Problem. Let A, B and C be matrices of sizes $m \times n$, $m \times m$, and $n \times m$, respectively, and suppose $C\,A = I$. Give a necessary and sufficient condition for the block matrix $M = \begin{bmatrix} A & B \\ 0 & C \end{bmatrix}$ to be invertible and find an expression for M^{-1} when this condition holds.

Solution. Put $D = I + (I - A\,C)(B - I)$. Then M is invertible if and only if D is invertible and

$$M^{-1} = \begin{bmatrix} C[I - (B - I)D^{-1}(I - A\,C)] & -[I + C(B - I)\,D^{-1}A] \\ D^{-1}(I - A\,C) & D^{-1}A \end{bmatrix}$$

Proof. Let N be M in the case where $B = I$. Then N is invertible and $N^{-1} = \begin{bmatrix} C & -I \\ I - A\,C & A \end{bmatrix}$ so $N^{-1}\,M = \begin{bmatrix} I & C(B - I) \\ 0 & D \end{bmatrix}$. Thus $N^{-1}\,M$ is invertible if and only if D is invertible and in that case $M^{-1}\,N = \begin{bmatrix} I & -C(B - I)\,D^{-1} \\ 0 & D^{-1} \end{bmatrix}$. The solution follows from this.