A Generalized Sewing Construction for Polytopes

Carl W. Lee, University of Kentucky
Matthew M. Menzel, Marietta College

Two major combinatorial problems are to characterize the f-vectors and flag f-vectors of convex d-polytopes. For 3-polytopes these problems were solved by Steinitz [24, 25] nearly a century ago. They also were solved for the class of simplicial polytopes by Stanley [23] and Billera and Lee [10] more than 25 years ago. For $d \geq 4$, however, the problems of characterizing the f-vectors and flag f-vectors of general d-polytopes are unresolved. Several linear and non-linear inequalities for flag f-vectors of d-polytopes have been established, but in order to confirm whether a set of conditions is sufficient for describing the flag f-vectors of d-polytopes it is necessary to develop new methods for constructing classes of nonsimplicial polytopes. This paper will focus on a new technique for constructing d-polytopes, which is a generalization of Shemer's sewing construction for simplicial neighborly polytopes [22], and which has been modified to allow the construction of nonsimplicial polytopes as well. One motivation for this construction is that the ordinary polytopes of Bisztriczky [11, 13], a nice generalization of cyclic polytopes, can be constructed by generalized sewing. We also will construct several infinite families of polytopes in this manner, including one whose g-vectors satisfy the relation $g_2 = 0$, and we will consider bounds on the flag f-vectors of 4-polytopes that can be inductively constructed when beginning with the 4-simplex.

1 Introduction

The basic terms and ideas in this paper can be found in many standard sources on convex polytopes such as Ziegler [26], Grünbaum [18], or Bayer and Lee [8].

A *convex polytope*, or polytope for short, is defined to be the convex hull of a finite number of points in Euclidean space. The *dimension* of a polytope is one less than the maximum number of affinely independent points contained therein, and a polytope of dimension d often is referred to as a d-polytope. Given a d-polytope P, the *f-vector* of P is defined by $f(P) := (f_0(P), f_1(P), \ldots, f_{d-1}(P))$, where $f_j(P)$ is the number of j-dimensional faces of P. Faces of dimensions 0, 1, and $d-1$ are often referred to as *vertices*, *edges*, and *facets*, respectively.

In order to obtain a more complete description of the combinatorial structure of an arbitrary d-polytope, attention has been focused on both the numbers of faces of all possible dimensions and the numbers of chains of faces of the polytope. A *flag* is a strictly increasing sequence of faces $T_1 \subset T_2 \subset \cdots \subset T_q$. Given a set $S \subseteq \{0, \ldots, d-1\}$, an S-*flag* is a flag $\{T_j\}_{j=1}^q$ for which $S = \{\dim(T_j) : j = 1, \ldots, q\}$. The *flag f-vector* of a d-polytope P is defined by

$$f_S = |\{\{T_j\}_{j=1}^q : \{T_j\}_{j=1}^q \text{ is an } S\text{-flag of } P\}|,$$

where S ranges over all subsets of $\{1, \ldots, d-1\}$.

Let $\mathbf{a}$ and $\mathbf{b}$ be noncommuting indeterminates. For $S \subseteq \{0, \ldots, d-1\}$, define $w_S = w_0 \cdots w_{d-1}$, where $w_i = \mathbf{a} - \mathbf{b}$ if $i \notin S$, and $w_i = \mathbf{b}$ if $i \in S$. The **ab**-*index* of P is then

$$\Psi(P) = \sum_S f_S w_S,$$

where the sum is taken over all subsets $S \subseteq \{0, \ldots, d-1\}$. Bayer-Klapper [7] proved that the **ab**-index can be written as a polynomial in the indeterminates $\mathbf{c} = \mathbf{a}+\mathbf{b}$ and $\mathbf{d} = \mathbf{ab}+\mathbf{ba}$. In this form, $\Psi(P)$ is known as the **cd**-*index*, which is known to concisely capture a basis for the set of **ab**-indices of convex d-polytopes.

2 4-Polytopes

For $d \geq 4$, the problem of characterizing flag f-vectors, and equivalently *cd*-polynomials [7], of general d-polytopes is unresolved. Bayer [2] and Ziegler and Höppner [19] provide overviews of what is currently known in the $d = 4$ case.

The generalized Dehn-Sommerville equations [5] imply that the dimension of the affine span of the flag vectors of 4-polytopes is four, and hence any four linearly independent components of the flag f-vector of a 4-polytope determine the remaining components. We will use the components (f_0, f_1, f_2, f_{02}), and we henceforth will refer to such a 4-tuple as the flag f-vector of a 4-polytope.

Bayer observed that the set of all flag f-vectors of 4-polytopes is not the intersection of the integer lattice with a convex set, nor is its convex hull closed.

Theorem 2.1 (Bayer) *If* $(f_0, f_1, f_2, f_{02}) = (f_0(P), f_1(P), f_2(P), f_{02}(P))$ *for some 4-polytope* P*, then*

1. $f_{02} - 3f_2 \geq 0$
2. $f_{02} - 3f_1 \geq 0$
3. $f_{02} - 3f_2 + f_1 - 4f_0 + 10 \geq 0$
4. $6f_1 - 6f_0 - f_{02} \geq 0$
5. $f_0 - 5 \geq 0$
6. $f_2 - f_1 + f_0 - 5 \geq 0$

The closed convex set $\mathcal{N}$ as determined by the known linear inequalities listed in Theorem 2.1 is a 4-dimensional cone with apex $(5, 10, 10, 30)$, the flag f-vector of the 4-simplex. If we let $\mathcal{M} \subset \mathbb{R}^4$ denote the convex hull of flag vectors $(f_0(P), f_1(P), f_2(P), f_{02}(P))$, where P ranges over all 4-polytopes, then the cone $\mathcal{N}$ has six facets and seven extreme rays and contains $\mathcal{M}$.

Ziegler and Höppner [19] enumerated the 4-tuples with $f_0 \leq 8$ that satisfy the known linear and quadratic inequalities but are not the flag f-vectors of any 4-polytope.

3 Sewing and A-Sewing

We say that a point $x \in \mathbb{R}^d$ outside a d-polytope P is *beneath* a facet F of P provided that x belongs to the open half-space determined by the supporting hyperplane $\operatorname{aff} F$ (the affine span of F) and containing $\operatorname{int} P$. We say that x is *beyond* F if x belongs to the open half-space determined by $\operatorname{aff} F$ that does not contain $\operatorname{int} P$. If x is an element of $\operatorname{aff} F$, we say that x is *on* F.

Given a d-polytope P and a point x, we can construct a new d-polytope, $Q := [P, x]$, the convex hull of P together with the point x. The following theorem of Grünbaum, as formulated by Altshuler and Shemer [1], describes the facial structure of Q.

Theorem 3.1 (Grünbaum) *Let $P \subset \mathbb{R}^d$ be a d-polytope, and let $x \in \mathbb{R}^d$ be a point outside P. Let $\mathcal{A}$, $\mathcal{B}$, $\mathcal{C}$ be the partition of the facets $\mathcal{F}$ of P such that x lies in the affine hull of every $A \in \mathcal{A}$, beyond every $B \in \mathcal{B}$, and beneath every $C \in \mathcal{C}$. Define three types of sets G:*

(i) G is a face of a member of $\mathcal{C}$.

(ii) $G = \operatorname{conv}(F \cup \{x\})$, where F is the intersection of a subset of $\mathcal{A}$ (or, equivalently, F is a face of P and $x \in \operatorname{aff} F$). ($\cap\varnothing = P$.)

(iii) $G = \operatorname{conv}(F \cup \{x\})$, where F is a face of a member of $\mathcal{B}$ and also a face of a member of $\mathcal{C}$.

Then the sets of types (i), (ii), and (iii) are faces of $Q := [P, x]$, and each face of Q is of exactly one of the above types.

This theorem provides a mechanism for creating new polytopes and new classes of polytopes by choosing x to be on, beyond, and beneath certain collections of facets of P. We also can reverse the process by partitioning the facets $\mathcal{F}$ of P into $\mathcal{A}, \mathcal{B}, \mathcal{C}$ and trying to determine whether or not there exists an actual point that would yield the same partition. To address the latter question Altshuler and Shemer [1] defined a pair $\mathcal{B} \mid \mathcal{A}$ to be *coverable* if there exists a point $x \in \mathbb{R}^d$ such that $x \in \bigcap_{F \in \mathcal{A}} \operatorname{aff} F$; x lies beyond all members of $\mathcal{B}$; and x lies beneath all members of $\mathcal{C} =: \mathcal{F} \setminus (\mathcal{A} \cup \mathcal{B})$. We say that the point x *covers* $\mathcal{B} \mid \mathcal{A}$, and if x covers $\mathcal{B} \mid \varnothing$ then we say that x lies *exactly beyond* $\mathcal{B}$.

Although Shemer [22] used the term *tower* to refer to a particular type of flag, his results show that any flag determines a certain partition of $\mathcal{F}$ that is always coverable. Given a flag $\mathcal{T} = \{T_j\}_{j=1}^q$, we let $\mathcal{F}_j := \{F \in \mathcal{F} : T_j \subseteq F\}$.

Proposition 3.2 (Shemer) *Let $\mathcal{T} = \{T_j\}_{j=1}^q$ be a flag of a d-polytope P, and let $\mathcal{B} = \mathcal{B}(P, \mathcal{T}) := \mathcal{F}_1 \setminus (\mathcal{F}_2 \setminus (\cdots \setminus \mathcal{F}_q) \cdots)$. Then there is a point $x \in \mathbb{R}^d$ that lies exactly beyond $\mathcal{B}$.*

Proof: Inducting on q, we define $\mathcal{B} := \varnothing$ for $q = 0$, and observe that every point $x \in \operatorname{int} P$ lies exactly beyond $\mathcal{B}$. If $q \geq 1$, we let $\mathcal{T}' := \mathcal{T} \setminus T_1$ and $\mathcal{B}' := \mathcal{B}(P, \mathcal{T}')$. The induction hypothesis guarantees the existence of a point $x' \in \mathbb{R}^d$, which lies exactly beyond $\mathcal{B}'$. Observing that $\mathcal{B}' \subset \mathcal{F}_1$ and $\mathcal{B} = \mathcal{F}_1 \setminus \mathcal{B}'$, we choose a point $p \in \operatorname{relint} F_1$, and let $x := (1 + \epsilon)p - \epsilon x'$. For a sufficiently small and positive ϵ, x lies exactly beyond $\mathcal{B}$. $\square$

We will say that the point x provided by Proposition 3.2 is *exactly beyond* $\mathcal{T}$. We also observe that

$$\mathcal{F}_1 \backslash (\mathcal{F}_2 \backslash (\cdots \backslash \mathcal{F}_q) \cdots) = (\mathcal{F}_1 \setminus \mathcal{F}_2) \cup (\mathcal{F}_3 \setminus \mathcal{F}_4) \cup \cdots ,$$

where the last term in the union is $\mathcal{F}_{q-1} \setminus \mathcal{F}_q$ if q is even and $\mathcal{F}_q$ if q is odd. We hence can view the aforementioned process as choosing x to be beyond all facets of P that contain T_1, except beneath those that contain T_2, except beyond those that contain $T_3, \ldots$, except beyond/beneath those that contain T_q.

Shemer [22] developed a construction called *sewing a vertex onto a polytope*, or sewing for short, which when applied to a neighborly $2m$-polytope, yields a neighborly $2m$-polytope with one more vertex. He began with a cyclic $2m$-polytope and sequentially "sewed" on new vertices that were chosen to be exactly beyond specific flags of length m. We will generalize Shemer's concept of sewing to include choosing the new point x to be exactly beyond an arbitrary flag. To simplify the wording, we often will say that we are sewing a vertex x onto a polytope P over a flag $\mathcal{T}$.

Figure 1 illustrates sewing a new point x onto a pentagonal prism P over the flag $\{a\} \subset [a, b] \subset [a, b, c, d, e]$ (pictured on the left). The point u is in the interior of the prism. We sew outward through a point in the relative interior of the pentagon $[a, b, c, d, e]$ to arrive at the point v, which is beyond precisely the top pentagon. Then we sew through a point in the relative interior of the edge $[a, b]$ to arrive at the point w, which is now beneath the top pentagon, beyond the right-front rectangle, and beneath the remaining facets. Finally we sew through the vertex a to arrive at the point x, which is now beneath the right-front rectangle, beyond the top pentagon and the left-front rectangle, and beneath the remaining facets. The polytope pictured on the right is then $Q := [P, x]$.

All new facets, and hence all new proper faces, obtained by sewing a new vertex onto a simplicial polytope must be simplices, as any *new* k-face is the convex hull of a $(k-1)$-simplex and a point outside of its affine span. If we wish to create a non-simplicial polytope by adding a new vertex to a given polytope, we must either begin with a non-simplicial polytope or modify the sewing process so that $\mathcal{A}$ is non-empty. We will do the latter by creating a process that we will refer to as *A-sewing a vertex onto a polytope*, or A-sewing. We again choose a flag $\mathcal{T} = \{T_j\}_{j=1}^q$, but we now choose x to be in

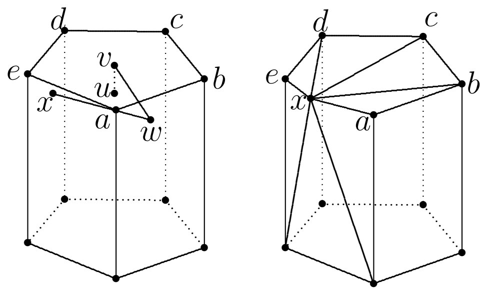

Figure 1: Sewing a point x onto a pentagonal prism

the affine span of T_q and exactly beyond $\mathcal{T}' = \{T_j\}_{j=1}^{q-1}$. The "$A$" in A-sewing is chosen to represent the fact that $\mathcal{A} \neq \varnothing$.

Proposition 3.3 *Let P be a d-polytope, and let $\mathcal{T} = \{T_j\}_{j=1}^{q}$ be a flag of P. We partition the facets $\mathcal{F}$ of P into $\mathcal{A}, \mathcal{B}, \mathcal{C}$, where*

- $\mathcal{A} := \mathcal{F}_q$
- $\mathcal{B} := \mathcal{F}_1 \backslash (\mathcal{F}_2 \backslash (\mathcal{F}_3 \backslash (\cdots \backslash \mathcal{F}_{q-1}) \cdots)) \backslash \mathcal{F}_q$
  $= ((\mathcal{F}_1 \setminus \mathcal{F}_2) \cup (\mathcal{F}_3 \setminus \mathcal{F}_4) \cup \cdots) \setminus \mathcal{F}_q$
- $\mathcal{C} := \mathcal{F} \backslash (\mathcal{A} \ \cup \ \mathcal{B})$

Then, $\mathcal{B} \mid \mathcal{A}$ and $\mathcal{B} \mid \varnothing$ are coverable.

Proof: To prove that $\mathcal{B} \mid \mathcal{A}$ is coverable, we consider the polytope T_q sitting in the ambient space $\operatorname{aff} T_q$. We make the following assignments:

- $\mathcal{F}' := \{F' : F' \text{ is a facet of } T_q\}$,
- $\mathcal{T}' := \{T_j\}_{j=1}^{q-1}$, (note that $T_j \subset T_q$ for $j = 1, \ldots, q-1$), and

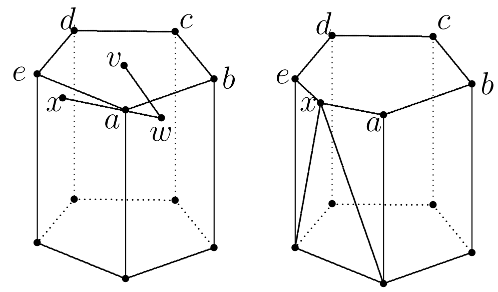

Figure 2: A-sewing a point x onto a pentagonal prism

- $\mathcal{F}'_j := \{F' \in \mathcal{F}' : T_j \subseteq F'\}$ for $j = 1, \ldots, q-1$.

By Proposition 3.2, there exists a point $x \in \operatorname{aff} T_q$ that covers $\mathcal{B}' \mid \varnothing$, where $\mathcal{B}' := \mathcal{B}(T_q, \mathcal{T}') = \mathcal{F}'_1 \backslash (\mathcal{F}'_2 \backslash (\cdots \backslash \mathcal{F}'_{q-1}) \cdots)$. By construction, this point x covers $\mathcal{B} \mid \mathcal{A}$.

We now will verify that $\mathcal{B} \mid \varnothing$ is coverable. If q is odd, then we define $\mathcal{T}' := \{T_j\}_{j=1}^{q-1}$, and if q is even we define $\mathcal{T}' := \mathcal{T}$. In either case, Proposition 3.2 implies that $\mathcal{B} \mid \varnothing = \mathcal{B}(P, \mathcal{T}') \mid \varnothing$ is coverable. □

We will say that the point x that covers $\mathcal{B} \mid \mathcal{A}$ in Proposition 3.3 is *almost exactly beyond* the flag $\mathcal{T}$. Figure 2 illustrates A-sewing a new point x onto a pentagonal prism P over the flag $\{a\} \subset [a, b] \subset [a, b, c, d, e]$ (pictured on the left). We begin with a point v in the relative interior of the pentagon $[a, b, c, d, e]$ (and beneath the remaining facets). Then we sew through a point in the relative interior of the edge $[a, b]$ to arrive at the point w, which is in the affine span of the top pentagon, beyond the right-front rectangle and beneath the remaining facets. Finally we sew through the vertex a to arrive at the point x, which is now beneath the right-front rectangle, beyond the left-front rectangle, still in the affine span of the top pentagon, and beneath the remaining facets. The polytope pictured on the right is then $Q := [P, x]$.

We also note that when A-sewing a point x onto P over $\mathcal{T} = T_1$, we have $Q := [P, x] = P$. Hence, when A-sewing we will assume that the flag length is at least 2.

3.1 Sewing and A-sewing 3-polytopes

We define $\mathcal{P}^d_\sigma$ to be the set of all d-polytopes that can be obtained by performing a sequence of sewing and A-sewing operations starting with the d-simplex. We hence will let $f(\mathcal{P}^d_\sigma)$ denote the set of all f-vectors of polytopes in $\mathcal{P}^d_\sigma$. The following theorem characterizes the f-vectors in $f(\mathcal{P}^3_\sigma)$.

Theorem 3.4 *A vector (f_0, f_1, f_2) of nonnegative integers is the f-vector of a 3-polytope that can be obtained by performing a sequence of sewing and A-sewing operations beginning with the 3-simplex if and only if the following conditions hold.*

(i) $f_1 = f_0 + f_2 - 2$.

(ii) $f_2 \leq 2f_0 - 4$.

(iii) $f_0 \leq f_2$.

Proof: Steinitz' Theorem implies that $f(P)$ must satisfy conditions (i) and (ii) for any polytope $P \in \mathcal{P}^3_\sigma$.

We will verify (iii) by induction on $f_0 \geq 4$. The 3-simplex establishes our basis. We choose $P \in \mathcal{P}^3_\sigma$ and inductively assume that

$$f_0(P) \leq f_2(P).$$

We let Q be the 3-polytope obtained by sewing or A-sewing x onto P over $\mathcal{T} = \{T_j\}_{j=1}^q$, and we define $\Delta f_i = f_i(Q) - f_i(P)$. It is sufficient to verify that $\Delta f_0 \leq \Delta f_2$. We observe that x is the only new vertex, and T_1 is the only possibility for a vertex that is destroyed by the construction. It follows that $\Delta f_0 \in \{0, 1\}$. We will consider two cases:

- Case 1: $\Delta f_0 = 0$. In this case, we must either be sewing over a $\{0\}$-flag or A-sewing over a $\{0, p\}$-flag, where $p \in \{1, 2\}$. In either case, any destroyed facet must contain T_1. Any such facet must have at least one edge, e, that does not contain T_1 and hence also is contained in a facet in $\mathcal{C}$. This edge e will correspond to a new facet, $[e, x]$, of Q and it follows that
$$\Delta f_2 \geq 0 = \Delta f_0.$$

- Case 2: $\Delta f_0 = 1$. The above argument implies that $\Delta f_2 \geq 0$, as every destroyed facet still corresponds to at least one facet of Q that was not a facet of P. If T_1 is a vertex, then the assumption that $\Delta f_0 = 1$ implies that T_1 is not destroyed, and hence T_1 must be contained in both a facet in $\mathcal{B}$ and a facet in $\mathcal{C}$. It follows that there must be at least one edge, e, that contains T_1 and also is contained in both of facet in $\mathcal{B}$ and one in $\mathcal{C}$. Any such new edge will create a new facet, $[e, x]$, of Q, and it follows that $\Delta f_2 \geq 1$.

  If T_1 is an edge or a facet, however, then any destroyed facet must have at least two edges that do not contain T_1 and hence also are contained in a facet in $\mathcal{C}$. Each destroyed facet, of which there must be at least one, consequently corresponds a minimum of two new facets, and it follows that
$$\Delta f_2 \geq 1 = \Delta f_0.$$

In either case, we may conclude that $f_0(P) \leq f_2(P)$ for all polytopes $P \in \mathcal{P}^3_\sigma$.

We now must verify that every integer vector (f_0, f_1, f_2) satisfying conditions $(i) - (iii)$ belongs to $f(\mathcal{P}^3_\sigma)$. We first observe that we can construct a pyramid over an n-gon from a pyramid over an $(n-1)$-gon by A-sewing over a $\{1,2\}$-flag $\mathcal{T} = T_1 \subset T_2$, where T_1 is any edge of the $(n-1)$-gon T_2. It follows that we can obtain a pyramid over an n-gon ($n \geq 3$), with f-vector $(n+1, 2n, n+1)$ by performing a sequence of $n-3$ A-sewing operations starting with the 3-simplex.

A traditional technique for constructing 3-polytopes whose f-vectors lie in Steinitz' cone and satisfy both Euler's relation and the inequality $f_0 < f_2$ involves sequentially making shallow pyramids over triangular faces, beginning with a pyramid over an n-gon. Such f-vectors belong to $f(\mathcal{P}^3_\sigma)$, as making a shallow pyramid over a triangle T is equivalent to sewing over the flag $\mathcal{T} = T$. □

We observe that the f-vectors of polytopes in $\mathcal{P}^3_\sigma$ correspond to "half" of the f-vectors of all 3-polytopes, and we can obtain f-vectors for the other half by considering polytopes that are dual to those in $\mathcal{P}^3_\sigma$. Although Theorem 3.4 characterizes the f-vectors of polytopes in $\mathcal{P}^3_\sigma$, there exist 3-polytopes that do not belong to $\mathcal{P}^3_\sigma$, although their f-vectors lie in $f(\mathcal{P}^3_\sigma)$. An open problem hence would be to determine a set of necessary and sufficient properties that a polytope in $\mathcal{P}^3_\sigma$ (or $\mathcal{P}^d_\sigma$) must satisfy.

3.2 Sewing/A-sewing and proper faces

We now will consider the proper faces of a d-polytope in $\mathcal{P}^d_\sigma$.

Remark 3.5 *If P_2 is the polytope obtained by sewing x onto the d-polytope P_1 over $\mathcal{T} = \{T_j\}_{j=1}^q$, then* $\mathrm{Pyr}\,(P_2)$ *is combinatorially equivalent to the polytope obtained by A-sewing x onto* $\mathrm{Pyr}\,(P_1)$ *over $\mathcal{T}' := T_1 \subset \cdots \subset T_q \subset P_1$.*

Remark 3.6 *If P_2 is obtained by A-sewing x onto P_1 over $\mathcal{T} = \{T_j\}_{j=1}^q$, then* $\mathrm{Pyr}\,(P_2)$ *is combinatorially equivalent to the polytope obtained by A-sewing x onto* $\mathrm{Pyr}\,(P_1)$ *over $\mathcal{T}$.*

The following theorem demonstrates that the property of being sewn/A-sewn is inherited by the proper faces of a polytope.

Theorem 3.7 *Let P be a d-polytope in $\mathcal{P}^d_\sigma$, and let F be a proper k-face of P. Then $F \in \mathcal{P}^k_\sigma$.*

Proof: We will prove this by induction on the number of sewing/A-sewing operations, ℓ, that are performed in sequence starting with the d-simplex. If P_1 is the d-polytope obtained by sewing or A-sewing x_1 onto the d-simplex, then any k-face that remains unchanged by the construction trivially satisfies the desired property. Any new k-face of P_1 is of the form $[S, x_1] = \mathrm{Pyr}\,(S)$, where S is a $(k-1)$-face of the d-simplex, and hence all new k-faces of P_1 are k-simplices. It remains only to consider k-faces of P_1 that are of the form $F = [G, x_1]$, where G is a k-simplex that is the intersection of facets belonging to $\mathcal{A}$. In this case, we must be A-sewing, and G must contain T_q. It follows that F is combinatorially equivalent to the polytope obtained by A-sewing x_1 onto the k-simplex G over $\mathcal{T}$.

We inductively assume that the desired result holds for any k-face of a d-polytope P_ℓ that is obtained by performing a sequence of ℓ sewing/A-sewing operations starting with the d-simplex, and we let $P_{\ell+1}$ be the polytope obtained by sewing or A-sewing $x_{\ell+1}$ onto P_ℓ over $\mathcal{T} = \{T_j\}_{j=1}^q$. Any k-face of $P_{\ell+1}$ that was a face of P_ℓ and remained unchanged by the sewing/A-sewing belongs to $\mathcal{P}^k_\sigma$ by the induction hypothesis. For k-faces of $P_{\ell+1}$ that are of the form $F = [S, x_{\ell+1}] = \mathrm{Pyr}\,(S)$, where S is a $(k-1)$-face of P_ℓ, the induction hypothesis hence implies that $S \in \mathcal{P}^k_\sigma$. As stated in Remark 3.5, the sequence of sewing and A-sewing operations used to construct S starting with the $(k-1)$-simplex corresponds to a sequence of sewing/A-sewing operations that will construct $F = \mathrm{Pyr}\,(S)$ when starting with the k-simplex.

It remains only to consider k-faces of $P_{\ell+1}$ that are of the form $F = [G, x]$, where G is a k-face of P_ℓ that is the intersection of facets contained in $\mathcal{A}$. In this case, we must be A-sewing, and G must contain T_q. As stated in Remark 3.6, it follows that F is combinatorially equivalent to the polytope obtained by A-sewing $x_{\ell+1}$ onto G over $\mathcal{T}$. Since the induction hypothesis assumes that G can be obtained by performing a sequence of sewing and A-sewing operations starting with the k-simplex, the desired result follows. □

4 Sewing and A-Sewing 4-Polytopes

The following lemmas and theorem investigate which vectors (f_0, f_1, f_2, f_{02}) belong to $f(\mathcal{P}^4_\sigma)$. The inequalities contained therein arose by using Komei Fukuda's *cdd* program [16] to determine the equations for the hyperplanes bounding the region of flag f-vectors of known polytopes in $\mathcal{P}^4_\sigma$.

4.1 Inequalities for Sewn and A-Sewn 4-Polytopes

Lemma 4.1 *Any flag f-vector (f_0, f_1, f_2, f_{02}) in $f(\mathcal{P}^4_\sigma)$ must satisfy the linear inequality*

$$-2f_0 + 2f_1 + 2f_2 - f_{02} \geq 0.$$

Proof: The Generalized Dehn-Sommerville Equations for 4-polytopes imply that $f_{03} = f_{02} + 2f_0 - 2f_1$ and $f_{23} = 2f_2$. The desired inequality hence is equivalent to

$$f_{23} - f_{03} \geq 0.$$

It is sufficient to show that $f_2(F) - f_0(F) \geq 0$ for every facet, F, of a polytope $P \in \mathcal{P}^4_\sigma$. Theorem 3.7 implies that $F \in \mathcal{P}^3_\sigma$, and Theorem 3.4 consequently implies that

$$f_2(F) - f_0(F) \geq 0.$$

□

Theorem 4.2 *Any flag f-vector (f_0, f_1, f_2, f_{02}) in $f(\mathcal{P}^4_\sigma)$ must satisfy the linear inequality*

$$3f_0 - 2f_1 + f_2 \geq 5.$$

Proof: Euler's relation for 4-polytopes implies that the desired inequality is equivalent to

$$f_1 - 2f_2 + 3f_3 \geq 5,$$

which clearly is satisfied by the f-vector of the 4-simplex.

We let P be an arbitrary 4-polytope in $\mathcal{P}^4_\sigma$; we let $Q := [P, x]$ be the 4-polytope obtained by sewing or A-sewing x onto P over $\mathcal{T} = \{T_j\}_{j=1}^q$; and we define $\Delta f_j := f_j(Q) - f_j(P)$ for $j \in \{0,1,2,3\}$. It hence is sufficient to verify that

$$\Delta f_1 - 2\Delta f_2 + 3\Delta f_3 \geq 0.$$

We let $\mathcal{A}, \mathcal{B}, \mathcal{C}$ be the partition of the facets $\mathcal{F}$ of P determined by sewing/A-sewing x onto P over $\mathcal{T}$, and we let $\mathcal{K}(\mathcal{A}), \mathcal{K}(\mathcal{B}), \mathcal{K}(\mathcal{C}), \mathcal{K}(\mathcal{F})$ be the polytopal complexes that arise from the respective collections of facets. We make the following assignments with respect to $f(P)$.

- Let f_j^b denote the number of j-faces that are contained in $\mathcal{K}(\mathcal{B}) \cap \mathcal{K}(\mathcal{C})$. These are faces that lie on the **b**oundaries of both $\mathcal{K}(\mathcal{B})$ and $\mathcal{K}(\mathcal{C})$, and each will be joined to x to create a $(j+1)$-face of Q.

- Let f_j^a denote the number of j-faces that are contained in $(\mathcal{K}(\mathcal{B}) \cap \mathcal{K}(\mathcal{A})) \setminus \mathcal{K}(\mathcal{C})$. These j-faces of P only arise when A-sewing, and they will be destroyed by the construction.

- Let f_j^i denote the number of j-faces that are contained in $\mathcal{K}(\mathcal{B}) \setminus \mathcal{K}(\mathcal{A} \cup \mathcal{C})$. These are the faces that lie on the **i**nterior of $\mathcal{K}(\mathcal{B})$, and they also will be destroyed by the construction.

Observe that $\mathcal{K}(\mathcal{B}) \cap \mathcal{K}(\mathcal{C})$, $(\mathcal{K}(\mathcal{B}) \cap \mathcal{K}(\mathcal{A})) \setminus \mathcal{K}(\mathcal{C})$, and $\mathcal{K}(\mathcal{B}) \setminus \mathcal{K}(\mathcal{A} \cup \mathcal{C})$ partition $\mathcal{K}(\mathcal{B})$. Since any face that is created by the construction must extend from a face on the boundary of $\mathcal{K}(\mathcal{B})$, and any destroyed face must lie in the interior of $\mathcal{K}(\mathcal{B}) \cap \mathcal{K}(\mathcal{A})$, it follows that

$$\begin{aligned} \Delta f_1 - 2\Delta f_2 + 3\Delta f_3 &= (f_0^b - f_1^i - f_1^a) - 2(f_1^b - f_2^i - f_2^a) + 3(f_2^b - f_3^a) \\ &= (f_0^b - f_1^b + f_2^b) \\ &\quad +[-f_1(\mathcal{K}(\mathcal{B})) + 2f_2(\mathcal{K}(\mathcal{B})) - 3f_3(\mathcal{K}(\mathcal{B}))] \end{aligned} \tag{1}$$

We observe that for sewing, we have $f_j^a = 0$ for $0 \leq j \leq 3$. We now will verify that

$$-f_1(\mathcal{K}(\mathcal{B})) + 2f_2(\mathcal{K}(\mathcal{B})) - 3f_3(\mathcal{K}(\mathcal{B})) \geq -2 \quad \text{for sewing, and}$$

$$-f_1(\mathcal{K}(\mathcal{B})) + 2f_2(\mathcal{K}(\mathcal{B})) - 3f_3(\mathcal{K}(\mathcal{B})) \geq -1 \quad \text{for } A\text{-sewing},$$

by induction on $f_3(\mathcal{K}(\mathcal{B})) \geq 1$.

Propositions 3.2 and 3.3 imply that some ordering of the facets belonging to $\mathcal{B}$ form the initial segment of a Bruggesser-Mani line shelling of ∂P. It hence is sufficient to consider

$$-f_1(\mathcal{K}) + 2f_2(\mathcal{K}) - 3f_3(\mathcal{K})$$

for any shellable, 3-dimensional polytopal complex $\mathcal{K}$, whose facets contain a common face, T_1, and belong to $\mathcal{P}^3_\sigma$.

If $f_3(\mathcal{K}) = 1$, then Theorem 3.4 and Euler's relation imply that

$$f_2(\mathcal{K}) \geq f_0(\mathcal{K}) = f_1(\mathcal{K}) - f_2(\mathcal{K}) + 2.$$

It follows that

$$[-f_1(\mathcal{K}) + 2f_2(\mathcal{K})] - 3f_3(\mathcal{K}) \geq 2 - 3 = -1,$$

and the basis for induction is established.

We inductively assume that for any shellable, 3-dimensional polytopal complex $\mathcal{K}$ consisting of $f_3(\mathcal{K}) < k$ facets, all of which can be obtained by performing a sequence of sewing and A-sewing operations starting with the 3-simplex, we have

$$-f_1(\mathcal{K}) + 2f_2(\mathcal{K}) - 3f_3(\mathcal{K}) \geq -1.$$

We now let $\mathcal{K}'$ be a shellable, 3-dimensional polytopal complex consisting of k such facets; we let F be the last facet of $\mathcal{K}'$ in a shelling order; and we define $\mathcal{K} := (\mathcal{K}' \setminus \mathcal{K}(F)) \cup (\mathcal{K}' \cap \mathcal{K}(F))$. We observe that $\mathcal{K}$ is a shellable, 3-dimensional polytopal complex for which the induction hypothesis holds, and consequently we have

$$f_j(\mathcal{K}') = f_j(\mathcal{K}) + f_j(\mathcal{K}(F)) - f_j\left(\mathcal{K} \cap \mathcal{K}(F)\right),$$

for $j \in \{1, 2, 3\}$. It follows by taking a linear combination of these three equations that

$$\begin{aligned}
-f_1(\mathcal{K}') + 2f_2(\mathcal{K}') - 3f_3(\mathcal{K}') \quad &= \quad [-f_1(\mathcal{K}) + 2f_2(\mathcal{K}) - 3f_3(\mathcal{K})] \\
&\quad + [-f_1(\mathcal{K}(F)) + 2f_2(\mathcal{K}(F)) - 3f_3(\mathcal{K}(F))] \\
&\quad - [-f_1(\mathcal{K} \cap \mathcal{K}(F)) + 2f_2(\mathcal{K} \cap \mathcal{K}(F)) \\
&\qquad -3f_3(\mathcal{K} \cap \mathcal{K}(F))] \\
&\geq \quad (-1) + (-1) + f_1(\mathcal{K} \cap \mathcal{K}(F)) - 2f_2(\mathcal{K} \cap \mathcal{K}(F)) \\
&\quad +3f_3(\mathcal{K} \cap \mathcal{K}(F)),
\end{aligned}$$

where the inequality is true by the induction hypothesis applied to $\mathcal{K}$ and $\mathcal{K}(\mathcal{F})$. Since $\mathcal{K} \cap \mathcal{K}(F)$ is a shellable, 2-dimensional polytopal complex, it follows that $f_3(\mathcal{K} \cap \mathcal{K}(F)) = 0$. It thus remains to verify that

$$f_1(\mathcal{K} \cap \mathcal{K}(F)) - 2f_2(\mathcal{K} \cap \mathcal{K}(F)) \geq 0,$$

if $\mathcal{K}'$ arises as the set of facets in $\mathcal{B}$ determined by a sewing operation, and

$$f_1(\mathcal{K} \cap \mathcal{K}(F)) - 2f_2(\mathcal{K} \cap \mathcal{K}(F)) \geq 1,$$

if $\mathcal{K}'$ arises as the set of facets in $\mathcal{B}$ determined by an A-sewing operation. If T_1 is a facet, then the desired result holds trivially as no 2-face can contain T_1. Otherwise, all 2-faces in $\mathcal{K} \cap \mathcal{K}(F)$ must contain T_1, and we consider the following three cases.

- T_1 is a 2-face. In this case $\mathcal{K} \cap \mathcal{K}(F) = \mathcal{K}(T_1)$, and it follows that

  $$f_1(\mathcal{K} \cap \mathcal{K}(F)) - 2f_2(\mathcal{K} \cap \mathcal{K}(F)) \geq 1.$$

- T_1 is an edge. In this case $T_1 \subseteq \mathcal{K} \cap \mathcal{K}(F)$ and consequently $\mathcal{K} \cap \mathcal{K}(F)$ contains either a single 2-face or two 2-faces sharing the edge T_1. In either case, it is apparent that

  $$f_1(\mathcal{K} \cap \mathcal{K}(F)) - 2f_2(\mathcal{K} \cap \mathcal{K}(F)) \geq 1.$$

- T_1 is a vertex. Again, if $\mathcal{K} \cap \mathcal{K}(F)$ consists of a single 2-face, then it is apparent that

  $$f_1(\mathcal{K} \cap \mathcal{K}(F)) - 2f_2(\mathcal{K} \cap \mathcal{K}(F)) \geq 1.$$

  Adding an additional 2-face (n-gon) to $\mathcal{K} \cap \mathcal{K}(F)$ increases $f_2(\mathcal{K} \cap \mathcal{K}(F))$ by one and $f_1(\mathcal{K} \cap \mathcal{K}(F))$ by $n \geq 2$. It follows that

  $$f_1(\mathcal{K} \cap \mathcal{K}(F)) - 2f_2(\mathcal{K} \cap \mathcal{K}(F)) \geq 1.$$

  The only exception to this occurs when two edges of the additional 2-face are adjacent to T_1, and in this case $f_2(\mathcal{K} \cap \mathcal{K}(F))$ and $f_1(\mathcal{K} \cap \mathcal{K}(F))$ both increase by one. It hence follows that

  $$f_1(\mathcal{K} \cap \mathcal{K}(F)) - 2f_2(\mathcal{K} \cap \mathcal{K}(F)) \geq 0.$$

  This can only happen when sewing over a $\{0\}$-flag, however, and this presents the reason for distinguishing between sewing and A-sewing.

It hence follows that

$$-f_1(\mathcal{K}(\mathcal{B})) + 2f_2(\mathcal{K}(\mathcal{B})) - 3f_3(\mathcal{K}(\mathcal{B})) \geq -2 \quad \text{for sewing, and}$$

$$-f_1(\mathcal{K}(\mathcal{B})) + 2f_2(\mathcal{K}(\mathcal{B})) - 3f_3(\mathcal{K}(\mathcal{B})) \geq -1 \quad \text{for } A\text{-sewing.}$$

To complete the proof of this theorem, we again will consider the cases of sewing and A-sewing separately.

- Sewing. In this case Theorem 3.1(iii) implies that the faces of $[P, x]$ containing x are of the form $[F, x]$, where $F \in \mathcal{K}(\mathcal{B}) \cap \mathcal{K}(\mathcal{C})$. Thus $\mathcal{K}(\mathcal{B}) \cap \mathcal{K}(\mathcal{C})$ is combinatorially equivalent to the vertex figure of x in $[P, x]$ and hence is a 2-dimensional sphere. It consequently has Euler characteristic 2 (i.e. $f_0^b - f_1^b + f_2^b = 2$), so equation (1) implies that

$$\begin{aligned}\Delta f_1 - 2\Delta f_2 + 3\Delta f_3 &= (f_0^b - f_1^b + f_2^b) + [-f_1(\mathcal{B}) + 2f_2(\mathcal{B}) - 3f_3(\mathcal{B})] \\ &\geq 2 + -2 = 0.\end{aligned}$$

- A-sewing. The faces of $Q := [P, x]$ are of types (i), (ii), and (iii), as given in Theorem 3.1. Choose a point y beyond precisely the facets containing x, and another point z beyond precisely the facets containing the face $[T_q, x]$. By ensuring that y and z are in sufficiently general position and that z is sufficiently close to Q, the line through y and z induces a line shelling of Q that first shells the facets of Q of type (ii), then those of type (iii), and finally those of type (i). The reversal of this shelling induces a shelling of the facets containing x that first shells those of type (iii), then those of type (ii). The facets of type (iii) form a proper subset of the facets of Q, and these facets are pyramids over the maximal faces of $\mathcal{K}(\mathcal{B}) \cap \mathcal{K}(\mathcal{C})$. Thus shelling only the facets of Q of type (iii) induces a shelling of a proper collection of facets of the vertex figure of x. We conclude that $\mathcal{K}(\mathcal{B}) \cap \mathcal{K}(\mathcal{C})$ is a 2-dimensional ball and hence has Euler characteristic 1 (i.e. $f_0^b - f_1^b + f_2^b = 1$). It follows from equation (1) that

$$\begin{aligned}\Delta f_1 - 2\Delta f_2 + 3\Delta f_3 &= (f_0^b - f_1^b + f_2^b) + [-f_1(\mathcal{B}) + 2f_2(\mathcal{B}) - 3f_3(\mathcal{B})] \\ &\geq 1 + -1 = 0.\end{aligned}$$

□

Using Maple we were able to verify which flag f-vectors correspond to polytopes in $\mathcal{P}^4_\sigma$ with at most eight vertices.

Theorem 4.3 *Any flag f-vector (f_0, f_1, f_2, f_{02}) in $f(\mathcal{P}^4_\sigma)$ must satisfy the following linear inequalities:*

1. $-3f_2 + f_{02} \geq 0$
2. $-4f_0 + f_1 - 3f_2 + f_{02} \geq -10$
3. $-2f_0 + 2f_1 + 2f_2 - f_{02} \geq 0$
4. $3f_0 - 2f_1 + f_2 \geq 5$

It also must also satisfy the following quadratic inequality:

$$f_{02} - 4f_2 + 3f_1 - 2f_0 \leq \binom{f_0}{2}.$$

Furthermore, all flag f-vectors that satisfy these inequalities and for which $f_0 \leq 8$ do correspond to flag f-vectors of polytopes in $\mathcal{P}^4_\sigma$ with the following exceptions:

$$(8, 22, 25, 78) \qquad (8, 23, 27, 83) \qquad (8, 24, 29, 88) \qquad (8, 25, 31, 93).$$

The first two linear inequalities in Theorem 4.3, as well as the quadratic inequality, hold for flag f-vectors of all 4-polytopes and were recorded by Bayer [2]. The third linear inequality was proved in Lemma 4.1 and the fourth in Theorem 4.2.

The four vectors listed as exceptions in Theorem 4.3 lie on the common line

$$\ell = \{(8, f_1, 2f_1 - 19, 5f_1 - 32)\}.$$

If we let $\mathcal{Q}$ denote the 4-dimensional cone that is described by the linear inequalities of Theorem 4.3, then a cross section of $\mathcal{Q}$ is a tetrahedron. The rays

$$\ell'_1 = \{(f_0, 3f_0 - 5, 3f_0 - 5, 10f_0 - 20) \colon f_0 \geq 5\},$$
$$\ell'_2 = \{(f_0, 4f_0 - 10, 5f_0 - 15, 15f_0 - 45) \colon f_0 \geq 5\}, \text{ and}$$

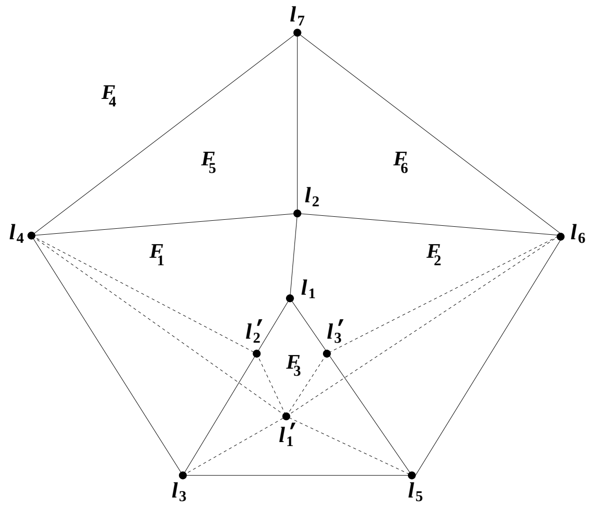

Figure 3: Cross section of $\mathcal{Q}$, and the duals of these flag vectors, inside a cross section of the polyhedron determined by all known linear inequalities for 4-polytope flag f-vectors [2].

$$\ell_3 = \{(f_0, 4f_0 - 10, 6f_0 - 20, 18f_0 - 60)\colon f_0 \geq 5\}$$

are extreme rays of $\mathcal{Q}$, as they each contain an infinite sequence of flag f-vectors of polytopes in $\mathcal{P}^4_\sigma$. The ray

$$\ell_4 = \{(5, f_1, 2f_1 - 10, 6f_1 - 30)\colon f_1 \geq 10\}$$

is an extreme ray of $\mathcal{Q}$, but it does not contain any flag f-vector in $f(\mathcal{P}^4_\sigma)$ other than that of the 4-simplex.

The flag f-vector of the 4-simplex satisfies the four linear inequalities of Theorem 4.3 tightly, and we observe that all flag f-vectors on ℓ_1' satisfy (2), (3) and (4) tightly; all flag f-vectors on ℓ_2' satisfy (1), (2) and (4) tightly; and all flag f-vectors on ℓ_3 satisfy (1), (2) and (3) tightly. All flag f-vectors on the ray ℓ_4, which lies in the closure of $\mathrm{conv}(f(\mathcal{P}^4_\sigma))$ but not in $\mathrm{conv}(f(\mathcal{P}^4_\sigma))$ itself, satisfy linear inequalities (1) and (3) of Theorem 4.3 with equality.

Figure 3 provides a Schlegel diagram of the cross section of the polyhedral 4-cone determined by the linear inequalities known to be satisfied by the flag f-vectors of all 4-polytopes [2]. The facets are labeled in accordance with the corresponding linear inequalities of Theorem 2.1, and the tetrahedron illustrated by the dashed lines and determined by vertices ℓ_1', ℓ_2', ℓ_3, and ℓ_4 is a cross section of $\mathcal{Q}$. The tetrahedron determined by vertices ℓ_1', ℓ_3', ℓ_5, and ℓ_6 is a cross section of the polyhedral 4-cone that is determined by dualizing the linear flag f-vector inequalities of Theorem 4.3. Note in particular that $\mathcal{Q}$ is full dimensional within the cone of flag f-vectors.

4.2 Extremal Families in $\mathcal{P}_\sigma^4$

Four sewing/A-sewing operations can be applied to the 4-simplex to obtain the four flag f-vectors of 4-polytopes with six vertices. Each of these operations is repeatable, and the first three provide infinite sequences of 4-polytopes that verify that ℓ_1', ℓ_2', and ℓ_3, as pictured in Figure 3, are extreme rays of $f(\mathcal{P}_\sigma^4)$. The fourth sequence determines a ray that passes through $(6, 15, 18, 54)$, which lies on the boundary of the quadratic inequality of Theorem 4.3. In the following discussion, we will let Δf and $\Delta\Psi$ represent, respectively, the changes in the flag f-vector and the **cd**-index that occur as a result of the indicated sewing or A-sewing operation.

1. $\Delta f = (1, 3, 3, 10)$ and $\Delta\Psi = dc^2 + c^2d + 2cdc + 2d^2$.
   Consider A-sewing over a $\{1, 2\}$-flag $\mathcal{T} = T_1 \subset T_2$, where T_1 is an edge that is contained in exactly three facets, at least one of which is a 3-simplex. We pick T_2 to be the 2-face that is the intersection of the other two facets. We must show that such a flag exists at each iteration of our sequential A-sewing and that A-sewing over the flag will produce the desired changes in the flag f-vector.

   Assuming that such a flag exists, the facets of P are partitioned by A-sewing x onto P over $\mathcal{T}$ in the following manner:

   - $\mathcal{A} = \{F_1, F_2\}$, where F_1 and F_2 are the two facets that contain T_2,
   - $\mathcal{B} = \{F_3\}$, where F_3 is the 3-simplex that contains T_1 but not T_2.
   - $\mathcal{C} = \mathcal{F} \setminus (\mathcal{A} \cup \mathcal{B})$.

   Theorem 3.1 implies that the facet F_3, the 2-faces $F_1 \cap F_3$ and $F_2 \cap F_3$, and the edge T_1 will be destroyed by the A-sewing. Each of the four vertices and five edges that are contained in $\mathcal{K}(\mathcal{B}) \cap \mathcal{K}(\mathcal{C})$ will correspond to a new face of one greater dimension that contains x. It follows that

   $$\Delta f_0 = 1, \qquad \Delta f_1 = 4 - 1 = 3, \text{ and} \qquad \Delta f_2 = 5 - 2 = 3.$$

The n-gon $T_2 = F_1 \cap F_2$ is precisely the intersection of facets in $\mathcal{A}$, and consequently it will be stretched to an $(n+1)$-gon containing x. This accounts for one additional $\{0,2\}$-flag than was present in P. Since all new and destroyed 2-faces are 2-simplices, it follows that

$$\Delta f_{02} = 3\Delta f_2 + 1 = 3(3) + 1 = 10.$$

We now will inductively prove that at each iteration of the sequential A-sewing, there is at least one flag that possesses the properties described above. The 4-simplex provides the basis for induction, as any $\{1,2\}$-flag satisfies the desired conditions. Assuming that the 4-polytope P, which has been constructed in this manner, possesses such a flag $\mathcal{T}$, we A-sew x onto P over $\mathcal{T}$ to obtain Q. Our assumptions regarding $\mathcal{T}$ imply that either vertex of T_1, say x_1, is contained in precisely three 2-faces of $\mathcal{K}(\mathcal{B})$. Two of these 2-faces are destroyed by the A-sewing and the other corresponds to a new facet, G, of Q. The edge $[x_1, x]$ is contained in the 3-simplex G, as well as the two stretched facets, $[F_1, x]$ and $[F_2, x]$. The flag $\mathcal{T}' = [x_1, x] \subset [T_2, x]$ thus satisfies the specified conditions.

2. $\Delta f = (1, 4, 5, 15)$ and $\Delta\Psi = dc^2 + 2c^2d + 3cdc + 3d^2$.

   We now will sequentially A-sew over a $\{1,3\}$-flag $\mathcal{T} = T_1 \subset T_2$, where T_1 is an edge that is contained in exactly three facets, at least two of which are 3-simplices. We pick T_2 to be the third facet, which may or may not be a 3-simplex. It can easily be verified that such a flag will exist at each iteration of our sequential A-sewing and that A-sewing over such a flag will produce the desired changes in the flag f-vector.

3. $\Delta f = (1, 4, 6, 18)$ and $\Delta\Psi = dc^2 + 3c^2d + 3cdc + 4d^2$.

   For this family, we iteratively sew over a flag $\mathcal{T} = T_1$, where T_1 is a 3-simplex. Since sewing over a simplicial polytope always results in a simplicial polytope, such a flag always will exist. The only face that will be destroyed by the sewing is T_1, while its four vertices and six edges will provide four new edges and six new 2-faces for the new polytope. Since no 2-faces will be stretched or destroyed, and all new 2-faces are 2-simplices, the desired changes result.

4. $\Delta f = (1, 5, 8, 24)$ and $\Delta\Psi = dc^2 + 4c^2d + 4cdc + 6d^2$.

   To obtain these changes, we sew over a $\{2\}$-flag $\mathcal{T} = T_1$, where T_1 is a 2-simplex that is the intersection of two 3-simplices, F_1 and F_2. As the polytope obtained at each iteration must be simplicial, it again follows that such a flag always will exist.

   The facets F_1 and F_2 and the 2-face T_1 will be destroyed by the sewing, and the five vertices and nine edges of $\mathcal{K}(\mathcal{B})$ will provide five new edges and nine new 2-faces. Since no 2-faces will be stretched by the sewing and all 2-faces that are either created or destroyed are 2-simplices, the desired changes follow.

We also can combine two sewing or A-sewing operations, and in so doing, we can obtain several repeatable, two-step constructions. We will consider one such combination.

5. $\Delta f = (2, 8, 10, 31)$ and $\Delta\Psi = 2dc^2 + 4c^2d + 6cdc + 7d^2$.

   In (2) and (3), we established sewing and A-sewing constructions that produce changes of $(1, 4, 5, 15)$ and $(1, 4, 5, 16)$, respectively, in the flag f-vectors of 4-polytopes. It can be inductively verified that when beginning with the 4-simplex and alternating in either order, the flags necessary for the changes described in (2) and (3) will be successively present. It follows that after each iteration of this 2-step process, a change of $(2, 8, 10, 31)$ in the flag f-vector will occur.

The aforementioned 2-step construction creates an infinite family of polytopes whose flag f-vectors lie on the edge connecting ℓ_2' and ℓ_3 and separating the facets F_1 and F_3 in Figure 3. The g-vectors [26] of these 2-simplicial polytopes all satisfy the equation $g_2 = 0$.

5 Cyclic and Ordinary Polytopes

The *cyclic polytope* $C(n, d)$, $n > d \geq 2$ is defined to be the convex hull of n points on the moment curve $(t, t^2, \ldots, t^d)$. It also can be described it combinatorially in the following manner. Let $V(P) = \{x_0, \ldots, x_{n-1}\}$ denote the set of vertices of P, and define a *vertex array* to be a total ordering of

$V(P)$, $x_0 < x_1 < \cdots < x_{n-1}$. A collection of vertices $X \subseteq V(P)$ is said to satisfy *Gale's Evenness Condition* [17] if any pair of vertices in $V(P) \setminus X$ has an even number of vertices from X between them in the vertex array. Recalling that $[X] := \text{conv } X$, $C(n,d)$ can be defined as the d-polytope with vertex array $x_0 < x_1 < \ldots < x_{n-1}$ whose facets are of the form $[X]$, where $X \subset V(P)$, $|X| = d$, and X satisfies Gale's Evenness Condition. Beyond realizing the bounds of the Upper Bound Theorem [20], cyclic polytopes play a crucial role in the construction of polytopes for the g-Theorem [10]. Introduced by Bisztriczky [11, 13] and proved realizable by Dinh [14], the class of *ordinary polytopes* provide a nonsimplicial analog to cyclic polytopes. Their interesting structure and flag f-vectors have been studied by Bayer [3, 4] and Bayer-Bruening-Stewart [6]. In this section we show that both cylic and ordinary polytopes can be constructed by generalized sewing.

5.1 Cyclic Polytopes

Theorem 5.1 *The cyclic polytope $C(n,d)$, $d \geq 2$, is achievable by sequentially performing $n-d-1$ sewing operations starting with the d-simplex.*

Proof: We will prove this by induction on $\ell := n-d-1$.

The desired result holds trivially for the case $\ell = 0$, as $C(d+1,d)$ is the d-simplex. We assume that the cyclic polytope $P_\ell := C(d+\ell+1,d)$ is achievable in this manner with vertices $V(P_\ell) := \{x_0, \ldots, x_{d+\ell}\}$. We then sew the vertex $x_{d+\ell+1}$ onto P_ℓ over the flag

$$\mathcal{T} = \{x_{d+\ell}\} \subset [x_{d+\ell-1}, x_{d+\ell}] \subset [x_{d+\ell-2}, x_{d+\ell-1}, x_{d+\ell}] \subset \cdots \subset [x_{\ell+1}, \ldots, x_{d+\ell}]$$

to create the polytope

$$P_{\ell+1} := [P_\ell, x_{d+\ell+1}].$$

Recall that

$$\mathcal{F}_1 \backslash (\mathcal{F}_2 \backslash (\cdots \backslash \mathcal{F}_d) \cdots) = (\mathcal{F}_1 \setminus \mathcal{F}_2) \cup (\mathcal{F}_3 \setminus \mathcal{F}_4) \cup \cdots ,$$

where the last term is $\mathcal{F}_{d-1} \setminus \mathcal{F}_d$ if d is even and $\mathcal{F}_d$ if d is odd. We observe that any facet of P_ℓ that does not contain $x_{d+\ell}$ lies in $\mathcal{C}$, and hence it will remain a facet of $P_{\ell+1}$. Furthermore, F is a facet in $\mathcal{C}$ that contains $x_{d+\ell}$ if and only if

$$V(F) = \{x_0, \ldots, x_i\} \cup Y \cup \{x_{d+\ell-j}, \ldots, x_{d+\ell}\}, \tag{2}$$

where Y is a paired subset of $\{x_{i+2}, \ldots, x_{d+\ell-j-2}\}$, and j is a positive odd integer. By defining $\{x_{d+\ell-j}, \ldots, x_{d+\ell}\} := \varnothing$ when $j < 0$, we can write $V(F)$ in form (2) for all facets $F \in \mathcal{C}$. Our induction hypothesis thus implies that the facets of $P_{\ell+1}$ not containing $x_{d+\ell+1}$ correspond precisely to those d-subsets of $V(P_\ell)$ that satisfy Gale's Evenness Condition when considered as subsets of $V(P_{\ell+1})$. It remains only to verify that F is a facet of $P_{\ell+1}$ that contains $x_{d+\ell+1}$ if and only if $V(F)$ is a d-subset of $V(P_{\ell+1})$ that contains $x_{d+\ell+1}$ and satisfies Gale's Evenness Condition.

The facets in $\mathcal{B}$ determined by the sewing operation are precisely those facets F for which $V(F)$ can be written in form (2), where Y is a paired subset of $\{x_{i+2}, \ldots, x_{d+\ell-j-2}\}$, and j is a non-negative even integer.

Let X be a d-subset of $V(P_{\ell+1})$ that contains $x_{d+\ell+1}$ and satisfies Gale's Evenness Condition. This implies that

$$X = \{x_0, \ldots, x_i\} \cup Y \cup \{x_{d+\ell-j+1}, \ldots, x_{d+\ell+1}\},$$

where Y is a paired subset of $\{x_{i+2}, \ldots, x_{d+\ell-j-1}\}$, and $j \geq 0$. It follows that $\{x_0, \ldots, x_i\}$ and Y may be empty, but $\{x_{d+\ell+1-j}, \ldots, x_{d+\ell+1}\}$ cannot be. Hence, $X = (X_1 \cap X_2) \cup \{x_{d+\ell+1}\}$, where

$$X_1 = \{x_0, \ldots, x_{i+1}\} \cup Y \cup \{x_{d+\ell-j+1}, \ldots, x_{d+\ell}\}, \text{ and}$$

$$X_2 = \{x_0, \ldots, x_i\} \cup Y \cup \{x_{d+\ell-j}, \ldots, x_{d+\ell}\}.$$

Since X_1 and X_2 are subsets of $V(P_\ell)$ that satisfy Gale's Evenness Condition and $|X_1| = |X_2| = d$, it follows that $F_1 = [X_1]$ and $F_2 = [X_2]$ must be facets of P_ℓ. If j is even, then F_1 belongs to $\mathcal{C}$ and F_2 belongs to $\mathcal{B}$, while if j is odd, then F_1 belongs to $\mathcal{B}$ and F_2 belongs to $\mathcal{C}$. In either case, Theorem 3.1 implies that $[X]$ is a facet of $P_{\ell+1}$.

We now let F be a facet of $P_{\ell+1}$ that contains $x_{d+\ell+1}$. Then, $F = [G_1 \cap G_2, x_{d+\ell+1}]$, for some facets $G_1 \in \mathcal{C}$ and $G_2 \in \mathcal{B}$. It follows that

$$V(G_1) = \{x_0, \ldots, x_{i_1}\} \cup Y_1 \cup \{x_{d+\ell-j_1}, \ldots, x_{d+\ell}\}, \text{ and}$$

$$V(G_2) = \{x_0, \ldots, x_{i_2}\} \cup Y_2 \cup \{x_{d+\ell-j_2}, \ldots, x_{d+\ell}\},$$

where Y_1 and Y_2 are paired subsets, j_1 is odd (possibly negative), and j_2 is even. Since $|G_1 \cap G_2| = d - 1$, we must have $|i_1 - i_2| = 1$, $Y_1 = Y_2$, and $|j_1 - j_2| = 1$. Assuming without loss of generality that $i_1 > i_2$ and $j_1 < j_2$, we have that

$$V(F) = \{x_0, \ldots, x_{i_2}\} \cup Y_1 \cup \{x_{d+\ell-j_1}, \ldots, x_{d+\ell+1}\},$$

where Y_1 is a paired subset. It follows that $V(F)$ is a d-subset of $V(P_{\ell+1})$ that contains $x_{d+\ell+1}$ and satisfies Gale's Evenness Condition. □

We now will introduce some notation and terminology that was developed by Bisztriczky [11, 13] and Dinh [14]. We let P be a d-polytope with $n+1$ vertices that satisfies the necessary part of Gale's Evenness Condition, and we call such a polytope P a *Gale Polytope.*

Definition 5.2 Let P be a d-polytope, $d \geq 2$, with vertex array $x_0 < x_1 < \cdots < x_n$. Notationally, we define $x_i := x_0$ for $i < 0$ and $x_i := x_n$ for $i > n$. We say that P is a *d-multiplex* if the facets of P are precisely $F_i = [x_{i-d+1}, \ldots, x_{i-1}, x_{i+1}, \ldots, x_{i+d-1}]$ for $i = 0, 1, \ldots, n$.

Bisztriczky [12] developed multiplices as a generalization of simplices and observed that they behave like simplices in several important ways. For odd values of d, any d-multiplex is a Gale polytope; and for even values of d, the only d-multiplices that are Gale polytopes are the d-simplices.

5.2 Ordinary Polytopes

In 1994, Bisztriczky [11] generalized cyclic 3-polytopes to a class of polytopes that he named ordinary 3-polytopes, and he further generalized this concept to create the notion of ordinary d-polytopes, $d \geq 3$.

Definition 5.3 Let P be a d-polytope, $d \geq 3$. If there is a vertex array $x_0 < x_1 < \cdots < x_n$ of P, $n \geq d$ such that

1. P is a Gale polytope with this vertex array, and
2. each facet of P is a $(d-1)$-multiplex (with the induced vertex array),

then we say that P is an *ordinary polytope.*

Bisztriczky established that if P is an ordinary $d = (2m+1)$-polytope, $m \geq 2$, then there is an integer $k \geq d$ such that the vertices sharing an edge with x_0 are precisely are precisely $x_1, x_2, \ldots, x_k$, and the vertices sharing an edge with x_n are precisely $x_{n-k}, x_{n-k+1}, \ldots, x_{n-1}$. This number k is called the *characteristic* of P. For odd $d \geq 5$, the combinatorial type of an ordinary d-polytope, $P^{d,k,n}$, is completely determined by its dimension d, the cardinality of its vertex set $n+1$, and its characteristic k. Dinh [14] provided a simple description of the facets of an ordinary $(2m+1)$-polytope, $m \geq 2$, in the following theorem.

Theorem 5.4 (Dinh) *Let n, k, d, m be integers so that $n \geq k \geq d = 2m + 1 \geq 5$, and let P be a d-polytope with $n + 1$ vertices. Then P is an ordinary d-polytope with characteristic k if and only if there is a vertex array $x_0, x_1, \ldots, x_n$ of P so that the facets of P are* conv X, *where*

$$X = \{x_i, \ldots, x_{i+2r-1}\} \cup Y \cup \{x_{i+k}, \ldots, x_{i+k+2r-1}\},$$

$i \in \mathbb{Z}$, $r = 1, 2, \ldots, m$, and Y is a paired $(d - 2r - 1)$-subset of $\{x_{i+2r+1}, \ldots, x_{i+k-2}\}$ for which $|X| \geq d$.

Dinh used this characterization of the facets of $P^{d,k,n}$ to prove that the ordinary d-polytopes are realizable. He began with the cyclic polytope $C(k+1, d) = P^{d,k,k}$. He then demonstrated that given an ordinary d-polytope, $P_n := P^{d,k,n}$, we can find a point x_{n+1} such that $P_{n+1} := [P_n, x_{n+1}]$ is an ordinary d-polytope with characteristic k and $n + 2$ vertices. We will reconstruct Dinh's argument using the terminology of A-sewing.

Lemma 5.5 *Let $x_0, x_1, \ldots, x_n \in \mathbb{R}^{\mathbf{d}}$, and let $P_n := [x_0, \ldots, x_n]$ be an ordinary d-polytope with characteristic k and vertex array $x_0 < x_1 < \cdots < x_n$. Then, $[x_{n-k}, x_{n-k+1}, x_n]$ is a 2-face of P_n.*

Proof: We first note that it is enough to show that $[x_{n-k}, x_{n-k+1}, x_n]$ is a face of P_n, as by assumption x_{n-k}, x_{n-k+1}, and x_n are all vertices of P_n, and hence dim $[x_{n-k}, x_{n-k+1}, x_n] = 2$. For each $j \in \{-2m + 2, \ldots, 0\}$, we define

$$X_j = \{x_{n-k+j}, \ldots, x_{n-k+j+2m-1}\} \cup \varnothing \cup \{x_{n+j}, \ldots, x_{n+j+2m-1}\}$$

and observe that $x_n \in X_j$ since $n + j + 2m - 1 \geq n + 1$. It follows that

(i) If $n - k + j \geq 0$, then

$$\begin{aligned} |X_j| &= [(n - k + j + 2m - 1) - (n - k + j) + 1] \\ &\quad + 0 + [n - (n + j) + 1] \\ &= 2m - j + 1 \\ &\geq 2m + 1 = d. \end{aligned}$$

(ii) If $n - k + j < 0$, then

$$\begin{aligned} |X_j| &= [(n - k + j + 2m - 1) + 1] + 0 + [n - (n + j) + 1] \\ &= 2m + 1 + n - k \\ &\geq 2m + 1 = d. \end{aligned}$$

Theorem 5.4 thus implies that $[X_j]$ is a facet of P_n with $i = n - k + j$; $r = m$; and $Y = \varnothing$. We still must demonstrate that $\cap_{j=-2m+2}^{0} X_j = \{x_{n-k}, x_{n-k+1}, x_n\}$.
Since $-2m + 2 \le j \le 0$, we have

$$n - k + j \;\le\; n - k \;<\; n - k + 1 \;\le\; n - k + j + 2m - 1,$$

which implies that

$$x_{n-k}, x_{n-k+1} \in \{x_{n-k+j}, \cdots, x_{n-k+j+2m-1}\} \subseteq X_j,$$

for each $j \in \{-2m + 2, \ldots, 0\}$.

We also saw above that $x_n \in X_j$ for each $j \in \{-2m + 2, \ldots, 0\}$, and it follows that $\{x_{n-k}, x_{n-k+1}, x_n\} \subseteq \cap_{j=-2m+2}^{0} X_j$. It remains only to show that $x_\ell \notin \{x_{n-k}, x_{n-k+1}, x_n\}$ implies that $x_\ell \notin X_j$ for some j. If $\ell < n - k$, then $x_\ell \notin X_0$, and if $n - k + 1 < \ell < n$, then

$$x_\ell \notin X_{1+\ell-n} = \{x_{\ell-k+1}, \ldots, x_{\ell-k+2m}\} \cup \varnothing \cup \{x_{\ell+1}, \ldots, x_{\ell+2m}\}.$$

□

The following proposition was established by Dinh [14], but we will provide an alternate proof using A-sewing.

Proposition 5.6 (Dinh) *Let $x_0, x_1, \ldots, x_n \in \mathbb{R}^{\mathbf{d}}$ such that $P_n = [x_0, \ldots, x_n]$ is an ordinary d-polytope with characteristic k and vertex array $x_0 < x_1 < \cdots < x_n$. Then there exists a point $x_{n+1} \in \mathbb{R}^{\mathbf{d}}$ such that:*

(i) $[x_{n-k}, x_{n-k+1}, x_n, x_{n+1}]$ *is a convex 4-gon where* $[x_{n-k}, x_{n+1}]$ *is one of its diagonals,*

(ii) x_{n+1} *is beyond all facets F of P_n with the property that $x_n \in F$ and $x_{n-k} \notin F$, and*

(iii) x_{n+1} *is beneath all facets F of P_n with the property that $x_n \notin F$.*

Proof: We consider the flag

$$\mathcal{T} = \{x_n\} \subset [x_{n-k}, x_n] \subset [x_{n-k}, x_{n-k+1}, x_n],$$

observing that x_n is a vertex of P_n; $[x_{n-k}, x_n]$ is an edge of P_n by definition of characteristic k; and $[x_{n-k}, x_{n-k+1}, x_n]$ is a 2-face of P_n by Lemma 5.5.

Proposition 3.3 hence guarantees the existence of a point x_{n+1} that is almost exactly beyond $\mathcal{T}$. We will A-sew x_{n+1} onto P_n and verify that x_{n+1} satisfies properties $(i) - (iii)$.

Since $[x_{n-k}, x_{n-k+1}, x_n] \in \mathcal{A}$, Theorem 3.1 implies that it will be stretched to become the 2-face $[x_{n-k}, x_{n-k+1}, x_n, x_{n+1}]$. The choice of $\mathcal{T}$, in combination with the A-sewing construction, guarantees that x_{n-k}, x_{n-k+1}, and x_n are each contained in a facet belonging to $\mathcal{C}$ and will remain vertices of P_{n+1}.

In order to verify that $[x_{n-k}, x_{n+1}]$ is not an edge of P_{n+1}, we first observe that all facets of P_n not containing x_n belong to $\mathcal{C}$. Furthermore, any facet that contains both x_{n-k} and x_n belongs to either $\mathcal{C}$ or $\mathcal{A}$. It follows that no facets containing x_{n-k} are elements of $\mathcal{B}$, so Theorem 3.1 implies that $[x_{n-k}, x_{n+1}]$ will be a diagonal of the newly created 4-gon. This verifies that x_{n+1} satisfies (i). Properties (ii) and (iii) are also clearly satisfied by x_{n+1} by definition of being almost exactly beyond $\mathcal{T}$. $\square$

Dinh also proved that the polytope constructed in Proposition 5.6 is of combinatorial type $P^{d,k,n+1}$. Beginning with a cyclic polytope and proceeding by induction on n, he hence concluded that the ordinary d-polytopes are realizable. We consequently have verified that starting with the d-simplex, we can apply $k-d$ sewing operations to arrive at a cyclic polytope $C(k+1, d) = P^{d,k,k}$. We then can apply $n-k$ A-sewing operations to obtain a polytope of combinatorial type $P^{d,k,n}$. It follows that all polytopes of types $C(n, d)$ and $P^{d,k,n}$ belong to $\mathcal{P}^d_\sigma$.

6 Open Problems

We have approached the problem of generating flag f-vectors of 4-polytopes through sewing and A-sewing, starting from a simplex. What are the (say, linear) constraints on higher dimensional flag f-vectors in $\mathcal{P}^d_\sigma$? Theorem 3.7 demonstrated that all proper k-faces of a polytope in $\mathcal{P}^d_\sigma$ can be realized by applying a related sequence of sewings and A-sewings starting with the k-simplex. The proofs of Lemma 4.1 and Theorem 4.2 then used Theorem 3.7 and an inequality satisfied by all f-vectors in $f(\mathcal{P}^3_\sigma)$ to verify an inequality for all flag f-vectors in $f(\mathcal{P}^4_\sigma)$. It is reasonable to assume that such a technique will work in higher dimensions as well, but such an extension has not yet been considered.

It can be shown that the **cd**-index of a 4-polytope is monotonically nondecreasing under sewing or A-sewing [21]. Does this hold in higher dimensions as well? This would generalize a result of Billera and Ehrenborg [9].

What are some of the combinatorial properties of polytopes in $\mathcal{P}^d_\sigma$? Finbow-Singh [15] (using the term *tailoring* instead of *sewing*) constructed seventy-eight combinatorial types of neighborly 5-polytopes with nine vertices. Can one characterize the polytopes in $\mathcal{P}^d_\sigma$?

The ordinary polytopes provide an infinite family that includes many nonsimplicial polytopes and is achievable using the sewing and A-sewing constructions. Are there other combinatorially nice families of nonsimplicial polytopes that can be described by iterated sewing and A-sewing?

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