A bivariate Markov inequality for Chebyshev polynomials of the second kind

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Abstract

This note presents a Markov-type inequality for polynomials in two variables where the Chebyshev polynomials of the second kind in either one of the variables are extremal. We assume a bound on a polynomial at the set of even or odd Chebyshev nodes with the boundary nodes omitted and obtain bounds on its even or odd order directional derivatives in a critical direction. Previously, the author has given a corresponding inequality for Chebyshev polynomials of the first kind and has obtained the extension of V. A. Markov's theorem to real normed linear spaces as an easy corollary.

To prove our inequality we construct Lagrange polynomials for the new class of nodes we consider and give a corresponding Christoffel-Darboux formula. It is enough to determine the sign of the directional derivatives of the Lagrange polynomials.

1. Introduction. Let $\mathcal{P}_m(\mathbb{R}^2)$ denote the space of all real-valued polynomials of degree at most m in two variables. Given $p \in \mathcal{P}_m(\mathbb{R}^2)$ and $x \in \mathbb{R}^2$, let

$$\hat{D}^k p(x)y = \frac{d^k}{dt^k} p(x+ty)\bigg|_{t=0}$$

for all $y \in \mathbb{R}^2$ and $k \geq 0$. Then $\hat{D}^k p(x)$ is the homogeneous polynomial associated with the Fréchet derivative $D^k p(x)$ when $k \geq 1$. In a previous paper [3] the author proved the following inequality for polynomials in two variables and deduced an extension of V. A. Markov's theorem to real normed linear spaces. Forms of this inequality have also been given in [9] and [10].

Theorem 1. *If $p \in \mathcal{P}_m(\mathbb{R}^2)$ where $m \geq 1$ and if*

$$\left| p\left(\cos\frac{n\pi}{m}, \cos\frac{q\pi}{m}\right)\right| \leq 1 \tag{1}$$

whenever $0 \leq n, q \leq m$ and $n - q = k \bmod 2$, then

$$\left|\hat{D}^k p(r,r)(1,-1)\right| \leq T_m^{(k)}(r) \quad \text{for } r \geq 1. \tag{2}$$

Equality holds in (2) when $p(s,t) = T_m(s)$.

For example, to deduce V. A. Markov's classical theorem, let $p(x)$ be a polynomial of degree at most m satisfying $|p(x)| \leq 1$ whenever $-1 \leq x \leq 1$. For a fixed x, put $\alpha = (x+1)/2$ and note that $0 \leq \alpha \leq 1$. Then the polynomial

$$\tilde{p}(s,t) = p(\alpha s + (1-\alpha)(-t))$$

satisfies the hypotheses of Theorem 1 for any $k \geq 0$. Since $\tilde{p}((1,1) + t(1,-1)) = p(x+t)$ for all $t \in \mathbb{R}$, we have

$$\hat{D}^k \tilde{p}(1,1)(1,-1) = p^{(k)}(x).$$

Thus $|p^{(k)}(x)| \leq T_m^{(k)}(1)$, as required. We show later in Section 5 that (2) holds more generally for $r \geq \cos\frac{\pi}{2m}$ when k is odd.

Our main result is an analogous theorem for the case where the "boundary" nodes (i.e., nodes where one of the coordinates is ± 1) are omitted in (1) and where Chebyshev polynomials of the first kind are replaced by Chebyshev polynomials of the second kind.

Theorem 2. *If $p \in \mathcal{P}_{m-2}(\mathbb{R}^2)$ where $m \geq 2$ and if (1) holds whenever $0 < n, q < m$ and $n - q = k \bmod 2$, then*

$$\left|\hat{D}^k p(r,r)(1,-1)\right| \leq U_{m-2}^{(k)}(r) \quad \text{for } r \geq \cos\frac{a\pi}{2m}, \tag{3}$$

where $a = 2$ when k is even and $a = 1$ when k is odd. Equality holds in (3) when $p(s,t) = U_{m-2}(s)$.

As usual, T_m and U_m denote the Chebyshev polynomials of the first and second kind, respectively. Thus if $x = \cos\theta$, then

$$T_m(x) = \cos m\theta \quad \text{and} \quad U_m(x) = \frac{\sin(m+1)\theta}{\sin\theta}$$

if $\sin\theta \neq 0$. Note that unlike the case of T_m, when $m > 2$ the function U_{m-2} does not have a local extremum at any value of $\cos(n\pi/m)$, $0 \leq n \leq m$, except when this value is zero. For the reasons given in [2], inequalities (2) and (3) do not hold with any bound in directions other than $(1,1)$ and $(1,-1)$ nor do they hold if p is allowed to be of larger degree.

When p is a polynomial of a single variable, Theorem 1 and Theorem 2 with $a=2$ are consequences of a result of Rogosinski [8, Theorem I]. See also [1] for a related result in this case.

2. Main results. We define nodes in $\mathbb{R}^2$ for U_m analogous to the Chebyshev nodes for T_m given in [2]. First put

$$h_n = \cos\frac{(n+1)\pi}{m+2}, \quad 0 \le n \le m, \tag{4}$$

and note that $U_m(h_n) = (-1)^n$ when $0 \le n \le m$. Define $\mathcal{N}_0$ to be the set of ordered pairs (h_n, h_q), $0 \le n, q \le m$, where n and q are both even or both odd and define $\mathcal{N}_1$ to be the set of ordered pairs (h_n, h_q), $0 \le n, q \le m$, where n is even and q is odd or n is odd and q is even. Thus, if $k=0$ or $k=1$, then

$$\mathcal{N}_k = \{(h_n, h_q) : \ (n,q) \in Q_k\}, \quad \text{where} \tag{5}$$

$$Q_k = \{(n,q) : \ 0 \le n, q \le m, \ n-q = k \bmod 2\}.$$

Note that $\mathcal{N}_k = \mathcal{N}_0$ when k is even and $\mathcal{N}_k = \mathcal{N}_1$ when k is odd. With this notation, our main result becomes the following:

Theorem 3. *Let $0 \le k \le m$. If $p \in \mathcal{P}_m(\mathbb{R}^2)$ and if $|p(x)| \le 1$ whenever $x \in \mathcal{N}_k$, then*

$$|\hat{D}^k p(r,r)(1,-1)| \le U_m^{(k)}(r) \quad \text{for } r \ge r_k, \tag{6}$$

where $r_k = \cos\dfrac{\pi}{m+2}$ when k is even and $r_k = \cos\dfrac{\pi}{2(m+2)}$ when k is odd. Equality holds in (6) when $p(s,t) = U_m(s)$.

Let P_m be a Chebyshev polynomial of the second, third or fourth kind with degree m (see [5, Chapter 1]) and suppose there is a decreasing sequence $h_0, \ldots, h_m$ satisfying $P_m(h_n) = \pm 1$ when $0 \le n \le m$ and such that

$$|\hat{D}^k p(1,1)(1,-1)| \le P_m^{(k)}(1), \quad 0 \le k \le m,$$

for all $p \in \mathcal{P}_m(\mathbb{R}^2)$ satisfying $|p(x)| \le 1$ whenever $x \in \mathcal{N}_k$, where $\mathcal{N}_k$ is defined by (5). If $2 \le m \le 8$, then it can be verified by computer computation that $P_m = U_m$ and that if $m \ne 3$ then the terms of the sequence are given by (4). When $m=3$, the only other such sequence is $(\sqrt{5}+1)/4, 1/2, -1/2, -(\sqrt{5}+1)/4$.

As in the case of Theorem 1, we prove Theorem 3 by extending Rogosinski's Lagrange interpolation argument [8] to two dimensions. Lagrange polynomials for each of the sets $\mathcal{N}_0$ and $\mathcal{N}_1$ are given by

$$P_{n,q}(s,t) = c_n c_q G_m(s,t,h_n,h_q), \quad 0 \le n,q \le m,$$

where

$$c_n = \frac{2(1-h_n^2)}{m+2}, \quad G_m = K_{m-1} + K_m,$$

and

$$K_m(s,t,u,v) = \sum_{i=0}^{m} \sum_{j=0}^{i} U_{i-j}(s)U_j(t)U_{i-j}(u)U_j(v).$$

Note that K_m is the reproducing kernel for the space $\mathcal{P}_m(\mathbb{R}^2)$ with the weight function $w(s)w(t)$ on the square $[-1,1] \times [-1,1]$, where

$$w(t) = \frac{2}{\pi}\sqrt{1-t^2}. \tag{7}$$

(Compare [12, §2.3].)

Theorem 4. *Given $k = 0$ or $k = 1$, let $(n,q) \in Q_k$. Then $P_{n,q}(h_n,h_q) = 1$ and $P_{n,q}(x) = 0$ whenever $x \in \mathcal{N}_k$ and $x \neq (h_n,h_q)$.*

Corollary 5. *$(-1)^n \hat{D}^k P_{n,q}(r,r)(1,-1) \ge 0$ whenever $r \ge r_k$, $(n,q) \in Q_k$ and $k \ge 0$.*

To deduce Corollary 5, let $(n,q) \in Q_k$ and take $p(s,t) = U_m(s) - (-1)^n P_{n,q}(s,t)$. By Theorem 4, $|p(x)| \le 1$ whenever $x \in \mathcal{N}_k$ so

$$\hat{D}^k p(r,r)(1,-1) = U_m^{(k)}(r) - (-1)^n \hat{D}^k P_{n,q}(r,r)(1,-1) \le U_m^{(k)}(r)$$

for $r \ge r_k$ by Theorem 3. We shall see in Section 4 that Theorem 3 is a consequence of this special case.

Given $k \ge 0$, define

$$V_i(s,t) = U_{m-i}(s)U_i(t) - (-1)^k U_i(s)U_{m-i}(t), \quad i = 0,\ldots,m, \tag{8}$$

$$W_i(s,t) = U_{m-i+1}(s)U_i(t) - (-1)^k U_{i-1}(s)U_{m-i}(t), \quad i = 0,\ldots,m+1, \tag{9}$$

where $U_{-1} \equiv 0$. Each V_i is a polynomial of degree m (except when $V_i \equiv 0$) and each W_i is a polynomial of degree $m+1$. It is easy to verify that these polynomials vanish on the nodes $\mathcal{N}_k$ since

$$U_{m-i}(h_n) = (-1)^n U_i(h_n), \quad i, n = 0, \dots, m. \tag{10}$$

A Lagrange interpolation formula can be deduced from Theorem 4 as in [2, Theorem 5].

Theorem 6. *Let $k = 0$ or $k = 1$. If $p \in \mathcal{P}_m(\mathbb{R}^2)$ then*

$$p = \sum_{(n,q)\in Q_k} p(h_n, h_q) P_{n,q} + \overline{p}_k,$$

where $\overline{p}_k$ is a linear combination of the polynomials (8).

The following Christoffel-Darboux formula is a key ingredient of the proofs of Theorem 3 and Theorem 4:

$$\begin{aligned} &2(s-u)G_m(s,t,u,v) = \\ &\quad \sum_{i=0}^{m} [W_i(s,t)U_{m-i}(u)U_i(v) - W_i(u,v)U_{m-i}(s)U_i(t)] \\ &\quad + \sum_{i=0}^{m-1} [V_i(s,t)U_{m-i-1}(u)U_i(v) - V_i(u,v)U_{m-i-1}(s)U_i(t)]. \end{aligned} \tag{11}$$

It can be derived from the classical Christoffel-Darboux formula as in [3, p. 380]. A highly general formula of this type is given in [11, §4.2]. The following related result can be deduced as in [3, Corollary 9]:

$$G_m(s,t,u,v) = \frac{1}{s-u} \sum_{i=0}^{m} [T_{i+1}(s)U_i(u) - T_{i+1}(u)U_i(s)]U_{m-i}(t)U_{m-i}(v).$$

3. Proof of Theorem 4. Let $x_0 \in \mathcal{N}_k$ and put $p(s,t) = G_m(s,t,h_{n_0},h_{q_0})$, where $x_0 = (h_{n_0}, h_{q_0})$ and $(n_0, q_0) \in Q_k$. It follows from (11) that $p(x) = 0$ whenever $x \in \mathcal{N}_k$ and $x \neq x_0$. Thus the same is true for P_{n_0,q_0}.

One can obtain a direct proof that $P_{n_0,q_0}(x_0) = 1$ by summing the expression defining $G_m(x_0, x_0)$. However, for simplicity, we choose to prove this instead by appealing to a cubature formula for the Chebyshev weight. Define $\tilde{p}(s,t) = (1-s^2)(1-t^2)p(s,t)$. Clearly $\tilde{p} \in \mathcal{P}_{m+4}(\mathbb{R}^2)$ and $\tilde{p}(s,t) = 0$ when $s = 1$ or $t = 1$.

Applying [4, Corollary 2.3] with p replaced by $\tilde{p}$ and m replaced by $m+2$, we obtain

$$\int_{-1}^{1}\int_{-1}^{1} p(s,t)\frac{w(s)w(t)}{4}\,ds\,dt = \frac{2}{(m+2)^2}\sum_{(n,q)\in Q_k} \tilde{p}(h_n,h_q),$$

where w is as in (7). By the orthonormality of the polynomials $\{U_i\}$ with respect to w and the first part of our argument, this equality becomes

$$\frac{1}{2} = \frac{2}{(m+2)^2}(1-h_{n_0}^2)(1-h_{q_0}^2)G_m(x_0,x_0),$$

which proves that $P_{n_0,q_0}(x_0) = 1$. (Compare [6, p. 964].)

4. Proof of Theorem 3. Given m, k and r, define a linear functional ℓ_k on $\mathcal{P}_m(\mathbb{R}^2)$ by

$$\ell_k(p) = \hat{D}^k p(r,r)(1,-1) = \frac{d^k}{dt^k}p(r+t,r-t)\Big|_{t=0}.$$

If $0 \le i \le m$, it follows from the identity $V_i(t,s) = -(-1)^k V_i(s,t)$ that $\ell_k(V_i) = 0$. Hence by Theorem 6,

$$\ell_k(p) = \sum_{(n,q)\in Q_k} \ell_k(P_{n,q})p(h_n,h_q) \tag{12}$$

for all $p \in \mathcal{P}_m(\mathbb{R}^2)$. If $(-1)^n\ell_k(P_{n,q}) \ge 0$ for all $(n,q)\in Q_k$ and if p satisfies the hypotheses of Theorem 3, it follows from (12) and the triangle inequality that

$$|\ell_k(p)| \le \sum_{(n,q)\in Q_k} (-1)^n \ell_k(P_{n,q}) = \ell_k(U_m) = U_m^{(k)}(r).$$

Thus to prove Theorem 3 it suffices to give a direct proof of Corollary 5. Given $(n,q)\in Q_k$, define

$$F_{n,q}(s,t) = \sum_{i=0}^{m} W_i(s,t)\,U_i(h_n)U_i(h_q). \tag{13}$$

It follows from (11) as in [3, p. 377] that

$$(2r-h_n-h_q)(-1)^n\ell_k(G_{n,q}) = \ell_k(F_{n,q}),$$

where $G_{n,q}(s,t) = G_m(s,t,h_n,h_q)$. Hence it suffices to prove that $\ell_k(F_{n,q}) \ge 0$ for $r \ge r_k$. Unfortunately, there is no apparent simple reduction formula as in the case of Chebyshev polynomials of the first kind. (See (20) below.)

To obtain an explicit formula for $F_{n,q}$, we begin by expanding the sum on the right hand side of (13) as the difference of two sums, reversing the order of summation in the first sum and applying the identities (10) and $2T_n = U_n - U_{n-2}$, to obtain

$$F_{n,q}(s,t) = 2(-1)^k \sum_{i=1}^{m+1} T_i(s)U_{m-i+1}(t)U_{i-1}(h_n)U_{i-1}(h_q), \tag{14}$$

where we have changed the index i of summation to $i-1$. Next we reduce the number of factors in the terms of the right hand side of (14) so that it can be summed. It follows from the identity

$$2\sin\theta\,\sin\phi = \cos(\theta-\phi) - \cos(\theta+\phi)$$

that

$$2\sqrt{(1-h_n^2)(1-h_q^2)}\,U_{i-1}(h_n)U_{i-1}(h_q) = T_i(h_{n-q-1}) - T_i(h_{n+q+1}),$$

and it follows from the identity

$$2\cos\theta\,\cos\phi = \cos(\theta+\phi) + \cos(\theta-\phi)$$

that

$$2T_i(s)T_i(h_n) = T_i(x(n)) + T_i(y(n)),$$

where

$$x(n) = sh_n - \sqrt{(1-s^2)(1-h_n^2)}, \quad y(n) = sh_n + \sqrt{(1-s^2)(1-h_n^2)}.$$

In these equations, h_n is still given by (4) when n is out of range. Put

$$\phi(s,t) = \sum_{i=1}^{m+1} U_{m-i+1}(t)T_i(s).$$

and let

$$n_1 = n-q-1, \quad n_2 = n+q+1.$$

Then

$$F_{n,q}(s,t) = (-1)^k \frac{\phi(x(n_1),t) + \phi(y(n_1),t) - \phi(x(n_2),t) - \phi(y(n_2),t)}{2\sqrt{(1-h_n^2)(1-h_q^2)}}. \tag{15}$$

By the Darboux-Christoffel method (see [3, p. 379]),

$$2\phi(s,t) = \frac{T_{m+2}(s) - T_{m+2}(t)}{s-t} - U_{m+1}(t).$$

Since

$$T_{m+2}(x(n)) = T_{m+2}(y(n)) = (-1)^{n+1} T_{m+2}(s),$$

we have

$$2[\phi(x(n),t) + \phi(y(n),t)] = X[(-1)^{n+1} T_{m+2}(s) - T_{m+2}(t)] - 2U_{m+2}(t), \quad (16)$$

where

$$X = \frac{1}{x(n)-t} + \frac{1}{y(n)-t}.$$

Since $x(n) + y(n) = 2sh_n$ and $x(n)\, y(n) = s^2 + h_n^2 - 1$, it follows that

$$X = 2\frac{sh_n - t}{D(s,t,h_n)}, \quad (17)$$

where

$$D(s,t,u) = s^2 + t^2 + u^2 - 2stu - 1.$$

Thus by (15), (16) and (17),

$$F_{n,q}(s,t) = \left[\frac{sh_{n_1} - t}{D(s,t,h_{n_1})} - \frac{sh_{n_2} - t}{D(s,t,h_{n_2})}\right] \frac{T_{m+2}(s) - (-1)^k T_{m+2}(t)}{2\sqrt{(1-h_n^2)(1-h_q^2)}}. \quad (18)$$

To obtain a function that is easier to factor, define

$$H_{n,q}(s,t) = F_{n,q}(s,t) + (-1)^k F_{n,q}(t,s)$$

and note that $\ell_k(H_{n,q}) = 2\ell_k(F_{n,q})$. Then applying (18) and the identities

$$h_{n_1} - h_{n_2} = 2\sqrt{(1-h_n^2)(1-h_q^2)}, \quad (1+h_{n_1})(1+h_{n_2}) = (h_n + h_q)^2,$$

we obtain

$$\begin{aligned} H_{n,q}(s,t) &= (s-t)\left[\frac{1+h_{n_1}}{D(s,t,h_{n_1})} - \frac{1+h_{n_2}}{D(s,t,h_{n_2})}\right] \frac{T_{m+2}(s) - (-1)^k T_{m+2}(t)}{2\sqrt{(1-h_n^2)(1-h_q^2)}} \\ &= \frac{(s-t)[(s+t)^2 - (h_n+h_q)^2]}{D(s,t,h_{n_1}) D(s,t,h_{n_2})} [T_{m+2}(s) - (-1)^k T_{m+2}(t)] \end{aligned}$$

It follows as in [3, Lemma 5] that $H_{n,q}(s,t)$ is 2^{m+1} times a product of factors $f(s,t)$ from the following list:

a) $f(s,t) = D(s,t,h_j)$, $0 \le j \le m$, $j+1 = k \bmod 2$,

b) $f(s,t) = s - t$,

c) $f(s,t) = s + t$,

d) $f(s,t) = s + t + h_n + h_q$,

e) $f(s,t) = s + t - h_n - h_q$.

Now by the Leibnitz rule for differentiation of products, it suffices to show that $\ell_i(f) \ge 0$ for each factor f in (a)-(e) when $r \ge r_k$ and $i \ge 0$. If f and j are as in (a), then $f(r,r) = (2r^2 - 1 - h_j)(1 - h_j)$ and

$$1 + h_j = 2\cos^2 \frac{(j+1)\pi}{2(m+2)}. \tag{19}$$

Hence $\ell_0(f) \ge 0$ when $r \ge r_k$. The remaining inequalities are easily verified under the less restrictive condition $r \ge h_0$. This completes the proof.

5. Case $r < 1$ in Theorem 1. The purpose of this section is to show that when k is odd the inequality (2) in Theorem 1 holds more generally for

$$r \ge R := \cos \frac{\pi}{2m} .$$

We also show that if k is even and $k < m$ then (2) does not hold for any r with $z_k < r < 1$, where z_k is the largest zero of $T_m^{(k+1)}$ when $k < m-1$ and $z_{m-1} = 0$.

Let k be odd and let $(n,q) \in Q_k$. Put

$$\alpha_{n,q}^{(k)}(r) = (-1)^n \ell_k(G_{n,q}),$$

where $G_{n,q}$ is as defined in [3, p. 376] with $h_n = \cos(n\pi/m)$. We may suppose that $q \le n$ since $\alpha_{n,q}^{(k)}(r) = \alpha_{q,n}^{(k)}(r)$ by [2, p. 354]. Then $1 \le n - q \le m$ and $1 \le n + q < 2m$. Hence all of the coefficients in the identity,

$$2(2r-h_n-h_q)\alpha_{n,q}^{(k)}(r) = (2r-h_{n+q}-1)\alpha_{n+q,0}^{(k)}(r)+(2r-h_{n-q}-1)\alpha_{n-q,0}^{(k)}(r), \tag{20}$$

are positive when $r > R$. The arguments given in [3, Lemma 5] and at the end of the proof of Theorem 3 show that $\alpha_{j,0}^{(k)}(r) \ge 0$ when $r \ge R$ and j is odd. Thus $\alpha_{n,q}^{(k)}(r) \ge 0$ when $r \ge R$ and this implies the asserted extension of Theorem 1 by the argument in [3].

To see that (2) does not hold when k is even and $z_k < r < 1$, let P be the polynomial given by

$$P(s,t) = \frac{1}{4m^2}(s+t-2)\frac{T_m(s) - T_m(t)}{s-t}, \quad s \ne t.$$

If $n = q \bmod 2$, then

$$P\left(\cos\frac{n\pi}{m}, \cos\frac{q\pi}{m}\right) = \begin{cases} 0 & \text{when}(n,q) \neq (m,m) \\ (-1)^m & \text{when}(n,q) = (m,m) \end{cases}.$$

Hence $p(s,t) = T_m(s) - 2P(s,t)$ satisfies the hypotheses of Theorem 1 but

$$\hat{D}^k p(r,r)(1,-1) = T_m^{(k)}(r) + \frac{1-r}{(k+1)m^2} T_m^{(k+1)}(r) > T_m^{(k)}(r)$$

by [2, p. 356].

A stronger result for the case of polynomials of a single variable is given in [7, 1.5.10].

References

[1] B. Bojanov and G. Nikolov, *Duffin and Schaeffer type inequality for Ultraspherical polynomials*, J. Approx Theory **84** (1996), 129–138.

[2] L. A. Harris, *Multivariate Markov polynomial inequalities and Chebyshev nodes*, J. Math. Anal. Appl. **338** (2008), 350–357.

[3] ———, *A proof of Markov's theorem for polynomials on Banach spaces*, J. Math. Anal. Appl. **368** (2010), 374–381.

[4] ———, *Bivariate Lagrange interpolation at the Chebyshev nodes*, Proc. Amer. Math. Soc. **138** (2010), 4447–4453.

[5] J. C. Mason and D. C. Handscomb, *Chebyshev Polynomials*, Chapman & Hall/CRC, Boca Raton, 2003.

[6] C. R. Morrow and T. N. L. Patterson, *Construction of algebraic cubature rules using polynomial ideal theory*, SIAM J. Numer. Anal. **15** (1978), 953–976.

[7] T. J. Rivlin, *Chebyshev Polynomials*, Second edition, John Wiley & Sons, Inc., New York, 1990.

[8] W. W. Rogosinski, *Some elementary inequalities for polynomials*, Math. Gaz. **39** (1955), 7–12.

[9] V. I. Skalyga, *Bounds for the derivatives of polynomials on centrally symmetric convex bodies*, Izv. Math. **69** (2005), no. 3, 607–621.

[10] ________, *Analogue of A. A. Markov's inequality for polynomials in two variables*, Sb. Math. **199** (2008), 1409-1420.

[11] Yuan Xu, *Common Zeros of Polynomials in Several Variables and Higher-dimensional Quadrature*, Pitman research notes in mathematics, Longman, Essex 1994.

[12] ________, *Lagrange interpolation on Chebyshev points of two variables*, J. Approx. Theory **87** (1996), 220–238.