Banach Algebras where the Singular Elements are Removable Singularities

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Let A be a C*-algebra with identity and suppose A has real rank 0. Suppose a complex-valued function is holomorphic and bounded on the intersection of the open unit ball of A and the identity component of the set of invertible elements of A. We show that then the function has a holomorphic extension to the entire open unit ball of A. Further, we show that this does not hold when $A = C(S)$, where S is any compact Hausdorff space that contains a homeomorphic image of the interval [0,1].

1. THE REMOVABLE SINGULARITY PROPERTY

Throughout, A denotes a Banach algebra with identity e, $A_{\rm inv}$ denotes the open set of invertible elements of A, and $A^e_{\rm inv}$ denotes the identity component of $A_{\rm inv}$. Also, A_0 denotes the open unit ball of A, i.e.,

$$A_0 = \{x \in A : \ \|x\| < 1\},$$

and $D(A)$ denotes $A_0 \cap A^e_{\rm inv}$. In particular, $D(\mathbb{C})$ is the open unit disc punctured at the origin.

Definition 1. A Banach algebra A has the *removable singularity property* if every bounded holomorphic function $f : D(A) \to \mathbb{C}$ has a holomorphic extension to A_0.

In this definition we consider the identity component of $A_{\rm inv}$ rather than $A_{\rm inv}$ since if $A_{\rm inv}$ is not connected the holomorphic function on $A_{\rm inv}$ which is 1 on one of the components and 0 on the others has no holomorphic extension to A_0. It follows from Theorem 3 given in the next section (or see [12, Theorem 3]) that our definition is no more stringent when the range of f is allowed to be any Banach space.

Every finite dimensional Banach algebra A has the removable singularity property by the classical Riemann removable singularities theorem [6, p. 30] and the characterization of the singular elements of A as those with determinant zero. (Define the determinant of an element of A as the determinant of its left regular representation [1, p. 3].)

Definition 1 was given previously in [10, p.460] for closed complex Jordan algebras of operators on a Hilbert space and used to determine the biholomorphic mappings of certain domains in these algebras.

Definition 2. The *holomorphic radius* of a Banach algebra A is the largest number $r \geq 0$ such that every holomorphic function $f : A \to \mathbb{C}$ satisfying $|f(x)| \leq 1$ for all $x \in D(A)$ also satisfies $|f(x)| \leq 1$ for all $x \in A$ with $\|x\| \leq r$. We denote the holomorphic radius of A by $r(A)$.

It is always true that $1/2 \leq r(A) \leq 1$. Indeed, let f be as in Definition 2 and let $x \in A$ with $\|x\| < 1/2$. Then $g(\mu) = f(\mu e + x)$ is holomorphic in $\mathbb{C}$ and $|g(\mu)| \leq 1$ when $|\mu| = 1/2$ since then $\mu e + x \in D(A)$. (See [1, p. 40].) By the maximum principle, $|g(0)| \leq 1$, i.e., $|f(x)| \leq 1$. Hence, $|f(x)| \leq 1$ for all $x \in A$ with $\|x\| \leq 1/2$ by the continuity of f. Thus $r(A) \geq 1/2$. It follows from the Hahn-Banach theorem that $r(A) \leq 1$.

Also, $r(A) = 1$ if A has the removable singularity property. Suppose on the contrary that $r(A) < 1$. Then there exist a holomorphic function $f : A \to \mathbb{C}$ and a $y \in A_0$ such that $|f(x)| \leq 1$ for all $x \in D(A)$ and $|f(y)| > 1$. Therefore

$$g(x) = \frac{1}{f(y) - f(x)}$$

is a bounded holomorphic function on $D(A)$ with no holomorphic extension to A_0.

Further, $r(A) = 1$ for any C*-algebra A with identity. This follows immediately from the first part of the maximum principle given in [8]. For C*-algebras, this asserts that if $f : A \to \mathbb{C}$ is a holomorphic function satisfying $|f(u)| \leq 1$ for all elements u in the identity component U_e of the set of all unitary elements of A, then $|f(x)| \leq 1$ for all $x \in A$ with $\|x\| \leq 1$.

Example 1. Let $A = H^\infty(\Delta)$, where Δ is the open unit disc. Clearly, A_{inv} is connected since each of its elements has a holomorphic logarithm. Define a continuous linear functional $f_1 : A \to \mathbb{C}$ by $f_1(x) = e\,x'(0)/2$. Then $|f_1(x)| \leq 1$ for all $x \in A_0 \cap A_{\text{inv}}$ by [3, Exercise 6.36] but $f_1(rj) = e\,r/2$ where $j(\lambda) = \lambda$ and $0 < r < 1$. Hence $r(A) \leq 2/e$. In particular, A does not have the removable singularity property.

Define a holomorphic 1-homogeneous function $f_2 : A_{\text{inv}} \to \mathbb{C}$ by

$$f_2(x) = \frac{x(\frac{1}{3})^2}{x(0)}.$$

Then $|f_2(x)| \leq 1$ for all $x \in A_0 \cap A_{\text{inv}}$ by [3, Corollary 6.32] but $f_2(x)$ does not have a holomorphic extension to a neighborhood of 0 since each such neighborhood contains a function rj, where $r > 0$.

Let A and B be Banach algebras with identity and let $\alpha : D(A) \to D(B)$ and $\beta : B_0 \to A_0$ be holomorphic functions satisfying

$$\begin{aligned} \beta(D(B)) &\subseteq D(A), \\ \alpha(\beta(x)) &= x \quad \text{for all } x \in D(B). \end{aligned}$$

It is easy to verify that if A has the removable singularity property then so does B. Thus, for example, if the product algebra $A \times B$ with the max norm has the removable singularity property then so do A and B.

Let S be a compact Hausdorff space and let $T \subseteq S$ be a retract of S, i.e., there exists a continuous map $\tau : S \to T$ with $\tau(t) = t$ for all $t \in T$. The above applies with $\alpha(x) = x|_T$ for $x \in C(S)$ and $\beta(x) = x \circ \tau$ for $x \in C(T)$. Thus if $C(S)$ has the removable singularity property, so does $C(T)$.

2. MAIN RESULTS

Definition 3 [2]. Let A be a C*-algebra and let S be the set of self-adjoint elements of A. Then A has *real rank 0* if the elements of S with finite spectra are dense in S.

As shown by Brown and Pedersen in [2], many interesting C*-algebras have real rank 0. For example, the C*-algebra $\mathcal{L}(H)$ of all bounded linear operators on a Hilbert space H has real rank 0. More generally, any von Neumann algebra has real rank 0. The space $C(S)$ of all continuous functions on a compact Hausdorff space S has real rank 0 if and only if S is totally disconnected. (It is a von Neumann algebra only if S is extremely disconnected.) Also, any AF-algebra has real rank 0. If $\mathcal{LC}(H)$ is the C*-algebra of all compact operators on H, then $A = \mathbb{C}I + \mathcal{LC}(H)$ has real rank 0 as does the Calkin algebra $\mathcal{L}(H)/\mathcal{LC}(H)$. Note that the set of invertible elements of the Calkin algebra has a different component for each value of the Fredholm index and thus is not connected. See [4] for further details and references.

THEOREM 1. *Every C*-algebra A with identity that has real rank 0 has the removable singularity property.*

EXAMPLE 2. Since the set of noninvertible elements of A may have nonempty interior in infinite dimensions, removable singularities for the domain A_0 may not be removable for other domains. Let $A = \mathcal{L}(H)$, where H is an infinite dimensional Hilbert space, and recall that A_{inv} is connected by [7, Prob. 141]. Let v be a nonunitary isometry in A (such as the shift operator on ℓ_2). Note that none of the operators in the ball

$$B(v;1) = \{x \in A : \ \|x - v\| < 1\}$$

is invertible by the identity $v^*(v - x) = e - v^*x$ since $\|v\| = 1$. Take $\mathcal{E} = A_0 \cup B(v;1)$ and let $1/2 < r < 1$. By the Hahn-Banach theorem, there is an $\ell \in A^*$ with $\|\ell\| = \ell(v) = 1$ so the function

$$f(x) = \frac{1}{1 - r\ell(x)}$$

is holomorphic and bounded in A_0. Hence $f : \mathcal{E} \cap A_{\text{inv}} \to \mathbb{C}$ is a bounded holomorphic function which has no holomorphic extension to $\mathcal{E}$ since $v/r \in B(v;1)$.

PROPOSITION 2. *Let S be a compact Hausdorff space that contains a homeomorphic image of* $[0,1]$. *Then the commutative C*-algebra $A = C(S)$ does not have the removable singularity property. In fact, there exists a bounded holomorphic function on $D(A)$ that does not have a holomorphic extension to a neighborhood of* 0.

In view of these results, it is natural to ask whether the removable singularity property characterizes those C*-algebras with real rank 0.

DEFINITION 4. A set $S \subseteq A$ is said to be a *thick* subset of A if the only homogeneous polynomial $p : A \to \mathbb{C}$ satisfying $p(x) = 0$ for all $x \in S$ is $p = 0$.

A more stringent definition of a "thick" subset is given in [9, p. 130] (where the open set Ω should be connected).

EXAMPLE 3. Let A be a C*-algebra and let S be the set of self-adjoint elements of A. Suppose $p : A \to \mathbb{C}$ is a homogeneous polynomial with

$p(x) = 0$ for all $x \in S$. If $z \in A$, then $z = x + i\,y$, where x and y are elements of S given by

$$x = \frac{z + z^*}{2}, \qquad y = \frac{z - z^*}{2\,i}.$$

Hence $g(\lambda) = p(x + \lambda y)$ is an entire function which vanishes on the real line so $g \equiv 0$. In particular, $p(z) = g(i) = 0$. Thus S is a thick subset of A.

Proof of Theorem 1. It follows from the remarks after Definition 2 that $r(A) = 1$ since A is a C*-algebra. Hence it suffices to show that part (c) of Theorem 3 (below) holds when S is the set of self-adjoint elements in A with finite spectrum. This set is thick by Example 3 since A has real rank 0.

Suppose $f : D(A) \to \mathbb{C}$ is a bounded holomorphic function. Let $x \in S$ with $x \neq 0$ and take $\delta = 1/(1 + \|x\|)$. Given $0 < |\mu| < \delta$, define $g(\lambda) = f(\mu e + \lambda x)$. Then g is bounded and holomorphic for $|\lambda| < \delta$ except at most finitely many points λ where $\mu e + \lambda x \notin A_{\text{inv}}$. Therefore, these singularities are removable. Thus part (c) holds. ▮

By the theorem below, each of the conditions (a)–(d) with the additional assumption that $r(A) = 1$ is equivalent to the assertion that A has the removable singularity property.

THEOREM 3. *Let A be a Banach algebra with identity. The following are equivalent:*

a) Every bounded holomorphic function $f : D(A) \to X$ has a holomorphic extension to $\Omega = \{x \in A : \; \|x\| < r(A)\}$, where X is any Banach space.

b) Every bounded holomorphic function $f : D(A) \to \mathbb{C}$ has a holomorphic extension to a neighborhood of 0 in A.

c) For every bounded holomorphic function $f : D(A) \to \mathbb{C}$ there is a thick subset S of A such that for each $x \in S$ the function $\lambda \to f(\mu e + \lambda x)$ has a holomorphic extension to a disc $|\lambda| < \delta$ when $0 < |\mu| < \delta$, where $\delta > 0$.

d) Every holomorphic function $f : A^e_{\text{inv}} \to \mathbb{C}$ satisfying

$$f(\lambda x) = \lambda^n f(x) \quad \text{and} \quad |f(x)| \leq \|x\|^n$$

for all $x \in A^e_{\text{inv}}$ and $\lambda \neq 0$ extends to a homogeneous polynomial on A of degree n, where n is any positive integer.

It is well known [5, Theorem 1.4] that a compact subset K of a domain in $\mathbb{C}$ is a set of removable singularities for bounded holomorphic functions when K is finite or countable (since then the 1-dimensional Hausdorff measure of K is zero.) Moreover, the complement of K is connected. Thus our proof of Theorem 1 shows that a Banach algebra A with identity has the removable singularity property if $r(A) = 1$ and if the set S of all elements of A with finite or countable spectrum is a thick subset of A.

3. REMAINING PROOFS

Proof of Theorem 3. (a) $\Rightarrow$ (b). This follows with $X = \mathbb{C}$ since $r(A) > 0$.
(b) $\Rightarrow$ (c). By (b), f has a holomorphic extension to $B(0;r) = \{x \in A : \|x\| < r\}$, where $r > 0$. Take $S = A$, let $x \in S$, and put $\delta = r/(1+\|x\|)$. Then $\mu e + \lambda x \in B(0;r)$ whenever $0 < |\mu| < \delta$ and $|\lambda| < \delta$. Hence (c) holds.
(c) $\Rightarrow$ (d). Let f satisfy the hypotheses of (d). Given a complex number $\mu \neq 0$ and $x \in A$ with $x \neq 0$, note that $\mu e + \lambda x \in A^e_{\mathrm{inv}}$ when $\|\lambda x\| < |\mu|$. Hence $g(\lambda) = f(\mu e + \lambda x)$ is holomorphic and satisfies $|g(\lambda)| \leq (2|\mu|)^n$ for $|\lambda| < |\mu|/\|x\|$. Applying the Cauchy estimates and the identity $g^{(k)}(0) = \hat{D}^k f(\mu e)x$, we obtain

$$|\hat{D}^k f(\mu e)x| \leq 2^n k! |\mu|^{n-k} \|x\|^k \tag{1}$$

for all $x \in A$ and $\mu \neq 0$.

By (c), if x is in some thick subset of A, there is a $\delta > 0$ such that $g(\lambda)$ has a holomorphic extension to the disc $|\lambda| < \delta$ when $0 < |\mu| < \delta$. By the Cauchy estimates,

$$|\hat{D}^k f(\mu e)x| \leq \frac{k!}{\delta^k}[(1 + \|x\|)\delta]^n.$$

Hence the function $\mu \to \hat{D}^k f(\mu e)x$ has a removable singularity at $\mu = 0$ and thus is entire. We obtain $\hat{D}^k f(e)x = 0$ for $k > n$ by applying the maximum principle to (1) for large disks. Therefore, $\hat{D}^k f(e) = 0$ for all $k > n$.

By the power series expansion of f,

$$f(x) = \sum_{k=0}^{n} \frac{1}{k!} \hat{D}^k f(e)(x - e)$$

for $\|x - e\| < 1$. Since f is n-homogeneous and $\hat{D}^k f(e)$ is k-homogeneous, it follows that $f(x) = \frac{1}{n!}\hat{D}^n f(e)x$. By the identity theorem, this holds for all $x \in A^e_{\mathrm{inv}}$.

(d) $\Rightarrow$ (a). Let $f : D(A) \to X$ be a bounded holomorphic function. Without loss of generality, we may suppose that $\|f(x)\| \leq 1$ for all $x \in D(A)$. Given $x \in A^e_{\mathrm{inv}}$, let $r > 0$ satisfy $r\|x\| < 1$. Then $g(\lambda) = f(\lambda x)$ is a vector-valued function that is holomorphic in an open set containing the disc $|\lambda| \leq r$ since $\lambda = 0$ is a removable singularity of g by [11, Theorem 3.13.3]. Hence, applying the Taylor expansion for g with the Cauchy integral form of the coefficients [11, p. 96-97], we obtain

$$f(\lambda x) = \sum_{n=0}^{\infty} f_n(x)\lambda^n \tag{2}$$

for $|\lambda| \leq r$, where

$$f_n(x) = \frac{1}{2\pi i} \int_{|\lambda|=r} \frac{f(\lambda x)}{\lambda^{n+1}}\, d\lambda. \tag{3}$$

Since r can be replaced by any smaller positive value in (3) without changing the value of $f_n(x)$, the formula (3) defines $f_n(x)$ for all $x \in A^e_{\mathrm{inv}}$. Moreover, f_n is holomorphic on A^e_{inv} since f is holomorphic on $D(A)$. If $x \in A^e_{\mathrm{inv}}$, then $\|f_n(x)\| \leq \|x\|^n$ by the Cauchy estimates and $f_n(\lambda x) = \lambda^n f_n(x)$ for all $\lambda \neq 0$ by a change of variable.

Note that f_0 extends to a constant function on A. Indeed, $f_0(e^{\lambda x})$ is a bounded entire function of λ and hence is constant [11, Theorem 3.13.2]. Then f_0 is constant on $S = \{e^x : x \in A\}$ so f_0 is constant on A^e_{inv} by the identity theorem [11, Theorem 3.16.4] since S contains a neighborhood of e.

Now for $n \geq 1$, let $\ell \in X^*$ be given with $\|\ell\| = 1$ and put $p = \ell \circ f_n$. By (d), p extends to a homogeneous polynomial on A of degree n. Hence if $x \in A$,

$$n!p(x) = \frac{d^n}{dt^n} p(e + t\,x)\bigg|_{t=0} = \ell(\hat{D}^n f_n(e)x),$$

so

$$\ell\Big(n!f_n(x) - \hat{D}^n f_n(e)x\Big) = 0.$$

It follows that $n!f_n(x) = \hat{D}^n f_n(e)x$ by the Hahn-Banach theorem. Thus, f_n extends to a homogeneous polynomial of degree n on A. Further, $|p(x)|\, r(A)^n \leq \|x\|^n$ since p is n-homogeneous, so

$$\|f_n(x)\| \leq \frac{\|x\|^n}{r(A)^n}$$

by the Hahn-Banach theorem.

Therefore, the series $\sum_{n=0}^{\infty} f_n(x)$ converges to a holomorphic function $\tilde{f}(x)$ for $x \in \Omega$ and for $x \in D(A)$ by the Weierstrass M-test and [11, Theorem 3.18.1]. Moreover, $\tilde{f}(x) = f(x)$ for all $x \in D(A)$ by (2). Thus $\tilde{f}$ is the required extension. ▮

Proof of Proposition 2. It follows from the Tietze extension theorem that the homeomorphic image of $[0,1]$ is a retract of S. Thus by the remarks at the end of Section 1, it suffices to consider the case $A = C[0,1]$.

We first show that there exists a holomorphic function $h : A_{\mathrm{inv}} \to \mathbb{C}$ satisfying

$$e^{h(\gamma)} = \frac{\gamma(1)}{\gamma(0)} \tag{4}$$

for all $\gamma \in A_{\mathrm{inv}}$.

Given $\gamma \in A_{\mathrm{inv}}$, it follows that $\gamma(t) \neq 0$ whenever $0 \leq t \leq 1$ so there exists a $\phi \in A$ with $e^{\phi} = \gamma$ by [3, Theorem 4.1]. Define $h(\gamma) = \phi(1) - \phi(0)$. To see that h is well defined, let $\psi \in A$ with $e^{\psi} = \gamma$. Then $e^{\phi - \psi} = 1$ so $(\phi - \psi)/(2\pi i)$ is an integer-valued continuous function on $[0,1]$. Hence ψ differs from ϕ by at most a constant and therefore $h(\gamma) = \psi(1) - \psi(0)$. Clearly (4) holds.

To show that h is holomorphic at each $\gamma_1 \in A_{\mathrm{inv}}$, let δ be the minimum of $|\gamma_1|$ and let $\gamma \in A_{\mathrm{inv}}$ with $\|\gamma - \gamma_1\| < \delta$. Define a continuous function $F : [0,1] \times [0,1] \to \mathbb{C}$ by

$$F(s,t) = (1-s)\gamma_1(t) + s\gamma(t)$$

and note that F has no zeros. Hence there exists a continuous function $f : [0,1] \times [0,1] \to \mathbb{C}$ with $F = e^f$ by [3, Example 4.21]. Clearly $F(0,t) = \gamma_1(t)$ and $F(1,t) = \gamma(t)$ whenever $0 \leq t \leq 1$. Define $\omega_0(s) = F(s,0)$ and $\omega_1(s) = F(s,1)$ for $0 \leq s \leq 1$. By definition,

$$\begin{aligned} h(\gamma_1) &= f(0,1) - f(0,0), \\ h(\gamma) &= f(1,1) - f(1,0), \\ h(\omega_0) &= f(1,0) - f(0,0), \\ h(\omega_1) &= f(1,1) - f(0,1), \end{aligned}$$

so $h(\gamma) = h(\omega_1) + h(\gamma_1) - h(\omega_0)$.

There exist holomorphic logarithms L_0 and L_1 in the discs

$$D_0 = \{\lambda \in \mathbb{C} : \ |\lambda - \gamma_1(0)| < \delta\}, \quad D_1 = \{\lambda \in \mathbb{C} : \ |\lambda - \gamma_1(1)| < \delta\}.$$

Since ω_0 and ω_1 are line segments in D_0 and D_1, respectively,

$$\begin{aligned} h(\omega_0) &= L_0(\omega_0(1)) - L_0(\omega_0(0)) = L_0(\gamma(0)) - L_0(\gamma_1(0)), \\ h(\omega_1) &= L_1(\omega_1(1)) - L_1(\omega_1(0)) = L_1(\gamma(1)) - L_1(\gamma_1(1)). \end{aligned}$$

Thus $h(\gamma)$ is holomorphic for $\|\gamma - \gamma_1\| < \delta$ by the result of the previous paragraph.

Now define

$$f(\gamma) = \gamma(0)e^{h(\gamma)/2}.$$

Then $f : A_{\text{inv}} \to \mathbb{C}$ is holomorphic and $f(\gamma)^2 = \gamma(1)\gamma(0)$ for all $\gamma \in A_{\text{inv}}$. In particular, $|f(\gamma)| < 1$ whenever $\gamma \in A_0 \cap A_{\text{inv}}$. Suppose there exists a function $\tilde{f}$ holomorphic in a neighborhood U of the origin of A and satisfying $\tilde{f}(\gamma) = f(\gamma)$ for all $\gamma \in A_0 \cap A^e_{\text{inv}}$. Then $\tilde{f}(\gamma)^2 = \gamma(1)\gamma(0)$ for all $\gamma \in U$ by the identity theorem [11, Theorem 3.16.4]. Define a holomorphic function $g : \Delta \to A_0$ by

$$g(\lambda)(t) = 1 + t(\lambda - 1)$$

and choose r with $rA_0 \subseteq U$ and $0 < r < 1$. Then $\tilde{f}(rg(\lambda))^2 = r^2\lambda$ for all $\lambda \in \Delta$. This contradicts the fact that the identity mapping on Δ has no holomorphic square root. ▮

APPENDIX

In this section we discuss some other removable singularity theorems. The first is an infinite dimensional extension of the classical Riemann removable singularities theorem [6, p. 30–31]. It asserts that the zeros of a holomorphic function are removable singularities for holomorphic functions that are locally bounded in an extended sense. (See [13, Theorem II.1.1.5].)

Let X be a (Hausdorff) topological vector space, let Y be a Banach space, and let W be a locally convex topological vector space.

Proposition 4. *Let D be an open set in X and let $S = \{x \in D : h(x) = 0\}$, where $h : D \to W$ is a holomorphic function with $h \not\equiv 0$ in D. Suppose $f : D \setminus S \to Y$ is a holomorphic function that is locally bounded on each (nonempty) intersection of a complex line with D. Then f has a holomorphic extension $\tilde{f} : D \to Y$. Moreover, $D \setminus S$ is a dense subset of D.*

Proof. We will show that for each $x_0 \in S$, there exists a holomorphic function f_V defined in a neighborhood V of x_0 such that f_V agrees with f at all points of V where f is defined.

By hypothesis, there exists an open balanced subset U of X with $x_0+U \subseteq D$. We first observe that there exists a $u \in U$ with $u \neq 0$ and an $r \in (0,1)$ such that $h(x_0+\lambda u) \neq 0$ whenever $\lambda \in C_r$, where $C_r = \{\lambda \in \mathbb{C} : |\lambda| = r\}$. Otherwise, for every $u \in U$, the function $g(\lambda) = h(x_0+\lambda u)$ is holomorphic at each point of $\overline{\Delta}$ and has a sequence of zeros with a limit point in Δ so $h(x_0+u) = 0$ by the identity theorem for vector-valued functions [11, Theorem 3.11.5]. It follows that $h \equiv 0$ on x_0+U and hence $h \equiv 0$ on D by the identity theorem for functions of vectors [11, Theorem 3.16.4].

Therefore, $x_0+\lambda u \in D \setminus S$ for all $\lambda \in C_r$. Since $D \setminus S$ is open and C_r is compact, there exists a neighborhood V of x_0 such that $V+\lambda u \subseteq D \setminus S$ whenever $\lambda \in C_r$. We may also choose V so that $V+\lambda u \subseteq D$ whenever $\lambda \in \overline{\Delta}$.

By hypothesis, the mapping $x \to f(x+\lambda u)$ is holomorphic on V for each $\lambda \in C_r$ and the mapping $\lambda \to f(x+\lambda u)$ is continuous on C_r for each $x \in V$. Define

$$f_V(x) = \frac{1}{2\pi i} \int_{|\lambda|=r} \frac{f(x+\lambda u)}{\lambda}\, d\lambda. \tag{A.1}$$

Then f_V is holomorphic in V since it is locally bounded (in the usual sense) and Gâteaux differentiable in V by the argument for [11, Theorem 3.11.2]. (See [11, Theorem 3.17.1].)

To show that f_V agrees with f on $V \cap (D \setminus S)$, let x be a member of this set. Since the holomorphic function $\lambda \to h(x+\lambda u)$ has only isolated zeros in Δ, the locally bounded function $\lambda \to f(x+\lambda u)$ has only isolated singularities in Δ. (This also shows that $D \setminus S$ is dense in D.) These singularities are removable by [11, Theorem 3.13.3] and thus the function is holomorphic in Δ. Therefore, $f(x) = f_V(x)$ by the Cauchy integral formula [11, Theorem 3.11.3].

Finally, we show that any two functions f_{V_1} and f_{V_2} agree on $V_1 \cap V_2$ when this set is nonempty. Clearly S is closed in D and has empty interior by the identity theorem. Hence each component of $V_1 \cap V_2$ contains a neighborhood disjoint from S. By what we have shown, the functions f_{V_1} and f_{V_2} agree with f on this neighborhood so $f_{V_1} = f_{V_2}$ on $V_1 \cap V_2$ by the identity theorem. Thus the required function $\tilde{f}$ is defined by $\tilde{f}(x) = f(x)$ for $x \in D \setminus S$ and $\tilde{f}(x) = f_V(x)$ for $x \in V$. ∎

Remark. The above proof shows that Proposition 4 holds for any set S closed in D with the following property: For each $x_0 \in S$ there exists a $u \in X$ and a neighborhood V of x_0 such that the set $\{\lambda \in \Delta : x+\lambda u \in S\}$ is finite for each $x \in V$.

Next, we establish a removable singularity theorem where the set S of singularities apparently does not have the property mentioned above. Our proposition replaces the trace class restriction of [10, Theorem 3] by a more general compactness condition. Let X and Y be normed linear spaces and let $\mathcal{L}(X,Y)$ denote the space of all bounded linear operators from X to Y.

PROPOSITION 5. *Let $\mathfrak{A}$ be a complex subspace of $\mathcal{L}(X,Y)$ and let*

$$\mathcal{D} = \{Z \in \mathfrak{A} : \ (CZ+D)^{-1} \text{ exists}\},$$

where $C \in \mathcal{L}(Y,X)$ and $D \in \mathcal{L}(X)$. Suppose C is compact or that $\mathfrak{A}$ consists only of compact operators and let W be a Banach space. If $\mathcal{E}$ is a domain in $\mathfrak{A}$ which intersects $\mathcal{D}$ and if $f : \mathcal{D} \cap \mathcal{E} \to W$ is a holomorphic function that is locally bounded, then f has a holomorphic extension $\tilde{f} : \mathcal{E} \to W$. Moreover, $\mathcal{D} \cap \mathcal{E}$ is dense in $\mathcal{E}$.

Proof. Our proof relies on the fact that the spectrum of a compact operator has no nonzero limit point. Adopt the notation

$$B(Z_0; r) = \{Z \in \mathfrak{A} : \ \|Z - Z_0\| < r\}.$$

We first show that if $Z_0 \in \overline{\mathcal{D}} \cap \mathcal{E}$, then f has a holomorphic extension to a neighborhood of Z_0.

Since f is locally bounded, there exists an $r > 0$ and a number M such that $B(Z_0; r) \subseteq \mathcal{E}$ and $\|f(Z)\| \leq M$ whenever $Z \in B(Z_0; r) \cap \mathcal{D}$. Given $\epsilon > 0$ with $2\epsilon < r$, by hypothesis there exists a $Z_1 \in \mathcal{D} \cap \mathcal{E}$ satisfying $\|Z_1 - Z_0\| < \epsilon$. Clearly $B(Z_1; r_1) \subseteq B(Z_0; r)$ and $Z_0 \in B(Z_1; r_1)$, where $r_1 = r - \epsilon$. Define

$$g(\lambda) = f(Z_1 + \lambda W),$$

where $W \in \mathfrak{A}$ with $\|W\| \leq r_1$. By hypothesis, $g(\lambda)$ is holomorphic at each $\lambda \in \Delta$ where $C(Z_1 + \lambda W) + D$ is invertible. Moreover, $\|g(\lambda)\| \leq M$ for these λ. Put $X_1 = (CZ_1 + D)^{-1}C$. Since

$$C(Z_1 + \lambda W) + D = (CZ_1 + D)(I + \lambda X_1 W)$$

and since $X_1 W$ is compact, it follows that $g(\lambda)$ is holomorphic at all but a finite set of points in Δ. (This also shows that $\mathcal{D} \cap B(Z_1; r_1)$ is connected and that $B(Z_1; r_1) \subseteq \overline{\mathcal{D}} \cap \mathcal{E}$.) The points of this finite set are removable singularities by [11, Theorem 3.13.3]. Hence by the Cauchy estimates [11, p. 97],

$$\|g^{(n)}(0)\| \leq n! M, \quad n = 0, 1, \ldots.$$

Since $g^{(n)}(0) = \hat{D}^n f(Z_1)W$, it follows that

$$\|\hat{D}^n f(Z_1)\| r_1^n \leq n!M, \quad n = 0, 1, \dots,$$

and hence by the Weierstrass M-test and [11, Theorem 3.18.1], the series

$$\sum_{n=0}^{\infty} \frac{1}{n!} \hat{D}^n f(Z_1)(Z - Z_1) \tag{A.2}$$

converges to a holomorphic function $f_\epsilon(Z)$ on $B(Z_1; r_1)$. Since (A.2) is the Taylor series expansion for f at Z_1, it follows that f_ϵ and f agree on a neighborhood of Z_1 by [11, Theorem 3.17.1]. Therefore, $\tilde{f} = f$ on $\mathcal{D} \cap B(Z_1; r_1)$ by the identity theorem. Thus, f_ϵ is the asserted extension of f to the neighborhood $B(Z_1; r_1)$ of Z_0.

Moreover, by what we have shown, the nonempty set $S = \overline{\mathcal{D}} \cap \mathcal{E}$ is both open and closed in $\mathcal{E}$, so $S = \mathcal{E}$, i.e. $\mathcal{D} \cap \mathcal{E}$ is dense in $\mathcal{E}$. Thus, to complete the proof, all we must show is that if f_1 and f_2 are the extensions of f defined above on the balls B_1 and B_2, respectively, then f_1 and f_2 agree on $B_1 \cap B_2$. This is a consequence of an argument in the proof of Proposition 4. ▌

REFERENCES

1. F. F. Bonsall and J. Duncan, *Complete Normed Algebras*, Ergebnisse der Math. Bd. 80, Springer-Verlag, New York, 1973.
2. L. G. Brown and G. K. Pedersen, *C*-algebras of real rank 0*, J. Funct. Anal. **99**(1991), 131–149.
3. R. B. Burckel, *An Introduction to Classical Complex Analysis* Vol. 1, Academic Press, New York, 1979.
4. K. R. Davidson, *C*-Algebras by Example*, Fields Institute Monographs, Vol. 6, AMS, Providence, 1996.
5. J. Garnett, *Analytic Capacity and Measure*, Lecture Notes in Mathematics, Vol. 297, Springer-Verlag, Berlin, 1972.
6. R. C. Gunning, *Introduction to Holomorphic Functions of Several Variables, Vol. I: Function Theory*, Brooks/Cole, Belmont, CA, 1990.
7. P. R. Halmos, *A Hilbert Space Problem Book*, Second edition, Graduate Texts in Mathematics, 19, Springer-Verlag, New York, 1982.
8. L. A. Harris, *Banach algebras with involution and Möbius transformations*, J. Funct. Anal. **11**(1972), 1–16.
9. L. A. Harris, *Linear fractional transformations of circular domains in operator spaces*, Indiana Univ. Math. J. **41**(1992), 125–147.
10. L. A. Harris, *Unbounded symmetric homogeneous domains in spaces of operators*, Ann. Scuola Norm. Sup. Pisa Cl. Sci. **22**(1995), 449–467.

11. E. Hille and R. S. Phillips, *Functional Analysis and Semi-Groups*, Amer. Math. Soc. Colloq. Publ., Vol. 31, AMS, Providence, 1957.

12. E. Ligocka and J. Siciak, *Weak analytic continuation*, Bull. Acad. Polon. Sci. Sér. Sci. Math. Astronom. Phys. **20**(1972), 461–466.

13. J.-P. Ramis, *Sous-ensembles Analytiques d'une Variété Banachique Complexe*, Ergebnisse der Math. No. 53, Springer-Verlag, Berlin, 1970.