SCHURIAN ALGEBRAS AND SPECTRAL ADDITIVITY

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A class of algebras is introduced that includes the unital Banach algebras over the complex numbers. Commutator results are proved for such algebras and used to establish spectral properties of certain elements of Banach algebras.

Dedicated to the memory of I. N. Herstein

1. **Introduction.** In this note, we introduce a condition on an algebra motivated by the situation in which Schur's lemma [S1] is applicable. We say that algebras satisfying this condition are Schurian. For such algebras, we prove some (purely algebraic) results characterizing those elements associated with inner derivations having range in the radical.

We note that, among others, each Banach algebra over the complex numbers $\mathbb{C}$ is Schurian. Our algebraic results are then applied, in conjunction with the theory of subharmonic functions — notably, Vesentini's result [V] on subharmonicity of the spectral radius (of holomorphic, Banach-algebra-valued functions from a domain in $\mathbb{C}$) — to determine those elements in a Banach algebra that exhibit certain spectral properties (*spectral additivity*). These spectral-additivity results are themselves applied, in [H-K], to a topic initiated by Frobenius [F] (and contributed to by Schur [S2]), the study of mappings that preserve invertible elements.

2. **Schurian algebras.** In the definition that follows, we describe the condition that plays the key role in our algebraic commutator results.

Definition 2.1. An algebra $\mathfrak{A}$ with unit I over a field $\mathbb{F}$ is said to be *Schurian* when $\mathcal{L}A \subseteq \mathcal{L}$ for a maximal left ideal $\mathcal{L}$ in $\mathfrak{A}$ and some A in $\mathfrak{A}$ implies that there is a z in $\mathbb{F}$ such that $A - zI \in \mathcal{L}$.

Schur's lemma, as commonly used in mathematical discussion, covers a range of commutation and intertwiner statements. The literature of functional analysis has versions of this lemma where the main point is an analytic argument. In practice this may be an approximation result as, for instance, in von Neumann's celebrated Double Commutant theorem [N]. For our purposes, we shall say that a (one-)transitive representation π of a unital algebra $\mathfrak{A}$ over a field $\mathbb{F}$ on a vector space $\mathcal{V}$ over $\mathbb{F}$ satisfies the *Schur condition* when each ($\mathbb{F}$-)linear transformation of $\mathcal{V}$ into itself that commutes with $\pi(\mathfrak{A})$ is multiplication by some element of $\mathbb{F}$. This concept is closely related to the notion of *absolute irreducibility* in the finite-dimensional case. (See the discussion of Section 3.) In place of "irreducible," we shall use "transitive," which better describes our situation in the infinite-dimensional case: no proper, non-zero, invariant (or "stable") linear submanifolds (rather than "reducing" linear submanifolds). Thus $\mathfrak{A}$ acts (one-)transitively on $\mathcal{V}$ when $\mathfrak{A}x$ $(= \{Ax : A \in \mathfrak{A}\})$

$= \mathcal{V}$ for each non-zero x in $\mathcal{V}$. We are indebted to Walter Feit for bringing the relevance of absolute irreducibility to our attention.

Theorem 2.2. *An algebra $\mathfrak{A}$ over a field $\mathbb{F}$ is Schurian if and only if each transitive representation of $\mathfrak{A}$ on a vector space over $\mathbb{F}$ satisfies the Schur condition.*

Proof. Let $\mathfrak{A}$ be an algebra represented transitively on a vector space $\mathcal{V}$ over $\mathbb{F}$. We may assume that $\mathfrak{A}$ is an algebra of linear transformations on $\mathcal{V}$ (and the unit I of $\mathfrak{A}$ is the identity transformation on $\mathcal{V}$). Let x be a non-zero element in $\mathcal{V}$ and let $\mathcal{L}_x$ be the left ideal of elements of $\mathfrak{A}$ that annihilate x. With A in $\mathfrak{A}$ not in $\mathcal{L}_x$, there is a B in $\mathfrak{A}$ such that $BAx = x$ (by transitivity). Thus $BA - I \in \mathcal{L}_x$. It follows that $\mathfrak{A}A + \mathcal{L}_x = \mathfrak{A}$, and $\mathcal{L}_x$ is a *maximal* left ideal in $\mathfrak{A}$.

Let T be a linear transformation on $\mathcal{V}$ that commutes with $\mathfrak{A}$. By transitivity, there is an A in $\mathfrak{A}$ such that $Ax = Tx$. For each B in $\mathfrak{A}$, $T(Bx) = BTx = BAx$. If $\mathcal{B} \in \mathcal{L}_x$, then $Bx = 0$ and $BAx = T(Bx) = 0$. Thus $BA \in \mathcal{L}_x$ in this case. It follows that $\mathcal{L}_x A \subseteq \mathcal{L}_x$. If we assume that $\mathfrak{A}$ is Schurian, there is an a in $\mathbb{F}$ such that $A - aI \in \mathcal{L}_x$. Hence $Ax = ax$ and $T(Sx) = STx = SAx = aSx$ for each S in $\mathfrak{A}$. But $\mathfrak{A}x = \mathcal{V}$, so $T = aI$, and the given representation of $\mathfrak{A}$ satisfies the Schur condition.

Suppose, next, that each transitive representation of $\mathfrak{A}$ satisfies the Schur condition. Let $\mathcal{L}$ be a maximal left ideal in $\mathfrak{A}$ and $\mathcal{V}$ be the quotient vector space $\mathfrak{A}/\mathcal{L}$ (over $\mathbb{F}$). With A in $\mathfrak{A}$ not in $\mathcal{L}$, $\mathfrak{A}A + \mathcal{L}$ is a left ideal containing $\mathcal{L}$ properly. Thus $\mathfrak{A}A + \mathcal{L} = \mathfrak{A}$ and $\mathfrak{A}(A + \mathcal{L}) = \mathcal{V}$, where we view $\mathfrak{A}$ as represented on $\mathcal{V}$ by its left-multiplication action on $\mathfrak{A}/\mathcal{L}$, and this representation is transitive. If A in $\mathfrak{A}$ is such that $\mathcal{L}A \subseteq \mathcal{L}$, the mapping T that maps $B + \mathcal{L}$ to $BA + \mathcal{L}$, for each B in $\mathfrak{A}$, is a well-defined linear mapping of $\mathcal{V}$ into itself that commutes with the action of $\mathfrak{A}$ on $\mathcal{V}$. By assumption $T = aI$ for some a in $\mathbb{F}$. Thus

$$aI + \mathcal{L} = T(I + \mathcal{L}) = I \cdot A + \mathcal{L} = A + \mathcal{L},$$

and $A - aI \in \mathcal{L}$. It follows that $\mathfrak{A}$ is Schurian.

Lemma 2.3. *Let $\mathfrak{A}$ be an algebra over $\mathbb{F}$ with unit I and radical $\mathcal{R}$. Then $\mathfrak{A}/\mathcal{R}$ $(= \tilde{\mathfrak{A}})$ is Schurian if and only if $\mathfrak{A}$ is Schurian.*

Proof. The quotient mapping $A \to \tilde{A}$ carries the set of maximal left ideals in $\mathfrak{A}$ onto the corresponding set in $\tilde{\mathfrak{A}}$. If $\mathfrak{A}$ is Schurian, $\tilde{\mathcal{L}}$ is a maximal left ideal in $\tilde{\mathfrak{A}}$, $\mathcal{L}$ its inverse image in $\mathfrak{A}$, and $\tilde{\mathcal{L}}\tilde{A} \subseteq \tilde{\mathcal{L}}$, then $\mathcal{L}A \subseteq \mathcal{L}$. By assumption, there is a z in $\mathbb{F}$ such that $A - zI \in \mathcal{L}$. Thus $\tilde{A} - z\tilde{I} \in \tilde{\mathcal{L}}$. It follows that $\tilde{\mathfrak{A}}$ is Schurian.

Suppose $\tilde{\mathfrak{A}}$ is Schurian, $\mathcal{L}$ is a maximal left ideal in $\mathfrak{A}$ and $\mathcal{L}A \subseteq \mathcal{L}$ for some A in $\mathfrak{A}$. Then $\tilde{\mathcal{L}}\tilde{A} \subseteq \tilde{\mathcal{L}}$. By assumption, there is a z in $\mathbb{F}$ such that $\tilde{A} - z\tilde{I} \in \tilde{\mathcal{L}}$. Since $\mathcal{R} \subseteq \mathcal{L}$, $A - zI \in \mathcal{L}$. Hence $\mathfrak{A}$ is Schurian.

Theorem 2.4. *Let $\mathfrak{A}$ be Schurian over a field $\mathbb{F}$ with radical $\mathcal{R}$ and unit I. Then A in $\mathfrak{A}$ has the property that $\mathcal{L}A \subseteq \mathcal{L}$ for every maximal left ideal $\mathcal{L}$ in $\mathfrak{A}$ if and only if* $(\operatorname{ad} A(B) =)\ AB - BA \in \mathcal{R}$ *for each B in $\mathfrak{A}$.*

Proof. Suppose $\operatorname{ad} A$ has range in $\mathcal{R}$ and B lies in some maximal left ideal $\mathcal{L}$ in $\mathfrak{A}$.

Then $AB \in \mathcal{L}$ and $AB - BA \in \mathcal{R} \subseteq \mathcal{L}$. Thus $BA \in \mathcal{L}$. Hence $\mathcal{L}A \subseteq \mathcal{L}$ for each maximal left ideal $\mathcal{L}$ in $\mathfrak{A}$.

Suppose, now, that $\mathcal{L}A \subseteq \mathcal{L}$ for each maximal left ideal $\mathcal{L}$ in $\mathfrak{A}$. If B is an element of $\mathfrak{A}$ and $\mathcal{L}$ is a maximal left ideal in $\mathfrak{A}$, then $\{T \in \mathfrak{A} : TB \in \mathcal{L}\}$ $(= \mathcal{L}_B)$ is a left ideal in $\mathfrak{A}$. If $B \in \mathcal{L}$, then $\mathcal{L}_B = \mathfrak{A}$. Suppose $S \notin \mathcal{L}_B$. Then $SB \notin \mathcal{L}$. Since $\mathcal{L}$ is a maximal left ideal in $\mathfrak{A}$, there is an A' in $\mathfrak{A}$ such that $A'SB - B \in \mathcal{L}$. Thus $A'S - I \in \mathcal{L}_B$. It follows that $\mathcal{L}_B$ is a maximal left ideal in $\mathfrak{A}$.

By assumption, $\mathcal{L}_B A \subseteq \mathcal{L}_B$. Since $\mathfrak{A}$ is Schurian, there is a z_B in $\mathbb{F}$ such that $A - z_B I \in \mathcal{L}_B$, equivalently, such that $(A - z_B I)B \in \mathcal{L}$. Thus $AB - z_B B \in \mathcal{L}$ for each B in $\mathfrak{A}$. The choice of z_B in $\mathbb{F}$ is unique when $B \notin \mathcal{L}$. If $B \in \mathcal{L}$, each z in $\mathbb{F}$ will serve as z_B.

Suppose $B \notin \mathcal{L}$ and $B - zB' \in \mathcal{L}$ for some z in $\mathbb{F}$ and B' in $\mathfrak{A}$. Then $AB - zAB'$, $AB - z_B B$, and $zAB' - zz_{B'}B'$ are in $\mathcal{L}$. Thus $z_B B - zz_{B'}B' \in \mathcal{L}$. But $z_B B - z_B zB' \in \mathcal{L}$. Hence $z(z_B - z_{B'})B' \in \mathcal{L}$. Now $z \neq 0$ since $B \notin \mathcal{L}$. Moreover, $B' \notin \mathcal{L}$ for the same reason. Thus $z_B = z_{B'}$.

Suppose B and B' are not in $\mathcal{L}$ and $B - zB' \notin \mathcal{L}$ for all z in $\mathbb{F}$. Then $A(B + B') - z_{B+B'}(B + B')$, $AB - z_B B$, and $AB' - z_{B'}B'$ are in $\mathcal{L}$. Thus

$$(z_B - z_{B+B'})B + (z_{B'} - z_{B+B'})B' \in \mathcal{L}.$$

By assumption on B, B', and $B - zB'$ with z in $\mathbb{F}$, we have that $z_B = z_{B+B'} = z_{B'}$.

It follows that there is a z in $\mathbb{F}$ such that $AB - zB \in \mathcal{L}$ for all B in $\mathfrak{A}$. In particular, $A - zI \in \mathcal{L}$ and $BA - zB \in \mathcal{L}$ for each B in $\mathfrak{A}$. Thus $AB - BA \in \mathcal{L}$. As this holds for each maximal left ideal $\mathcal{L}$ and each B in $\mathfrak{A}$, $\operatorname{ad} A$ has range in $\mathcal{R}$.

Theorem 2.5. *Let $\mathfrak{A}$ be Schurian over a field $\mathbb{F}$ with radical $\mathcal{R}$ and unit I. Then A in $\mathfrak{A}$ has the property that $AB - BA + I$ is invertible for all B in $\mathfrak{A}$ if and only if $\operatorname{ad} A$ has range in $\mathcal{R}$. If $\mathfrak{A}$ is semi-simple, as well (that is, $\mathcal{R} = (0)$), then $AB - BA + I$ is invertible for all B in $\mathfrak{A}$ if and only if A lies in the center of $\mathfrak{A}$.*

Proof. Suppose $\operatorname{ad} A$ has range in $\mathcal{R}$. Then $AB - BA$ lies in each maximal left and right ideal in $\mathfrak{A}$, and $AB - BA + I$ lies in no such maximal ideal. Hence $AB - BA + I$ lies in no proper left or right ideal. From this, we have that $\mathfrak{A}(AB - BA + I) = \mathfrak{A} = (AB - BA + I)\mathfrak{A}$. In particular, $AB - BA + I$ has a left and right inverse in $\mathfrak{A}$. It follows that $AB - BA + I$ is invertible in $\mathfrak{A}$.

Suppose $AB - BA + I$ is invertible for each B in $\mathfrak{A}$. If $\operatorname{ad} A$ does not have range in $\mathcal{R}$, then there is a maximal left ideal $\mathcal{L}$ in $\mathfrak{A}$ such that $\mathcal{L}A$ is not contained in $\mathcal{L}$, from Theorem 2.4. Thus there is a B' in $\mathcal{L}$ such that $B'A \notin \mathcal{L}$. Since $\mathcal{L}$ is a *maximal* left ideal, there is a T in $\mathfrak{A}$ such that $TB'A - I \in \mathcal{L}$. Now, $ATB' \in \mathcal{L}$, whence $ATB' - TB'A + I \in \mathcal{L}$. Letting B be TB', we have that $AB - BA + I$ is not invertible — contradicting our assumption. Thus $\operatorname{ad} A$ has range in $\mathcal{R}$. In the case where $\mathcal{R}$ is semi-simple, A lies in the center of $\mathfrak{A}$.

Corollary 2.6. *If $\mathfrak{A}$ is semi-simple, Schurian and such that $AB - BA + I$ is invertible for all A and B in $\mathfrak{A}$, then $\mathfrak{A}$ is commutative.*

Corollary 2.7. *If $\mathfrak{A}$ is semi-simple and Schurian, then each commutator is nilpotent if and only if $\mathfrak{A}$ is commutative.*

Proof. If $N^k = 0$ for some positive integer k, then $I - N + N^2 - \cdots (-1)^{k-1}N^{k-1}$ is inverse to $I + N$. In particular, $I + N$ is invertible for each nilpotent N. With the assumption on commutators, Corollary 2.6 applies and $\mathfrak{A}$ is commutative.

The algebra of upper triangular $n \times n$ matrices over $\mathbb{C}$ is Schurian (from Example 3.2) in which each commutator is nilpotent, yet it is not commutative. Its radical is the set of all elements with 0 diagonal. Thus the hypothesis of semi-simplicity is essential in Corollary 2.7. At the same time, Corollary 2.6 is strictly stronger than Corollary 2.7, as the next result shows. For it, we make use of Example 4.3.

Corollary 2.8. *If $\mathfrak{A}$ is a semi-simple Banach algebra over $\mathbb{C}$, then each commutator in $\mathfrak{A}$ is quasi-nilpotent if and only if $\mathfrak{A}$ is commutative.*

Corollary 2.7 is curiously akin to a commutativity theorem of Herstein. Theorem 3.1.3 of [H] states that if each commutator in a ring is equal to some power n of itself, where n is an integer greater than 1, then the ring is commutative. This result applies to all rings, while Corollary 2.7 holds for semi-simple Schurian algebras and may fail, as we saw, if semi-simplicity is not assumed.

3. **The finite-dimensional case.** We assume that $\mathfrak{A}$ is an n-dimensional algebra over a field $\mathbb{F}$. Suppose π is a representation of $\mathfrak{A}$ on an m-dimensional vector space $\mathcal{V}$ (over $\mathbb{F}$). Choosing a basis for $\mathcal{V}$ and representing each linear transformation of $\mathcal{V}$ into itself as an $m \times m$ matrix over $\mathbb{F}$ relative to this basis and elements of $\mathcal{V}$ as ordered m-tuples with entries from $\mathbb{F}$, we may consider π as a representation of $\mathfrak{A}$ in $\mathrm{M}_m(\mathbb{F})$ acting on column vectors in $\mathbb{F}^m$. Again, letting $\mathfrak{A}$ act on itself by left multiplication and choosing a linear basis for $\mathfrak{A}$, we may view $\mathfrak{A}$ as a subalgebra of $\mathrm{M}_n(\mathbb{F})$. Let $\mathbb{K}$ be a field containing $\mathbb{F}$ and $\tilde{\mathfrak{A}}$ the subalgebra of $\mathrm{M}_n(\mathbb{K})$ generated by $\mathfrak{A}$ and $\{aI : a \in \mathbb{K}\}$, where I denotes the unit matrix in $\mathrm{M}_n(\mathbb{K})$. Let $\tilde{\pi}$ be the (unique) ($\mathbb{K}$-)linear extension of π mapping $\tilde{\mathfrak{A}}$ into $\mathrm{M}_m(\mathbb{K})$ acting on $\mathbb{K}^m$. We say that π is *absolutely transitive* when $\tilde{\pi}$ is transitive for each field extension $\mathbb{K}$ of $\mathbb{F}$; when each transitive representation of $\mathfrak{A}$ is absolutely transitive, we say that the algebra $\mathfrak{A}$ (over $\mathbb{F}$) is absolutely transitive. Though $\mathfrak{A}$ (over $\mathbb{F}$) need not be absolutely transitive, $\tilde{\mathfrak{A}}$ (over $\mathbb{K}$) may be. In this case, $\mathbb{K}$ is said to be a *splitting field* for the algebra $\mathfrak{A}$ (over $\mathbb{F}$).

A basic result (proved by a dimension counting argument) states:

A representation π of $\mathfrak{A}$ on $\mathcal{V}$ is absolutely transitive if and only if it satisfies the Schur condition.

Applying this result and Theorem 2.2, we have the following corollary.

Corollary 3.1. *A finite-dimensional algebra $\mathfrak{A}$ over a field $\mathbb{F}$ is Schurian if and only if it is absolutely transitive.*

Example 3.2. It is known that a finite-dimensional algebra over an algebraically closed field is absolutely transitive. Thus the algebraic closure of $\mathbb{F}$ is a splitting field for each finite-dimensional algebra over $\mathbb{F}$. A proof is not difficult. If $\mathbb{F}$ is algebraically closed, $\mathfrak{A}$ is a finite-dimensional algebra over $\mathbb{F}$ and π is a transitive representation of $\mathfrak{A}$ on the vector space $\mathcal{V}$ over $\mathbb{F}$, then $\mathcal{V}$ is finite dimensional (has dimension not exceeding that of $\mathfrak{A}$). Let T be a linear transformation commuting with each $\pi(A)$ (A in $\mathfrak{A}$). Then the null space and range of T are stable under $\pi(\mathfrak{A})$. By transitivity, these subspaces are either (0) or $\mathcal{V}$. Thus T is either 0 or is invertible. Since the determinant of $T - xI$ is a polynomial in x over $\mathbb{F}$ and $\mathbb{F}$ is algebraically closed, this polynomial has a root a in $\mathbb{F}$. As $T - aI$ commutes with each $\pi(A)$ (A in $\mathfrak{A}$) and is not invertible, $T = aI$. It follows that π is absolutely transitive. Hence $\mathfrak{A}$ is absolutely transitive. In particular, each finite-dimensional algebra over $\mathbb{C}$ is Schurian. In Example 4.1, we shall see that this need not be the case if the assumption of finite dimensionality is not present. We shall also see (Example 4.3) that an analytic argument proves this same result when the field is $\mathbb{C}$ since each finite-dimensional algebra over $\mathbb{C}$ is a Banach algebra.

Example 3.3. The concepts of absolute transitivity and Schurian algebra are not as completely field dependent in finite dimensions as Example 3.2 might lead us to believe. We note that $\mathrm{M}_n(\mathbb{F})$ is Schurian over an arbitrary field $\mathbb{F}$. This follows from an appropriately stated version of Wedderburn's theorem, but is easily proved from the familiar ideal structure of $\mathrm{M}_n(\mathbb{F})$: the only proper two-sided ideal is (0) ($\mathrm{M}_n(\mathbb{F})$ is simple), and each maximal left ideal is similar to the left ideal of matrices with 0 at each first-column entry. If π is a transitive representation of $\mathrm{M}_n(\mathbb{F})$ on the vector space $\mathcal{V}$ (over $\mathbb{F}$) and x is a non-zero vector in $\mathcal{V}$, then $\mathcal{L}_x$, the set of matrices A in $\mathrm{M}_n(\mathbb{F})$ such that $\pi(A)x = 0$, is a maximal left ideal (as in the proof of Theorem 2.2). Thus $\mathrm{M}_n(\mathbb{F})/\mathcal{L}_x$ is a vector space of dimension n over $\mathbb{F}$ that is (linearly) isomorphic to $\mathcal{V}$. It follows that $\mathcal{L}(\mathcal{V})$, the linear space (over $\mathbb{F}$) of linear transformations of $\mathbb{F}$ into itself has dimension n^2. Since $\mathrm{M}_n(\mathbb{F})$ is simple and has dimension n^2 (over $\mathbb{F}$), π is an isomorphism of $\mathrm{M}_n(\mathbb{F})$ onto $\mathcal{L}(\mathcal{V})$. It remains to note that only scalars commute with $\mathcal{L}(\mathcal{V})$ and to apply Theorem 2.2 to conclude that $\mathrm{M}_n(\mathbb{F})$ is Schurian.

4. **Further examples.** We begin with a class of algebras that are Schurian.

Example 4.1. Let $\mathbb{K}$ be a *proper* field extension of the field $\mathbb{F}$. Viewed as an algebra over $\mathbb{F}$, the only (left) ideal, other than $\mathbb{K}$ is (0). This algebra is simple, and can be finite dimensional. Of course $(0)a = (0)$ for each a in $\mathbb{K}$, and $a - b \in (0)$ for some b in $\mathbb{F}$ if and only if $a = b \in \mathbb{F}$. Choosing a in $\mathbb{K} \setminus \mathbb{F}$, we see that, as an an algebra over $\mathbb{F}$, $\mathbb{K}$ is not Schurian. A proper field extension of another field is not Schurian over that field. A case in point is $\mathbb{C}$ as a two-dimensional, simple algebra over $\mathbb{R}$. This same argument applies when $\mathbb{K}$ is a division algebra over $\mathbb{F}$, and shows that $\mathbb{K}$ is Schurian in this case only when $\mathbb{K}$ is $\mathbb{F}$.

The field $\mathbb{C}(x)$ of rational functions over $\mathbb{C}$ provides us, now, with an example of (an infinite dimensional) algebra over the algebraically closed field $\mathbb{C}$, that is not Schurian (*cf.* Example 3.2).

Example 4.2. The algebra $\mathbb{C}[z]$ of complex polynomials in a single variable z is infinite-dimensional over $\mathbb{C}$; unlike $\mathbb{C}(z)$, it is Schurian. More generally, we show that the algebra of polynomials $\mathbb{F}[x]$ over each algebraically closed field $\mathbb{F}$ is Schurian. In this case, each maximal left ideal is a maximal two-sided ideal $\mathcal{M}$, and $\mathbb{F}[x]/\mathcal{M}$ is a field $\mathbb{K}$ extending $\mathbb{F}$. If k is the image of x under the quotient mapping, its inverse k^{-1} is the image of some polynomial $p(x)$ in $\mathbb{F}[x]$. It follows that $k^{-1} = p(k)$ and that k satisfies the polynomial equation $kp(k) - 1 = 0$ over $\mathbb{F}$. Since $\mathbb{F}$ is algebraically closed, $kp(k) - 1$ factors completely over $\mathbb{F}$. As $\mathbb{K}$ is a field, one of those factors is 0. Thus k lies in $\mathbb{F}$, and $\mathbb{F}[x]/\mathcal{M}$ is $\mathbb{F}$. It follows that $\mathbb{F}[x]$ is Schurian over $\mathbb{F}$.

Example 4.3. Each unital Banach algebra over the complex numbers is Schurian. To see this, we note, first, that if T' is a bounded operator on a Banach space $\mathfrak{Y}$ and T' commutes with each operator in a transitive family $\mathcal{F}$ of operators on $\mathfrak{Y}$, then T' is a scalar. If z is in the spectrum of T', then $T' - zI$ is not invertible. Now, the null space and the range of $T' - zI$ are both invariant under $\mathcal{F}$. Since $\mathcal{F}$ is assumed to be transitive, the range of $T' - zI$ is either (0), in which case, $T' = zI$, or the range is $\mathfrak{Y}$. We may assume that it is $\mathfrak{Y}$. Thus the null space of $T' - zI$ is not $\mathfrak{Y}$ and must be (0). It follows that $T' - zI$ is a continuous linear isomorphism of $\mathfrak{Y}$ onto $\mathfrak{Y}$. The Banach inversion theorem (*cf.* [K-R: Theorem 1.8.5]) applies, and $T' - zI$ is invertible — contradicting our choice of z. Hence $T' = zI$.

With $\mathcal{L}$ a maximal left ideal in the Banach algebra $\mathfrak{A}$ and A an element of $\mathfrak{A}$ such that $\mathcal{L}A \subseteq \mathcal{L}$, the mapping ρ_A of the quotient Banach space $\mathfrak{A}/\mathcal{L}$ into itself that assigns $BA + \mathcal{L}$ to $B + \mathcal{L}$ is a bounded linear transformation of the quotient into itself that commutes with the family $\{\lambda_B : B \in \mathfrak{A}\}$ $(= \mathcal{F})$, where λ_B assigns $BT + \mathcal{L}$ to $T + \mathcal{L}$. Since $\mathcal{L}$ is maximal, $\mathcal{F}$ is a transitive family of (bounded) linear transformations of $\mathfrak{A}/\mathcal{L}$ into itself. Thus ρ_A is some scalar multiple z of the identity transformation on $\mathfrak{A}/\mathcal{L}$. In particular, $A + \mathcal{L} = \rho_A(I + \mathcal{L}) = zI + \mathcal{L}$. Hence $\mathfrak{A}$ is Schurian.

Combining the results of the preceding example and Example 4.1, we see that $\mathbb{C}(z)$ cannot be normed as a Banach algebra over $\mathbb{C}$. This follows as well from the fact that a normed field over $\mathbb{C}$ must coincide with $\mathbb{C}$. From Example 4.2, $\mathbb{C}[z]$ is Schurian and can be normed as a *normed* algebra (for example, by restricting each polynomial to the closed unit disk in $\mathbb{C}$ and taking its supremum norm on that disk). As normed, it is not a Banach algebra nor can it be normed to be a Banach algebra. To see this, note that with t small and positive, $1 + tz$ will be near 1, the unit of $\mathbb{C}[z]$, relative to a given norm on $\mathbb{C}[z]$. If $\mathbb{C}[z]$ were a Banach algebra with that norm, then $1 + tz$ would have an inverse in $\mathbb{C}[z]$ which, of course, it does not. As remarked in Example 4.1, $\mathbb{C}$, as a two-dimensional algebra over $\mathbb{R}$, is not Schurian. It is, however, a Banach algebra over $\mathbb{R}$.

If $\mathfrak{A}$ is a (unital) commutative, normed algebra over $\mathbb{C}$ and $\mathcal{M}$ is a maximal ideal in $\mathfrak{A}$, the question of whether or not $\mathfrak{A}$ is Schurian amounts to the question of whether or not the field $\mathfrak{A}/\mathcal{M}$ is $\mathbb{C}$. If each such $\mathcal{M}$ is closed (as is the case when $\mathfrak{A}$ is a Banach algebra), then the quotient is a normed field over $\mathbb{C}$, and coincides with $\mathbb{C}$. But such ideals need not be closed. For an example of this, we may turn back to $\mathbb{C}[z]$ normed with the supremum norm of restriction to $[0, 1]$. Let ρ be the mapping that assigns to each polynomial in $\mathbb{C}[z]$ its value at 2. The kernel $\mathcal{M}$ of ρ is a maximal ideal since ρ is a homomorphism of $\mathbb{C}[z]$ onto a field (as it happens to be, $\mathbb{C}$). If $\mathcal{M}$ were closed in $\mathbb{C}[z]$, then ρ would be continuous (see [K-R: Corollary 1.2.5]) and extend, by uniform continuity, to a homomorphism of the algebra $\mathcal{C}$ of all continuous complex-valued functions on $[0,1]$. But such a homomorphism corresponds to evaluating each function in $\mathcal{C}$ at some point of $[0,1]$ (*cf.* [K-R: Corollary 3.4.2]). Thus ρ would assign to each polynomial, in particular, to z, its value at this point of $[0,1]$ and at 2 — a contradiction. Thus $\mathcal{M}$ is not closed. Nonetheless, $\mathbb{C}[z]$ is Schurian (Example 4.2) and the quotient by all maximal ideals is $\mathbb{C}$. Is this the case for each commutative normed algebra over $\mathbb{C}$? — for all normed algebras over $\mathbb{C}$?

Example 4.4. From Example 4.1, the quaternions $\mathcal{Q}$, as a 4-dimensional (associative, division) algebra over $\mathbb{R}$, is not Schurian.

At the same time, $\mathcal{Q}$ provides us with an example where the conclusions of Theorems 2.4 and 2.5 fail. For the conclusion of Theorem 2.4 to apply, since $(0)q \subseteq (0)$ for each q, $\operatorname{ad} q$ would have to map $\mathcal{Q}$ into (0) — that is, $\mathcal{Q}$ would have to be commutative. For Theorem 2.5, let e_0 be the unit and e_1, e_2, e_3 be elements of $\mathcal{Q}$ satisfying $e_1^2 = e_2^2 = e_3^2 = -e_0$ and e_ie_j is e_k or $-e_k$ as (i,j,k) is an even or an odd permutation of $(1,2,3)$. Then $e_1q - qe_1 + e_0$ is 0 for no q in $\mathcal{Q}$ and is, accordingly, invertible for all q. But e_1 is not in the center of $\mathcal{Q}$.

5. **Spectral additivity.** In this section, we study a spectral property, *spectral additivity*, of certain elements in a (unital) Banach algebra over $\mathbb{C}$. Making use of the fact that such Banach algebras are Schurian (see Example 4.3), the results of Section 2 on Schurian algebras, the subharmonicity of the spectral radius [V], and results for subharmonic functions [H-Ke], we identify the elements that are spectrally additive as those whose commutators lie in the radical of the Banach algebra (the central elements, when the algebra is semi-simple). We use the notation '$\operatorname{sp}(T)$' to denote the spectrum of the element T of a Banach algebra $\mathfrak{A}$ relative to $\mathfrak{A}$. When it is necessary to indicate the algebra relative to which the spectrum occurs, we use the notation '$\operatorname{sp}_{\mathfrak{A}}(T)$'.

Definition 5.1. An element A of a unital Banach algebra $\mathfrak{A}$ over $\mathbb{C}$ is said to be *spectrally additive* (in $\mathfrak{A}$) when $\operatorname{sp}(A+B) \subseteq \operatorname{sp}(A) + \operatorname{sp}(B)$ for each B in $\mathfrak{A}$.

We show, first, that central elements in a Banach algebra are spectrally additive.

Proposition 5.2. *If A and B are commuting elements of a Banach algebra $\mathfrak{A}$ over $\mathbb{C}$ with unit I, then $\operatorname{sp}(A+B) \subseteq \operatorname{sp}(A) + \operatorname{sp}(B)$.*

Proof. Let $\mathcal{A}$ be a maximal abelian subalgebra of $\mathfrak{A}$ containing A and B. If $TS = ST$ with T and S in $\mathfrak{A}$, and S is invertible, then $S^{-1}T = S^{-1}TSS^{-1} = S^{-1}STS^{-1} = TS^{-1}$. Thus the inverse of an invertible element in $\mathcal{A}$ commutes with $\mathcal{A}$, hence lies in $\mathcal{A}$. It follows that A, B, and $A+B$, have spectra relative to $\mathcal{A}$ identical with their spectra relative to $\mathfrak{A}$. With C in $\mathcal{A}$, $\lambda \in \operatorname{sp}_{\mathcal{A}}(C)$ if and only if there is a non-zero multiplicative linear functional ρ on $\mathcal{A}$ such that $\rho(C) = \lambda$ (since $C - \lambda I$ is in a maximal ideal of $\mathcal{A}$ if and only if there is such a ρ). If $\lambda \in \operatorname{sp}_{\mathcal{A}}(A+B)$, there is a multiplicative (non-zero) linear functional ρ on $\mathcal{A}$ such that $\lambda = \rho(A+B) = \rho(A) + \rho(B)$. Since $\rho(A) \in \operatorname{sp}_{\mathcal{A}}(A)$ and $\rho(B) \in \operatorname{sp}_{\mathcal{A}}(B)$, our assertion follows.

Corollary 5.3. *Each central element of a unital Banach algebra over $\mathbb{C}$ is spectrally additive.*

Lemma 5.4. *If $\mathfrak{A}$ is a Banach algebra over $\mathbb{C}$ with unit I and radical $\mathcal{R}$, then for each A in $\mathfrak{A}$, $\mathrm{sp}_{\mathfrak{A}}(A) = \mathrm{sp}_{\mathfrak{A}/\mathcal{R}}(\varphi(A))$, where φ is the quotient mapping of $\mathfrak{A}$ onto $\mathfrak{A}/\mathcal{R}$.*

Proof. We prove that B is invertible in $\mathfrak{A}$ if and only if $\varphi(B)$ is invertible in $\mathfrak{A}/\mathcal{R}$. Since $\varphi(I)$ is the unit of $\mathfrak{A}/\mathcal{R}$, if B is invertible in $\mathfrak{A}$, then $\varphi(B^{-1})$ is the inverse of $\varphi(B)$ in $\mathfrak{A}/\mathcal{R}$.

Suppose $\varphi(T)$ is inverse to $\varphi(B)$ in $\mathfrak{A}/\mathcal{R}$. Then $\varphi(TB - I) = 0 = \varphi(BT - I)$. Thus $TB - I$ and $BT - I$ are in $\mathcal{R}$. It follows that TB $(= TB - I + I)$ and BT are invertible. Since B has both a right and left inverse, B is invertible in $\mathfrak{A}$.

We conclude, now, that $A - zI$ is invertible in $\mathfrak{A}$ if and only if $\varphi(A) - z\varphi(I)$ is invertible in $\mathfrak{A}/\mathcal{R}$. Thus $\mathrm{sp}_{\mathfrak{A}}(A) = \mathrm{sp}_{\mathfrak{A}/\mathcal{R}}(\varphi(A))$.

Corollary 5.5. *An element A of a unital Banach algebra $\mathfrak{A}$ over $\mathbb{C}$ is spectrally additive in $\mathfrak{A}$ if and only if $\varphi(A)$ is spectrally additive in $\mathfrak{A}/\mathcal{R}$, where $\mathcal{R}$ is the radical of $\mathfrak{A}$ and φ is the quotient mapping of $\mathfrak{A}$ onto $\mathfrak{A}/\mathcal{R}$.*

Theorem 5.6. *An element A of a unital Banach algebra $\mathfrak{A}$ over $\mathbb{C}$ with radical $\mathcal{R}$ is spectrally additive in $\mathfrak{A}$ if and only if $AT - TA \in \mathcal{R}$ for each T in $\mathfrak{A}$. If $\mathfrak{A}$ is semi-simple, A is spectrally additive if and only if A lies in the center of $\mathfrak{A}$.*

Proof. If $AT - TA \in \mathcal{R}$ for each T in $\mathfrak{A}$ and φ is the quotient mapping of $\mathfrak{A}$ onto $\mathfrak{A}/\mathcal{R}$, then $\varphi(A)$ lies in the center of $\mathfrak{A}/\mathcal{R}$. From Corollary 5.3, $\varphi(A)$ is spectrally additive in $\mathfrak{A}/\mathcal{R}$. From Corollary 5.5, A is spectrally additive in $\mathfrak{A}$.

Suppose, next, that A is spectrally additive in $\mathfrak{A}$. Then $\mathrm{sp}(A+T) \subseteq \mathrm{sp}(A) + \mathrm{sp}(T)$, for each T in $\mathfrak{A}$. Thus $\mathrm{r}(A+T) \le \mathrm{r}(A) + \mathrm{r}(T)$, for each T in $\mathfrak{A}$, where $\mathrm{r}(S)$ denotes the spectral radius of S (that is, $\mathrm{r}(S) = \sup\{|z| : z \in \mathrm{sp}(S)\}$). The function f from $\mathbb{C}$ into $\mathfrak{A}$ defined at z as $\exp(-zT)\, A\, \exp(zT)$ is entire. Moreover, $f(z)$, the image of A under an automorphism of $\mathfrak{A}$, is spectrally additive, whence $\mathrm{r}(f(z) + T) \le \mathrm{r}(f(z)) + \mathrm{r}(T) = \mathrm{r}(A) + \mathrm{r}(T)$ for each z in $\mathbb{C}$ and T in $\mathfrak{A}$. Let $g(z)$ be $(f(z) - f(0))/z$ for z in $\mathbb{C} \setminus \{0\}$. Then g is holomorphic on $\mathbb{C} \setminus \{0\}$. At the same time, $g(z) \to AT - TA$ as $z \to 0$. If we define $g(0)$ to be $AT - TA$, then g, so extended, is entire. Now,

$$\mathrm{r}(g(z)) = \mathrm{r}\Big(\frac{f(z) - f(0)}{z}\Big) \le \frac{\mathrm{r}(f(z)) + \mathrm{r}(A)}{|z|} = \frac{2}{|z|}\mathrm{r}(A).$$

From [V], $z \to \mathrm{r}(g(z))$ is subharmonic. Liouville's theorem for subharmonic functions yields, now, that $z \to \mathrm{r}(g(z))$ is constant on $\mathbb{C}$. Since $\mathrm{r}(g(z)) \to 0$ as $z \to \infty$, we have that $\mathrm{r}(g(z)) = 0$ for all z in $\mathbb{C}$. In particular, $\mathrm{r}(AT - TA) = \mathrm{r}(g(0)) = 0$. Thus $AT - TA$ is quasi-nilpotent (has spectrum (0)) for each T in $\mathfrak{A}$. It follows that $AB - BA + I$ is invertible for each B in $\mathfrak{A}$. Since $\mathfrak{A}$ is Schurian, Theorem 2.5 applies, and $AT - TA \in \mathcal{R}$ for each T in $\mathfrak{A}$. If $\mathfrak{A}$ is semi-simple, $\mathcal{R} = (0)$, and A lies in the center of $\mathfrak{A}$.

Remark 5.7. The following argument allows us to avoid appealing to the theory of subharmonic functions in the preceding proof. At the point where we have shown that $\mathrm{r}(g(z)) \le 2\,\mathrm{r}(A)/|z|$, we prove that $\mathrm{r}(g(z)) = 0$ for all z in $\mathbb{C}$ by using the maximum modulus principle. Assume, on the contrary, that $\mathrm{r}(g(z_0)) \ne 0$ for some z_0 in $\mathbb{C}$. Suppose $c > 2\,\mathrm{r}(A)$,

$r > |z_0|$ and $c/r < \mathrm{r}(g(z_0))$. According to the spectral radius formula [K-R: Theorem 3.3.3], $\mathrm{r}(T) = \lim_{n\to\infty} \|T^n\|^{1/n}$ for each T in $\mathfrak{A}$. Hence for each z in $\mathbb{C}_r$ $(= \{z : |z| = r\})$, there is a positive integer m with $\|g(z)^m\|^{1/m} < c/r$. By continuity of g, this inequality holds for all complex numbers in some neighborhood of z. By compactness of $\mathbb{C}_r$, there is a finite set of such neighborhoods that covers $\mathbb{C}_r$. Let $n_1, \dots, n_k$ be the positive integers corresponding to these neighborhoods. If $|z| = r$, the inequality $\|g(z)^{n_j}\|^{1/n_j} < c/r$ holds for some j. Let n be $n_1 \cdots n_k$ and m_j be n/n_j. With z in $\mathbb{C}_r$, we have

$$\|g(z)^n\| = \|[g(z)^{n_j}]^{m_j}\| \le \|g(z)^{n_j}\|^{m_j} < (\tfrac{c}{r})^n.$$

By the Hahn-Banach Theorem [K-R: Corollary 1.6.2], there is a linear functional ρ on $\mathfrak{A}$ with norm 1 satisfying $\rho(g(z_0)^n) = \|g(z_0)^n\|$. Let $h(z)$ be $\rho(g(z)^n)$. Then h is entire and $|h(z)| \le \|g(z)^n\| < (c/r)^n$ for all z in $\mathbb{C}_r$. By the maximum principle, $\|g(z_0)^n\| = |h(z_0)| \le (c/r)^n$. If λ is in the spectrum of $g(z_0)$, then λ^n is in the spectrum of $g(z_0)^n$. Thus $|\lambda^n| \le (c/r)^n$ (since the spectral radius is bounded by the Banach algebra norm [K-R: Remark 3.2.7]). Therefore, $\mathrm{r}(g(z_0)) \le c/r$, contradicting our choice of r.

Remark 5.8. In a unital Banach algebra $\mathfrak{A}$ over $\mathbb{C}$, the elements of the form $C + R$ with C in $\mathcal{C}$ and R in $\mathcal{R}$, where $\mathcal{C}$ is the center of $\mathfrak{A}$ and $\mathcal{R}$ is the radical of $\mathfrak{A}$, are spectrally additive since they are the obvious elements that "derive" $\mathfrak{A}$ into $\mathcal{R}$. But not all elements that derive a Banach algebra into its radical need be such a sum (hence, not all spectrally additive elements of a Banach algebra are the sum of a central element and an element of the radical). For an example of this, we can take for $\mathfrak{A}$ the algebra of all upper triangular $n \times n$ complex matrices (those (a_{jk}) for which $a_{jk} = 0$ when $j > k$). In this case, the center $\mathcal{C}$ consists just of scalar multiples of I, the unit matrix, and the radical consists of those matrices (a_{jk}) for which $a_{jk} = 0$ when $j \ge k$. The elements that are sums of a central element and an element of the radical are those upper triangular matrices all of whose diagonal entries are equal. Thus a non-scalar diagonal matrix is an element of the algebra that is not such a sum. But each element of this algebra derives the algebra into the radical, thus all elements are spectrally additive in this algebra.

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