of $\omega_j(x, X)/(x - x_k)$ is $(-1)^k$ when $1 \leq j < k$ and $(-1)^{k-1}$ when $k < j \leq m$. Also, $|x_j - x_k| - |x_j - x|$ is $x - x_k$ when $1 \leq j < k$ and $-(x - x_k)$ when $k < j \leq m$. Hence,

$$R_j = (-1)^k \omega_j(x, X),$$

whenever $1 \leq j \leq m$ and $j \neq k$. Since the sign of $\omega_k(x, X)$ is $(-1)^{k-1}$, it follows that

$$R = \psi(x) + (-1)^k \sum_{j=1}^{m} \frac{\psi(x_j)\omega_j(x, X)}{|\omega_j(x, X)|}\ ,$$

and this holds for $x = x_k$ by continuity. Thus the asserted identity holds since the sign of $\omega_j(x, X)$ is $(-1)^{j-1}$. □

Lemma 11. *Suppose ϕ satisfies (ii). Let Y be a decreasing sequence of numbers $y_1, \ldots, y_{m-1}$, and let Q and g be defined by (7) and (8), respectively. Then*

$$\begin{aligned}&\phi(x)^m - (-1)^k Q(x) = \\ &\quad [g(y_1, \ldots, y_{k-1}, x, y_{k+1}, \ldots, y_{m-1}) - g(y_1, \ldots, y_{m-1})]\, |\omega_k(x, Y)| \qquad (20)\end{aligned}$$

for all x satisfying $y_{k+1} < x < y_{k-1}$, where $1 \leq k \leq m-1$. Here it is understood that y_{k-1} does not appear when $k = 1$ and y_{k+1} does not appear when $k = m-1$.

Proof. Let Y_k be the sequence of terms $w_1, \ldots, w_{m-1}$ obtained from Y by replacing the point y_k by an arbitrary number x, where $1 \leq k \leq m-1$. Let R be the right-hand side of (20), let S be the difference between the first term of g at the two values and let T be the difference between the remaining terms of g. Then $R = (S+T)|\omega_k(x, Y)|$, where

$$\begin{aligned} S &= \frac{A^m}{2} \left[(s_1 + x - y_k)^2 + (s_2 + x^2 - y_k^2) - s_1^2 - s_2 \right], \\ T &= \sum_{j=1}^{m-1} \frac{\phi(w_j)^m}{|\omega_j(w_j, Y_k)|} - \sum_{j=1}^{m-1} \frac{\phi(y_j)^m}{|\omega_j(y_j, Y)|} . \end{aligned}$$

By algebra, $S = A^m(x + s_1)(x - y_k)$. Suppose $y_{k+1} < x < y_{k-1}$. Then by the previous lemma,

$$T|\omega_k(x, Y)| = \phi(x)^m - (-1)^{k-1} \sum_{j=1}^{m-1} (-1)^{j-1} \phi(y_j)^m \frac{\omega_j(x, Y)}{\omega_j(y_j, Y)} ,$$

and clearly $\omega_k(x, Y)$ has sign $(-1)^{k-1}$. Hence,

$$R = \phi(x)^m + (-1)^{k-1} Q(x),$$

as required. □

Proof of Part (1). We first show that f assumes a minimum value at some point in its domain of definition. Let V be a value of f. By (ii), there is an $R > 0$ such that $\phi(x) \geq A|x|/2$ for all x with $|x| > R$. Let $p = (x_1, \ldots, x_m) \in \mathbb{R}^m$. If j is an index with $|x_j|$ maximum, then

$$\begin{aligned} f(p) &\geq \frac{\phi(x_j)^m}{|\omega_j(x_j, X)|} \geq \frac{\phi(x_j)^m}{(2|x_j|)^{m-1}} \\ &\geq \left(\frac{A}{4} \right)^m |x_j|, \end{aligned}$$

when $|x_j| > R$. Let $S = [-R, R]^m$ and take R larger if necessary so that $(A/4)^m R \geq V$. Hence $f(p) > V$ whenever $p \notin S$.

If $m > 1$ and $p \in S$, choose distinct indices i and j so that $|x_i - x_j|$ is minimum. Then

$$f(p) \geq \frac{B^m}{|x_i - x_j|(2R)^{m-2}}\ , \quad B = \min\{\phi(x) : -R \leq x \leq R\}.$$

By hypothesis, $B > 0$ so there is a $\delta > 0$ such that

$$\frac{B^m}{\delta(2R)^{m-2}} \geq V.$$

Put

$$K = \{p \in S :\ |x_i - x_j| \geq \delta \text{ when } i \neq j\}.$$

By what we have shown, $f(p) > V$ whenever $p \notin K$. Since f is continuous on K and K is compact, f assumes a minimum value V_1 on K and clearly $V_1 \leq V$. Hence V_1 is the minimum value of f.

A similar argument shows that g assumes a minimum value at some point in its domain of definition. (In fact, in each case these arguments can be used to obtain numbers R and δ growing and decreasing exponentially with m, respectively, such that the set K contains any optimal sequence of the corresponding type.)

Now arrange the coordinates of a minimum point for f into a decreasing sequence X of numbers $x_1, \ldots, x_m$. To prove that X is an optimal sequence of ϕ-oscillation points, we must show that $|P(x)| \leq \phi(x)^m$ for all real x. Let $1 \leq k < m$. It follows from Lemma 10 with $\psi(x) = \phi(x)^m$ that

$$(-1)^{k-1} P(x) \leq \phi(x)^m \tag{21}$$

whenever $x_{k+1} \leq x \leq x_{k-1}$ and

$$(-1)^k P(x) \leq \phi(x)^m \tag{22}$$

whenever $x_{k+2} \leq x \leq x_k$. Both these conditions are satisfied whenever $x_{k+1} \leq x \leq x_k$ and hence $|P(x)| \leq \phi(x)^m$ for all x in the interval $[x_{k+1}, x_k]$. Since these intervals cover $[x_m, x_1]$, we have $|P(x)| \leq \phi(x)^m$ for all x in $[x_m, x_1]$. It is easy to deduce from (2) that $P(x) \geq 0$ whenever $x > x_1$ and that $(-1)^{m-1}P(x) \geq 0$ whenever $x < x_m$. Hence, by applying (21) with

$k = 1$ and (22) with $k = m - 1$, we see that $|P(x)| \leq \phi(x)^m$ when either of these conditions hold. Thus, $|P(x)| \leq \phi(x)^m$ for all real x.

Similarly, arrange the coordinates of a minimum point for g into a decreasing sequence of numbers $y_1, \ldots, y_{m-1}$. To show that Y is an optimal sequence of (ϕ, ∞)-oscillation points, we need to show that $|Q(x)| \leq \phi(x)^m$ for all real x. Lemma 11 shows that $|Q(x)| \leq \phi(x)^m$ for all x in $[y_{m-1}, y_1]$; also,

$$-Q(x) \leq \phi(x)^m$$

whenever $x > y_1$ and

$$(-1)^{m-1}Q(x) \leq \phi(x)^m$$

whenever $x < y_{m-1}$. It suffices to show that the last two inequalities still hold when their left-hand sides have opposite signs. Put $\Omega(x) = (x + s_1)\omega(x, Y)$. It follows from (7) that

$$Q(x) \leq A^m \Omega(x) \tag{23}$$

whenever $x > y_1$ and

$$(-1)^m Q(x) \leq (-1)^m A^m \Omega(x) \tag{24}$$

whenever $x < y_{m-1}$.

Suppose $x > y_1$. If $x + s_1 \leq 0$, then $Q(x) \leq 0$ by (23). Otherwise,

$$\Omega(x) \leq \left[\frac{(x + s_1) + (x - y_1) + \cdots + (x - y_{m-1})}{m}\right]^m = x^m,$$

by the inequality between geometric and arithmetic means. Now suppose $x < y_{m-1}$. If $x + s_1 \geq 0$, then $(-1)^m Q(x) \leq 0$ by (24). Otherwise,

$$(-1)^m \Omega(x) \leq \left[\frac{(-x - s_1) + (y_1 - x) + \cdots + (y_{m-1} - x)}{m}\right]^m = (-x)^m,$$

by the inequality between geometric and arithmetic means. Hence by (23), (24) and (iii), we have that $Q(x) \leq \phi(x)^m$ for all $x > y_1$ and $(-1)^m Q(x) \leq \phi(x)^m$ for all $x < y_{m-1}$, as we wished to show. □

It remains to show that the sequences of oscillation points whose existence is proved in part (1) can be chosen to be symmetric about the origin. The following lemma will be needed.

Lemma 12. *Let $\psi : (-\infty, \infty) \to [0, \infty)$ be a positive even function with a minimum at $x = 0$ and let f be defined by (3) with $\phi(x)^m$ is replaced by $\psi(x)$. Then*

$$\begin{aligned} & f(x_1, \ldots, x_{n-1}, x_n, -x_{n-1}, \ldots, -x_1) \\ > \quad & f(x_1, \ldots, x_{n-1}, 0, -x_{n-1}, \ldots, -x_1), \end{aligned}$$

where $x_1, \ldots, x_n$ is a decreasing sequence of positive numbers. This inequality also holds when f is replaced by g.

Proof. Let X be the sequence

$$x_1, \ldots, x_{n-1}, x_n, -x_{n-1}, \ldots, -x_1, \tag{25}$$

and let U be the sequence $x_1^2, \ldots, x_{n-1}^2$. Put $m = 2n - 1$ and define a polynomial $\hat{P}$ of degree at most $m - 1$ by

$$\hat{P}(x) = \sum_{j=1}^{n-1} (-1)^{j-1} \psi(x_j) \frac{\omega_j(x^2, U)(x^2 - x_n^2)}{\omega_j(x_j^2, U)(x_j^2 - x_n^2)} + (-1)^{n-1} \psi(x_n) \frac{\omega(x^2, U)}{\omega(x_n^2, U)}.$$

Then $\hat{P}(\pm x_j) = (-1)^{j-1}\psi(x_j)$ for $j = 1, \ldots, n-1$ and $\hat{P}(x_n) = (-1)^{n-1}\psi(x_n)$. Hence,

$$\hat{P}(x_j) = (-1)^{j-1}\psi(x_j), \quad j = 1, \ldots, m.$$

Therefore, $\hat{P} = P$, where P is defined by (2) with $\phi(x)^m$ replaced by $\psi(x)$. Let p be the point with coordinates given in (25) and let p_0 be the same point but with x_n replaced by 0. It follows by comparison of coefficients that

$$f(p) = \sum_{j=1}^{n-1} \frac{\psi(x_j)}{|\omega_j(x_j^2, U)|(x_j^2 - x_n^2)} + \frac{\psi(x_n)}{|\omega(x_n^2, U)|}$$

since $f(p)$ is the coefficient of x^{m-1} in $P(x)$ and $\omega_j(x^2, U)(x^2 - x_n^2)$ and $\omega(x^2, U)$ are monic polynomials of degree $m - 1$ in $\hat{P}(x)$. (This identity can also be verified by a direct computation.) Hence $f(p) > f(p_0)$, since $x_j^2 > x_j^2 - x_n^2$, $\psi(x_n) \geq \psi(0)$ and $|\omega(0, U)| > |\omega(x_n^2, U)|$.

The corresponding inequality for g follows from the identity

$$\begin{aligned} g(p) &= \frac{A^m}{2}(s_1^2 + s_2) + f(p) \\ &= A^m \sum_{j=1}^{n} x_j^2 + f(p), \end{aligned}$$

where $f(p)$ is as above with $\psi(x) = \phi(x)^m$ and $m = 2n$. $\square$

Proof of Part (2). Let X be an optimal sequence of ϕ-oscillation points $x_1, \ldots, x_m$, and let P be the associated oscillating polynomial. We first show that $P(-x) = (-1)^{m-1}P(-x)$. Define $\hat{P}(x) = (-1)^{m-1}P(-x)$. Then

$$|\hat{P}(x)| = |P(-x)| \leq \phi(-x)^m = \phi(x)^m$$

for all real x. Reflect X in the origin by defining $\hat{x}_j = -x_{m-j+1}$ for $j = 1, \ldots, m$. Then $\hat{x}_1 > \cdots > \hat{x}_m$ and

$$\begin{aligned} \hat{P}(\hat{x}_j) &= (-1)^{m-1}P(x_{m-j+1}) \\ &= (-1)^{m-1}(-1)^{m-j}\phi(x_{m-j+1})^m \\ &= (-1)^{j-1}\phi(\hat{x}_j)^m, \end{aligned}$$

by (1). Hence $\hat{x}_1, \ldots, \hat{x}_m$ is an optimal sequence of ϕ-oscillation points and $\hat{P}$ is the associated oscillating polynomial. Therefore, $\hat{P} = P$ by Proposition 4.

Note that by reflecting the sequence X in the origin, if necessary, we may suppose that there are at least as many positive $x_j's$ as there are negative $x_j's$. Suppose m is even and let $n = m/2$. No x_j is 0 since $P(0) = 0$ and ϕ is positive. Hence there are at least n positive $x_j's$ and we reflect n of these in the origin to obtain a decreasing sequence $\hat{X}$ of m points which are symmetric with respect to the origin. Since P is odd, P oscillates on $\hat{X}$. Hence $\hat{X}$ is an optimal sequence of ϕ-oscillation points with m points and these points are symmetric with respect to the origin.

Now suppose m is odd and let $n = (m+1)/2$. There are at least n nonnegative $x_j's$. Reflect the first $n-1$ of these (which are positive) in the origin to obtain a decreasing sequence $\hat{X}$ of the form (25). Since P is even, P oscillates on $\hat{X}$. Thus $\hat{X}$ is an optimal sequence of ϕ-oscillation points with m points. Clearly, $\hat{X}$ is symmetric with respect to the origin if $x_n = 0$. This follows from Lemma 12 since f assumes its minimum at $\hat{X}$ by the remarks preceeding Proposition 3.

A similar proof starting with the identity $Q(-x) = (-1)^m Q(x)$ shows that an optimal sequence of (ϕ, ∞)-oscillation points can be converted to one that is symmetric with respect to the origin. $\square$

5. Proof of coefficient inequalities

Lemma 13. *Suppose* U *is a decreasing sequence of nonnegative numbers* $u_1, \ldots, u_n$ *and let* ψ *be a nonnegative function on* U. *If* p *is a polynomial of degree at most* n *satisfying*

$$|p(u_j)| \le \psi(u_j), \quad j = 1, \ldots, n,$$

and if $|a_n| \le C$, *where* a_n *is the coefficient of* x^n *in* $p(x)$, *then*

$$|p^{(k)}(0)| \le |R^{(k)}(0)|, \quad k = 0, \ldots, n, \tag{26}$$

where

$$R(x) = C\omega(x, U) + \sum_{j=1}^{n} (-1)^j \psi(u_j) \frac{\omega_j(x, U)}{\omega_j(u_j, U)}.$$

If equality holds in (26) for some k *with* $0 < k < n$, *then* $p = \pm R$.

This lemma in the case $C = 0$ is due to Rogosinski [10, Corollary (ii)]. (Other extensions are given in [5, Lemma 3.1] and [9, Prop. 4].) One can deduce (26) from [10, Theorem 1] by introducing an additional oscillation point and taking the limit as this point approaches infinity. However, because of difficulties with the case of equality and because of the elegance of Rogosinski's arguments, we give a direct proof.

Proof. Let $r(x) = p(x) - a_n\omega(x, U)$. By the Lagrange interpolation formula for r,

$$p(x) = a_n\omega(x, U) + \sum_{j=1}^{n} p(u_j) \frac{\omega_j(x, U)}{\omega_j(u_j, U)}. \tag{27}$$

Hence, by differentiation,

$$p^{(k)}(0) = a_n\omega^{(k)}(0, U) + \sum_{j=1}^{n} p(u_j) \frac{\omega_j^{(k)}(0, U)}{\omega_j(u_j, U)}, \tag{28}$$

so by hypothesis,

$$|p^{(k)}(0)| \le M_k, \tag{29}$$

where

$$M_k = C|\omega^{(k)}(0,U)| + \sum_{j=1}^{n} \psi(u_j) \frac{|\omega_j^{(k)}(0,U)|}{|\omega_j(u_j,U)|}.$$

Clearly, the sign of $\omega_j(u_j,U)$ is $(-1)^{j-1}$ for $j = 1,\ldots,n$. To determine the sign of $\omega_j^{(k)}(0,U)$, we first note that $\omega_j(x,U)$ is a polynomial of degree $n-1$ with $n-1$ distinct roots u_i, where $i \neq j$. Also, $\omega_j(x,U) > 0$ whenever $x > u_1$. By applying Rolle's theorem to each of the adjacent subintervals with endpoints at these roots, we see that $\omega_j'(x,U)$ is a polynomial of degree $n-2$ with $n-2$ distinct roots in (u_n,u_1). Also, $\omega_j'(x,U) > 0$ whenever $x \geq u_1$. Thus, by induction, $\omega_j^{(k)}(x,U)$ is a polynomial of degree $n-k-1$ with $n-k-1$ distinct roots in (u_n,u_1) and $\omega_j^{(k)}(x,U) > 0$ whenever $x \geq u_1$. It can be seen by factorization that $\omega_j^{(k)}(x,U)$ alternates in sign at each of its roots and hence $\omega_j^{(k)}(0,U)$ has sign $(-1)^{n-k-1}$. Similarly, $\omega^{(k)}(0,U)$ has sign $(-1)^{n-k}$. Therefore,

$$M_k = \epsilon_k R^{(k)}(0), \quad \epsilon_k = (-1)^{n-k},$$

and the desired inequality (26) follows from this and (29).

Suppose equality holds in (26) for p, where $0 < k < n$. By multiplying p by -1 if necessary, we may suppose that $p^{(k)}(0) \geq 0$. If

$$\epsilon_k(-1)^j p(u_j) < \psi(u_j)$$

for some j or if $\epsilon_k a_n < C$, then by (28),

$$\begin{aligned} p^{(k)}(0) &= \epsilon_k a_n |\omega^{(k)}(0,U)| + \sum_{j=1}^{n} \epsilon_k (-1)^j p(u_j) \frac{|\omega_j^{(k)}(0,U)|}{|\omega_j(u_j,U)|} \\ &< M_k, \end{aligned}$$

since $\omega^{(k)}(0,U) \neq 0$ and $\omega_j^{(k)}(0,U) \neq 0$ for $j = 1,\ldots,n$. (The restrictions on k are used here.) But then $|p^{(k)}(0)| < |R^{(k)}(0)|$, a contradiction. Hence,

$$\epsilon_k(-1)^j p(u_j) = \psi(u_j), \quad j = 1,\ldots,n,$$

and $\epsilon_k a_n = C$. Therefore, $\epsilon_k p = R$ by (27). $\square$

Suppose the u_j's are not required to be nonnegative. An examination of the above proof shows that Lemma 13 holds when 0 is replaced by any number $x \leq u_n$. Similarly, the lemma holds also when 0 is replaced by any number $x \geq u_1$ provided R is defined with the opposite sign before the summation.

Proof of Theorems 6 and 7. To prove Theorem 6, we first consider the case where m is odd. Write $m = 2n - 1$ and define

$$p_2(x) = \frac{p(x) + p(-x)}{2} .$$

Then there exists a polynomial r of degree at most $n-1$ with $p_2(x) = r(x^2)$ for all real x. Put $\psi(u) = \phi(\sqrt{u})^m$ and define $u_j = x_j^2$ for $1 \leq j \leq n$. By hypothesis,

$$|r(u_j)| \leq \psi(u_j), \quad j = 1, \ldots, n.$$

Hence, by Lemma 13 with $C = 0$ and R replaced by $-R$,

$$|r^{(\ell)}(0)| \leq |R^{(\ell)}(0)|, \quad \ell = 0, \ldots, n-1, \tag{30}$$

where R is the polynomial of degree at most $n-1$ satisfying

$$R(u_j) = (-1)^{j-1}\psi(u_j), \quad j = 1, \ldots, n.$$

Clearly, $P(x) = R(x^2)$ since both polynomials satisfy (1). Moreover, by Lemma 13, if $\ell > 0$ and equality holds in (30) then $r = \pm R$. Since the coefficients of r (resp., R) are the coefficients of the even powers of p (resp., P), the required inequality (14) follows from (30). Also, if equality holds in (14) for an even k with $0 < k < m$, then equality holds in (30) for $\ell = k/2$; therefore, $r = \pm R$ so $p_2 = \pm P$.

If m is even, write $m = 2n$ and define

$$p_1(x) = \frac{p(x) - p(-x)}{2} .$$

Then there exists a polynomial r of degree no greater than $n-1$ with $p_1(x) = xr(x^2)$ for all real x. An argument analogous to the preceding one with $\psi(u) = \phi(\sqrt{u})^m/\sqrt{u}$ and $P(x) = xR(x^2)$ establishes Theorem 6.

To prove Theorem 7, we first consider the case where m is even. Write $m = 2n$. The same argument as for the first case of Theorem 6 with $u_j = y_j^2$ for $j = 1, \ldots, n$, shows that r now has degree at most n and satisfies

$$|r^{(\ell)}(0)| \leq |R^{(\ell)}(0)|, \quad \ell = 0, \ldots, n, \tag{31}$$

where R is as in Lemma 13 with $C = A^m$. Clearly, $Q(x) = R(x^2)$ since both polynomials satisfy (4) and (5). Since the coefficients of r (resp., R) are the coefficients of the even powers of p (resp., Q), the required inequality (16) follows from (31). Moreover, if equality holds in (16) for an even k with $0 < k < m$, then equality holds in (31) for $\ell = k/2$; therefore, $r = \pm R$ so $p_2 = \pm Q$.

If m is odd, write $m = 2n + 1$ and apply the same argument as for the second case of Theorem 6 to show that Theorem 7 holds in this case. □

Proof of Proposition 9. The previous proof showed that if U is as in part (1) and if R is the negative of the polynomial R defined in Lemma 13 with $C = 0$, then $P(x) = R(x^2)$ when $m = 2n - 1$ and $P(x) = xR(x^2)$ when $m = 2n$. Part (1) of Proposition 9 follows by comparison of coefficients since $f(x_1, \ldots, x_m)$ is the coefficient of x^{m-1} in $P(x)$ and $\omega_j(x, U)$ is a monic polynomial of degree $n - 1$ in $R(x)$.

Similarly, the previous proof also showed that if U is as in part (2) and if R is as in Lemma 13 with $C = A^m$, then $Q(x) = R(x^2)$ when $m = 2n$ and $Q(x) = xR(x^2)$ when $m = 2n + 1$. Part (2) of Proposition 9 follows as before since $-g(y_1, \ldots, y_{m-1})$ is the coefficient of x^{m-2} in $Q(x)$ and

$$\omega(x, U) = x^n - \left(\sum_{j=1}^{n} y_j^2 \right) x^{n-1} + \cdots .$$

□

6. Further remarks

Put $\psi(x) = \phi(x)^m$. Let $x_1, \ldots, x_m$ and $y_1, \ldots, y_{m-1}$ be decreasing sequences and let P and Q be the associated oscillating polynomials. It follows easily from Lemmas 10 and 11 that

$$\begin{aligned} \psi'(x_j) - (-1)^{j-1} P'(x_j) &= \frac{\partial f}{\partial x_j} \, |\omega_j(x_j, X)|, \\ \psi'(y_j) - (-1)^{j} Q'(y_j) &= \frac{\partial g}{\partial y_j} \, |\omega_j(y_j, Y)|, \end{aligned}$$

for any j where $\psi'(x_j)$ and $\psi'(y_j)$ exist. For example, if $x_1, \ldots, x_m$ is an optimal sequence of ϕ-oscillation points, then

$$P(x_j) = (-1)^{j-1} \psi(x_j), \quad P'(x_j) = (-1)^{j-1} \psi'(x_j),$$

for any j where $\psi'(x_j)$ exists. Hence P can be obtained by Hermite interpolation [8, Sect. 1.4] at any $[(m+1)/2]$ oscillation points where the derivative of ϕ exists. (In particular, the Hermite interpolation polynomial for ψ oscillates between ψ and $-\psi$ when the interpolation points are chosen to be every other one of an optimal sequence of ϕ-oscillation points beginning at an endpoint.)

One can prove the first part of Theorems 6 and 7 in a different way by appealing to an argument of Kellogg [4]. For example, we show how to obtain (16) when m is even. Given a number λ with $-1 < \lambda < 1$, there is a $y_0 > y_1$ satisfying $|\lambda p_2(y_0)| < Q(y_0)$ by (15), (5) and (ii). Define $D = Q - \lambda p_2$ and write $m = 2n$. Then by (4) and hypothesis, D alternates in sign at $y_0, \ldots, y_n$, where $y_n = 0$. Hence D has at least n positive zeros. By Descartes' rule of signs, the coefficients of D have at least n changes in sign. Since D is an even polynomial of degree at most $2n$, each of the $n+1$ even coefficients of D must be nonzero. Hence if k even,

$$Q^{(k)}(0) - \lambda p_2^{(k)}(0) \neq 0, \qquad -1 < \lambda < 1,$$

so $|p^{(k)}(0)| \leq |Q^{(k)}(0)|$ since $p^{(k)}(0) = p_2^{(k)}(0)$.

A discussion of optimal estimates on the coefficients of polynomials of restricted growth has been given by Bernstein [1, §12]. Here it is assumed that there is a complex polynomial S of degree m satisfying $\phi(x)^m = |S(x)|$ for all real x and all the roots of S lie in the lower half-plane. (See also [7, p. 181].) Let A be the leading coefficient of S and suppose $A \geq 0$. In this case the extremal polynomials P and Q are given by the relation $S = Q + iP$. Moreover, the curve

$$z(x) = \frac{S(x)}{|S(x)|}$$

winds around the unit circle counterclockwise $m/2$ times as x traverses the real line from ∞ to $-\infty$. The reason for this is that $z(x)^2$ is a product of m terms of the form $(x - \lambda)/(x - \overline{\lambda})$, where $\operatorname{Im} \lambda < 0$. The solutions of the equation $z(y) = \pm 1$ give the y_j's and the solutions of the equation $z(x) = \pm i$ give the x_j's. A discussion of the case where $S(x) = (x+i)^m$ is given in [3, Example 1].

References

[1] S. N. Bernstein, *Lecons sur les Propriétés Extrémales et la Meilleure Approximation des Fonctions Analytiques d'une Variable Réele*, Gauthier-Villars, Paris, 1926.

[2] L. A. Harris, *A Bernstein-Markov theorem for normed spaces*, J. Math. Anal. Appl. **208**(1997), 476–486.

[3] ———, *Coefficients of polynomials of restricted growth on the real line*, (To appear in J. Approx. Theory in 1998.)

[4] O. D. Kellogg, *On bounded polynomials in several variables*, Math. Z. **27**(1928), 55–64.

[5] J. H. B. Kemperman, *Markov type inequalities for the derivatives of a polynomial*, in Aspects of Mathematics and its Applications, ed. by J. A. Barroso, North-Holland, Amsterdam-New York, 1986, p.465–476.

[6] D. Liu and J. Nocedal, *On the limited memory BFGS method for large scale optimization*, Math. Programming **45**(1989), 503–528.

[7] G. V. Milovanović, D. S. Mitrinović, Th. M. Rassias, *Topics In Polynomials: Extremal Problems, Inequalities, Zeros*, World Scientific, Singapore, 1994.

[8] T. J. Rivlin, *Chebyshev Polynomials*, 2nd ed. Wiley, New York, 1990.

[9] E. B. Saff and R. S. Varga, *On incomplete polynomials*, in Numerische Methoden der Approximationstheorie, Band 4, ed. by L. Collatz, G. Meinardus and H. Warner, ISNM 42, Birkhäuser Verlag, Basel and Stuttgart, 1978, p. 281–298.

[10] W. W. Rogosinski, *Some elementary inequalities for polynomials*, Math. Gaz. **39**(1955), 7–12.