Reflections on Transforming Mathematics

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KMED
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Prelude

Pipedream (by Animusic)

`https://www.youtube.com/watch?v=hyCIpKAIFyo`

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- Offer opportunities for strong connections to many concepts in math.
- Play significant roles in science, technology, engineering, and the arts.

Plan for Today

Offer "snapshots" of problems and applications involving transformations.

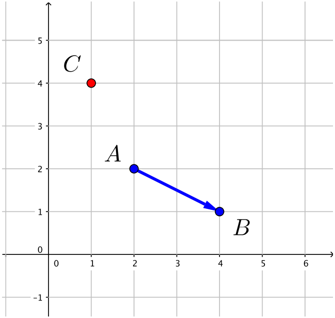
Rigid Motions

Rigid motions map the plane (or space) to itself without changing distances.

Translations

Translating a Point

Translate point C as indicated by vector $\overrightarrow{AB}$.



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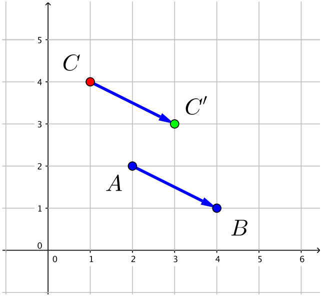
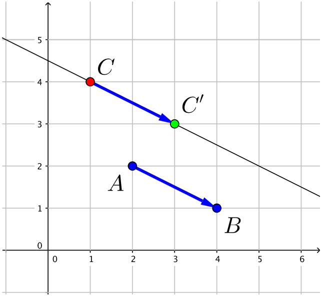


$C' = C + (B - A).$

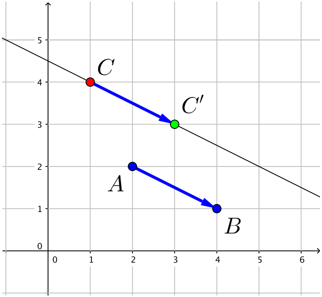
Lines



This connects to the parametric equation of a line $P(t) = C + t(B - A)$.

In this example, $P(t) = (1, 4) + t(2, -1)$.
$t = 1$ corresponds to the original translation.

Rectilinear Motion



This in turn connects to rectilinear motion — just vary t uniformly.

Make a slider for t in GeoGebra.
Type $(1,4) + t * (2,-1)$ in the input space.
Turn on animation.
See `translatepoint4.ggb`.

Rectilinear Motion

Animation software can make this look fancier. I used POV-Ray to create the images and Blender to make the movie. The key command is

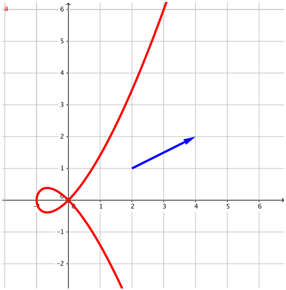
sphere$\{< 0, 1, 4 > +$clock$* < 0, 2, -1 >, 1$ texture$\{$T_Ruby_Glass$\}\}$.

(Note that we are looking directly at the x-axis towards the yz-plane.)

See `translatesphere.mov`.

Translating Curves

What if you want to translate a curve with a given equation?



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This connects to and explains the familiar "shifting" formulas that are seen in algebra, including shifting graphs of functions and writing equations of circles not centered at the origin.

Identifying Parabolas

Consider the parabola given by the equation $y = 2x^2 - 12x + 23$. How can we translate it so that the vertex is at the origin? If the translation is given by $\overline{x} = x + h$ and $\overline{y} = y + k$, then we have

$$\overline{y} - k = 2(\overline{x} - h)^2 - 12(\overline{x} - h) + 23$$

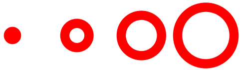
or

$$\overline{y} = 2\overline{x}^2 + (-4h - 12)\overline{x} + 2h^2 + 12h + 23 + k.$$

We want $-4h - 12$ to be zero, so $h = -3$.
We also want $2h^2 + 12h + 23 + k = 0$ so $k = -5$.
Then $\overline{y} = 2\overline{x}^2$ is the equation of the translated parabola.
So the equation of the original parabola is $y - 5 = 2(x - 3)^2$ which has vertex $(3, 5)$.

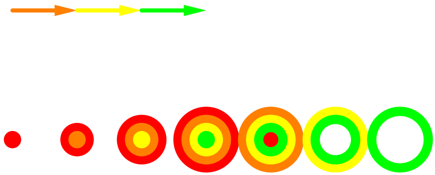
Using Translations to Make Patterns

Starting pattern



Using Translations to Make Patterns

Translate three times (images are in different colors to tell them apart)



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This is the structure of the round “Row, Row, Row Your Boat” with four voices, twice through. (Translation in time.)

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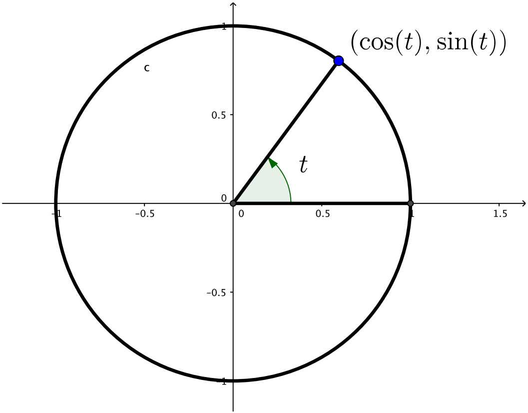
What does translation in (logarithm of) pitch do?

Translations in Time

For another entertaining example, see the video of Kylie Minogue's "Come into my World," `https://www.youtube.com/watch?v=63vqob-M1jQ`.

Rotations

The Power of Trig



So as t increases, the point rotates counterclockwise about the origin along the path of the circle.

The Power of Trig

Try this in GeoGebra.

Make a slider for t, selecting the "angle" option.

Type $(\cos(t), \sin(t))$ in the input space.

Turn on animation.

See `unitcircle2.ggb`.

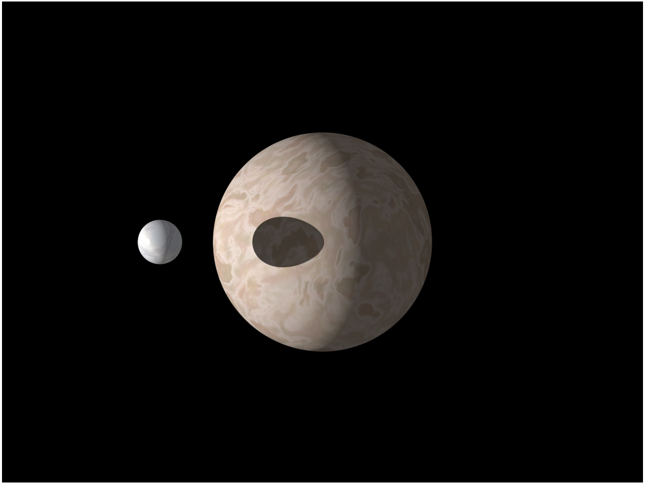
Simple Planetary Motion

A planet rotates counterclockwise around its axis three times while at the same time it revolves once counterclockwise around the sun. How many days do the inhabitants experience during the year?

See `planet.ggb`.

Simple Planetary Motion

A fancier planet and moon created with POV-Ray and Blender.



See `rotatesphere.mov`.

Adding and Multiplying Vectors

Add vectors in the usual way, placing them tail to head.
See `vectorsum.ggb`.

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Multiply vectors by multiplying their lengths and adding their angles.
See `vectorproduct.ggb`.

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Find a vector A such that $A^2 = -1$. Find another.

Find a vector A such that $A^3 = 1$. Find another.

Complex Numbers

We have just seen the geometric model for the complex numbers. "Under the hood" are the trig angle sum identities, which lead directly to the important rotation formula:

$$\begin{bmatrix} \overline{x} \\ \overline{y} \end{bmatrix} = \begin{bmatrix} \cos t & -\sin t \\ \sin t & \cos t \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}.$$

to rotate a point (x, y) counterclockwise about the origin by an angle with sine s and cosine c.

Rotating Conics

Problem from my High School course:
What is the resulting equation if the parabola described by $y = x^2$ is rotated counterclockwise about the origin by the angle t having $\sin t = \frac{7}{25}$ and $\cos t = \frac{24}{25}$?
We use the rotation formula:

$$\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} \cos t & \sin t \\ -\sin t & \cos t \end{bmatrix} \begin{bmatrix} \overline{x} \\ \overline{y} \end{bmatrix}.$$

Substituting, we have

$$-\frac{7}{25}\overline{x} + \frac{24}{25}\overline{y} = \left(\frac{24}{25}\overline{x} + \frac{7}{25}\overline{y}\right)^2$$

which simplifies to

$$576\overline{x}^2 + 336\overline{xy} + 49\overline{y}^2 + 175\overline{x} - 600\overline{y} = 0.$$

Rotating Conics

Problem from my High School course:
Analyze the conic given by the equation

$$73x^2 - 72xy + 52y^2 - 410x + 120y + 525 = 0.$$

We wish to apply a rotation by angle t that eliminates the xy term. We use the rotation formulas. Let's abbreviate $s = \sin t$ and $c = \cos t$.

$$\begin{aligned} x &= c\overline{x} + s\overline{y}, \\ y &= -s\overline{x} + c\overline{y}. \end{aligned}$$

After substitution and simplification we find that the coefficient of $\overline{xy}$ is

$$42sc - 72(c^2 - s^2).$$

Rotating Conics

We need an angle t so that this expression equals 0. Let $T = 2t$, $S = \sin T$, and $C = \cos T$. Then $S = 2sc$ and $C = c^2 - s^2$ by the Double Angle Formulas. So we want an angle T with

$$21S - 72C = 0.$$

But this means $\tan T = \frac{S}{C} = \frac{24}{7}$. From this (and the Pythagorean Theorem) we calculate $S = \frac{24}{25}$ and $C = \frac{7}{25}$.

Rotating Conics

Now we use the Half Angle Formulas to find s and c:

$$s = \sqrt{\frac{1-C}{2}} = \frac{3}{5},$$

$$c = \sqrt{\frac{1+C}{2}} = \frac{4}{5}.$$

Using these values of c and s, the rotated conic has equation

$$100\overline{x}^2 - 400\overline{x} + 25\overline{y}^2 - 150\overline{y} + 525 = 0.$$

Rotating Conics

Complete the two squares to get

$$100(\overline{x}^2 - 4\overline{x} + 4) + 25(\overline{y}^2 - 6\overline{y} + 9) = 100,$$

or

$$(\overline{x} - 2)^2 + \frac{(\overline{y} - 3)^2}{4} = 1.$$

This is an ellipse with center $(2, 3)$.

Rotating Conics

So we have deduced that the original ellipse can be obtained from the ellipse $\overline{x}^2 + \frac{\overline{y}^2}{4} = 1$ by first translating it by $(2, 3)$ and then rotating it clockwise by the angle t with $\sin t = \frac{3}{5}$ and $\cos t = \frac{4}{5}$.

Reflections

Reflections Real and Mathematical

Why does the apparent location of a reflected object match the defined location of the mathematical reflection of that object?

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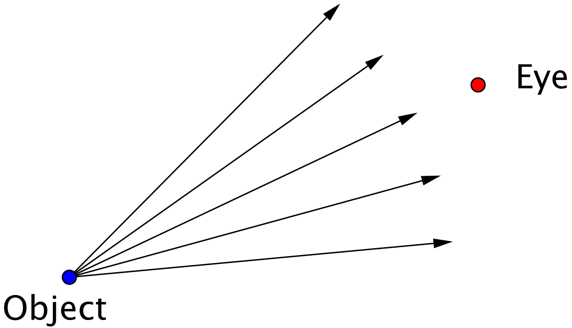
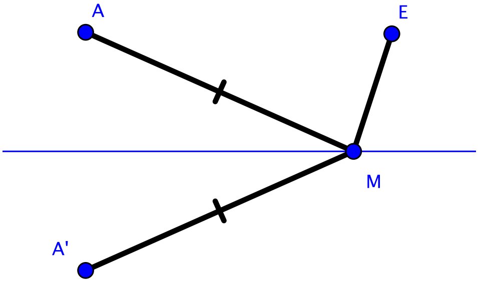
First Key Idea: The brain perceives the location of an object to be at the confluence of rays of light coming from that location.



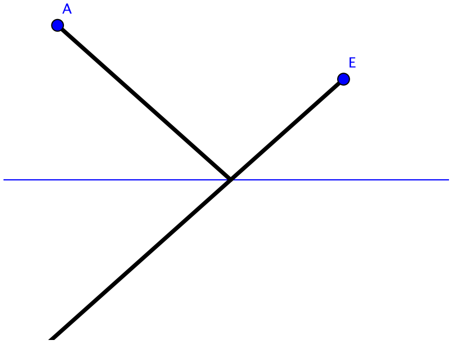
Reflections Real and Mathematical

Second Key Idea: Light travels the path of least time, and this implies the angle of incidence equals the angle of reflection. See `vision2.ggb`—move the point M.

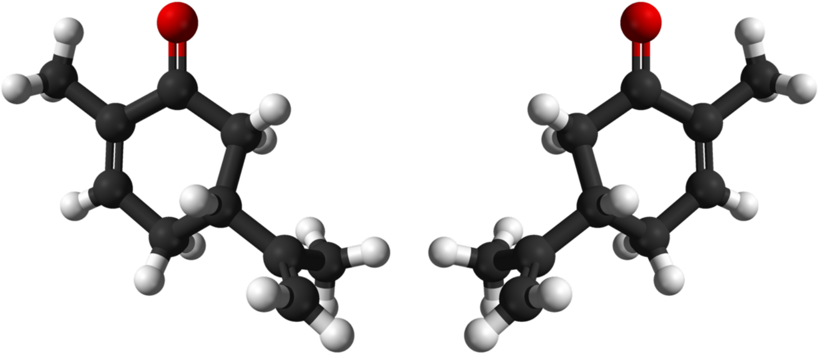


Reflections Real and Mathematical

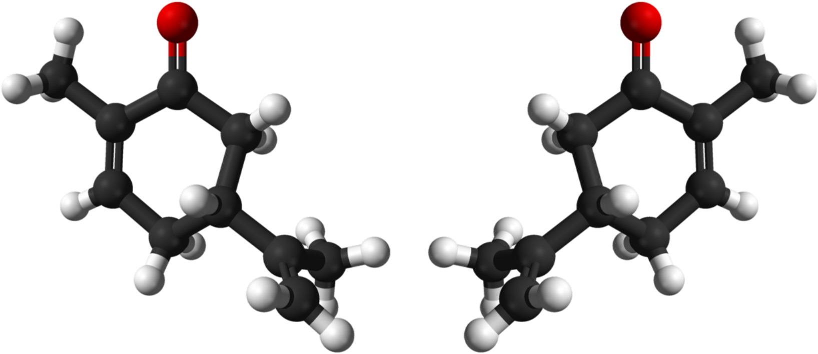
Now move the point E in `vision3.ggb` to see that the reflected rays reaching E appear to trace back and converge on the mathematical reflection of A in the mirror.



Reflections in Chemistry



Reflections in Chemistry



Spearmint and Caraway
(R-carvone and S-carvone)
Reflecting a compound may dramatically change its properties.

For Later Discussion

Why does a mirror reverse an image left and right, but not up and down?

Glide Reflections

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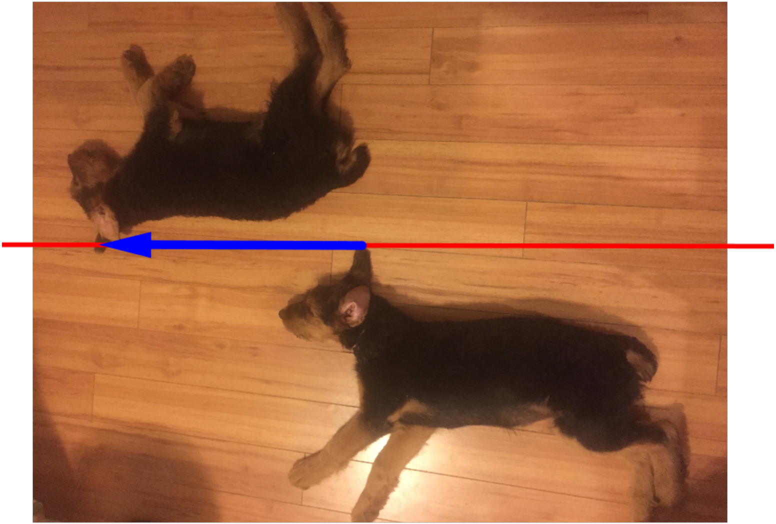

Classification

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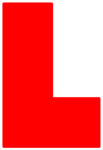
Every rigid motion of the plane is a combination of at most three reflections.

Throw two copies of the following shape on the floor and identify the isometry mapping one to the other.



Questions to Ask with Technology

For each of the following files, precisely determine what the rigid motion is. (Move the point A.)

`iso50.ggb`
`iso60.ggb`
`iso70.ggb`
`iso80.ggb`

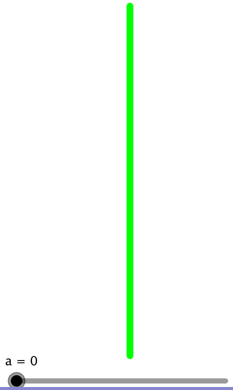
Dilations

Fractals

Start with a simple element; dilate, replicate, reposition, to make a new figure; repeat with the new figure.

See the file `fractal.ggb`.

Fractals



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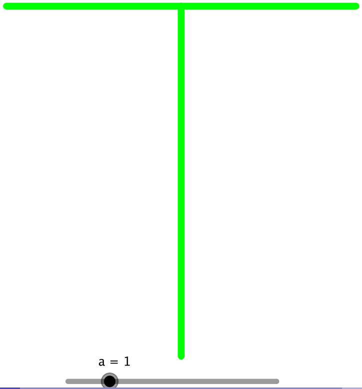

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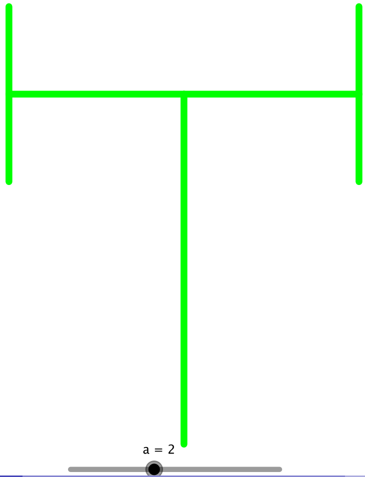

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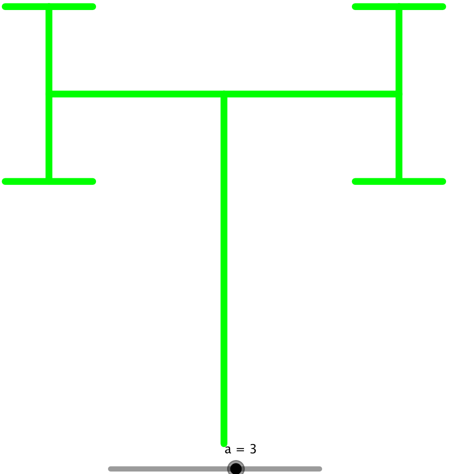

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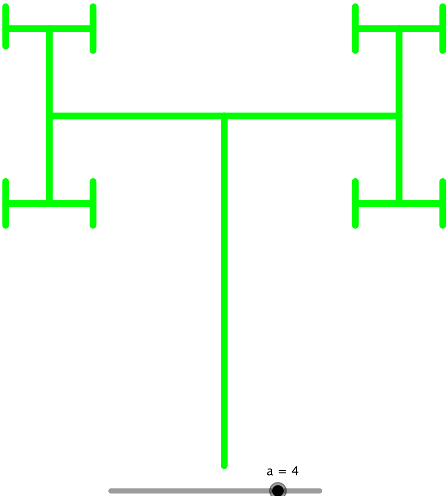


Fractals

A fancier version made with POV-Ray.

Classifying Molecules

Molecules can be classified according to their symmetries—what sets of 3D rigid motions leave the molecules unchanged in appearance. See, for example,
`https://en.wikipedia.org/wiki/Molecular_symmetry`.

Art — iOrnament

Powerful iPad app using transformations and symmetry systems to create beautiful images.

https://itunes.apple.com/us/app/iornament-draw-creative-geometry/id534529876?mt=8

Three-D Design — Jessie Clark Middle School

Programs like SketchUp and Blender are fundamentally based on transformations—some very sophisticated and powerful. Here is an example from an eighth grader at Jessie Clark Middle School.

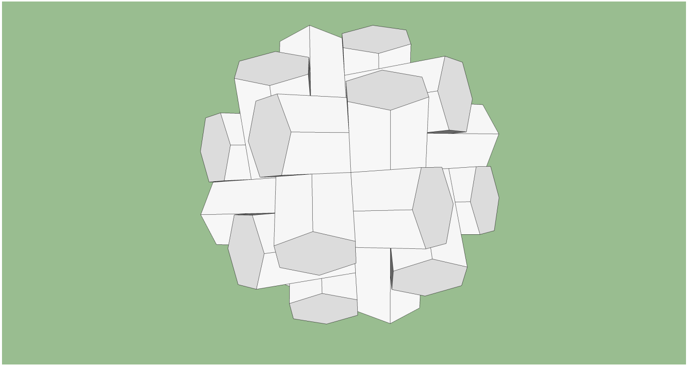
Here is the SketchUp file: `room1.skp`.

See also a summary of SketchUp and transformations, `SketchUp.pdf`.

Three-D Design — Jessie Clark Middle School

Three-D Design and Printing

An example of a puzzle constructed in SketchUp



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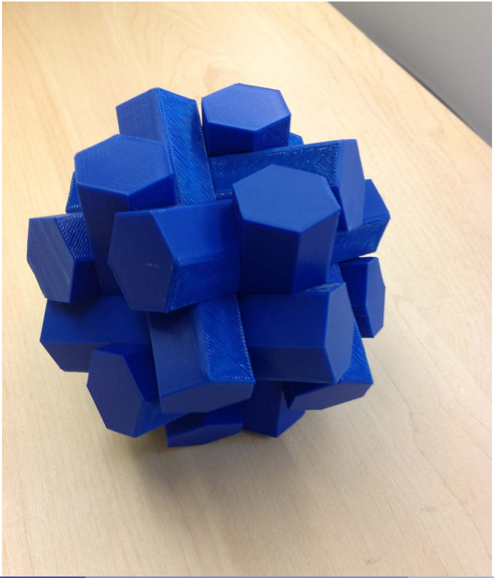

Parting Thoughts

- How can we make more of these rich, reinforcing connections among math, science, technology, engineering, and art in K–12 education?
- How can we better prepare our current future teachers to make such connections?

Thank you!

Images

http://www.spektrum.de/alias/dachzeile/ornament-wettbewerb/1223589