LU Decompositions

We seek a factorization of a square matrix $\boldsymbol{A}$ into the product of two matrices which yields an efficient method for solving the system $\boldsymbol{A}\,\boldsymbol{x} = \boldsymbol{b_k}$ where $\boldsymbol{A}$ is the coefficient matrix, $\boldsymbol{x}$ is our variable vector and $\boldsymbol{b_k}$ is a constant vector for $k = 1, 2, \ldots, \boldsymbol{n}$. The factorization of $\boldsymbol{A}$ into the product of two matrices is closely related to Gaussian elimination.

Definition

1. A square matrix $\boldsymbol{B} = (b_{ij})$ is said to be ***lower triangular*** if $b_{ij} = 0$ for all $i < j$.

2. A square matrix $\boldsymbol{B} = (b_{ij})$ is said to be ***unit lower triangular*** if it is lower triangular and each $b_{ii} = 1$.

3. A square matrix $\boldsymbol{B} = (b_{ij})$ is said to be ***upper triangular*** if $b_{ij} = 0$ for all $t > j$.

Examples

1. The following are all *lower triangular* matrices:

$$\begin{pmatrix} 2 & 0 \\ -1 & 7 \end{pmatrix}, \quad \begin{pmatrix} 1 & 0 & 0 \\ -2 & 1 & 0 \\ 5 & 3 & 1 \end{pmatrix}, \quad \begin{pmatrix} 3 & 0 & 0 & 0 \\ 9 & 7 & 0 & 0 \\ -5 & -8 & 0 & 0 \\ 3 & 8 & -7 & 8 \end{pmatrix}$$

2. The following are all *unit lower triangular* matrices:

$$\begin{pmatrix} 1 & 0 & 0 \\ -2 & 1 & 0 \\ 5 & 3 & 1 \end{pmatrix}, \quad \begin{pmatrix} 1 & 0 & 0 \\ 6 & 1 & 0 \\ -7 & 0 & 1 \end{pmatrix}, \quad \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & -2 & 1 & 0 \\ 0 & 6 & 0 & 1 \end{pmatrix}$$

3. The following are all *upper triangular* matrices:

$$\begin{pmatrix} 5 & 6 \\ 0 & 9 \end{pmatrix}, \quad \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \quad \begin{pmatrix} 10 & 3 & 0 & 6 \\ 0 & 4 & 3 & -4 \\ 0 & 0 & -3 & 6 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

We note that the $n \times n$ identity matrix I_n is only square matrix that is both unit lower triangular and upper triangular.

Example

Let

$$\boldsymbol{A} = \begin{pmatrix} 2 & 2 & 1 \\ 2 & 3 & -2 \\ 4 & 1 & -2 \end{pmatrix}.$$

For elementary matrices (See solv_lin_equ2.pdf) $E_{21} = \begin{pmatrix} 1 & 0 & 0 \\ -1 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$, $E_{31} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -2 & 0 & 1 \end{pmatrix}$,

and $E_{32} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 3 & 1 \end{pmatrix}$ we find that

$$E_{32}\left(E_{31}\left(E_{21}\, A\right)\right) = \begin{pmatrix} 2 & 2 & 1 \\ 0 & 1 & -3 \\ 0 & 0 & -13 \end{pmatrix} \equiv U.$$

Now, if $S \equiv E_{32}\, E_{31}\, E_{21}$, then direct computation yields

$$S = \begin{pmatrix} 1 & 0 & 0 \\ -1 & 1 & 0 \\ -5 & 3 & 1 \end{pmatrix}$$

and

$$S^{-1} = \begin{pmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 2 & -3 & 1 \end{pmatrix} \equiv L.$$

It follows that $\boldsymbol{S}\,\boldsymbol{A} = \boldsymbol{U}$ and, hence, that $\boldsymbol{A} = \boldsymbol{L}\,\boldsymbol{U}$ where $\boldsymbol{L}$ is unit lower triangular and $\boldsymbol{U}$ is upper triangular. That is,

$$\boldsymbol{A} = \begin{pmatrix} 2 & 2 & 1 \\ 2 & 3 & -2 \\ 4 & 1 & -2 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 2 & -3 & 1 \end{pmatrix} \begin{pmatrix} 2 & 2 & 1 \\ 0 & 1 & -3 \\ 0 & 0 & -13 \end{pmatrix} = \boldsymbol{L}\,\boldsymbol{U}.$$

Observe the key fact that the unit lower triangular matrix $\boldsymbol{L}$ "contains" the essential data of the three elementary matrices E_{21}, E_{31}, and E_{32}.

Definition

We say that the $\boldsymbol{n} \times \boldsymbol{n}$ matrix $\boldsymbol{A}$ has an ***LU decomposition*** if $\boldsymbol{A} = \boldsymbol{L}\,\boldsymbol{U}$ where $\boldsymbol{L}$ is unit lower triangular and $\boldsymbol{U}$ is upper triangular.

We also call the *LU decomposition* an *LU factorization*.

Example

1. $\boldsymbol{A} = \begin{pmatrix} 2 & 2 & 1 \\ 2 & 3 & -2 \\ 4 & 1 & -2 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 2 & -3 & 1 \end{pmatrix} \begin{pmatrix} 2 & 2 & 1 \\ 0 & 1 & -3 \\ 0 & 0 & -13 \end{pmatrix} = \boldsymbol{L}\,\boldsymbol{U}$ and so

$$A = \begin{pmatrix} 2 & 2 & 1 \\ 2 & 3 & -2 \\ 4 & 1 & -2 \end{pmatrix}$$ has an *LU decomposition*.

2. The matrix $B = \begin{pmatrix} 20 & 5 & 9 \\ 16 & 4 & 7 \\ 4 & 1 & 3 \end{pmatrix}$ has more than one *LU decomposition*. Two such LU factorizations are

$$\begin{pmatrix} 1 & 0 & 0 \\ 4/5 & 1 & 0 \\ 1/5 & 4 & 1 \end{pmatrix} \begin{pmatrix} 20 & 5 & 9 \\ 0 & 0 & -1/5 \\ 0 & 0 & 2 \end{pmatrix}$$

and

$$\begin{pmatrix} 1 & 0 & 0 \\ 4/5 & 1 & 0 \\ 1/5 & -6 & 1 \end{pmatrix} \begin{pmatrix} 20 & 5 & 9 \\ 0 & 0 & -1/5 \\ 0 & 0 & 0 \end{pmatrix}.$$

Theorem

1. The product of two lower triangular matrices is a lower triangular matrix.
2. The product of two unit lower triangular matrices is a unit lower triangular matrix.
3. Every unit lower triangular matrix is nonsingular and its inverse is also a unit lower triangular matrix.

Proof

2. Suppose M and N are unit lower triangular matrices. Let $L = MN$. We must show that $l_{ij} = 0$ for all $i < j$ and $l_{ii} = 1$ for each i. Let $i < j$ and consider:

$$l_{ij} = \sum_{k=1}^{n} m_{ik} n_{kj} = \sum_{k=1}^{j-1} m_{ik} n_{kj} + \sum_{k=j}^{n} m_{ik} n_{kj} = \sum_{k=1}^{j-1} m_{ik} 0 + \sum_{k=j}^{n} 0\, n_{kj} = 0.$$

Consider

$$\begin{aligned} l_{ii} &= \sum_{k=1}^{n} m_{ik} n_{ki} \\ &= \sum_{k=1}^{i-1} m_{ik} n_{ki} + m_{ii} n_{ii} + \sum_{k=i+1}^{n} m_{ik} n_{ki} \\ &= \sum_{k=1}^{i-1} m_{ik} 0 + 1(1) + \sum_{k=i+1}^{n} 0\, n_{kj} \\ &= 1 \end{aligned}$$

3. Let $\boldsymbol{L} = \left(l_{ij}\right)$ be an $\boldsymbol{n} \times \boldsymbol{n}$ unit lower triangular matrix. Now, define the $\boldsymbol{n} \times \boldsymbol{n}$ elementary matrix $E_{pq} = \left(e_{ij}^{pq}\right)$ where

$$e_{ij}^{pq} = \begin{cases} 1 & if \;\; i = j \\ l_{pq} & if \;\; p = i \;\&\; q = j. \\ 0 & otherwise \end{cases}$$

Then one can show that

$$L = \left(E_{2,1} E_{3,1} E_{4,1} \ldots E_{n,1}\right) \left(E_{3,2} E_{4,2} E_{5,2} \ldots E_{n,2}\right) \ldots \left(E_{n-1,n-2} E_{n,n-2}\right) \left(E_{n,n-1}\right).$$

(We note the order of the products within a given set of parenthesis above doesn't impact the product but the order of the groups of products does matter. We also note that each group in some sense yields a column of $\boldsymbol{L}$.) Hence, $\boldsymbol{L}$ is the product of elementary matrices and so it is nonsingular. It is now a relatively easy exercise to show that the inverse is unit lower triangular. (Hint: What is the inverse of $E_{pq} = \left(e_{ij}^{pq}\right)$?)

Theorem

A square matrix $\boldsymbol{A}$ has an *LU decomposition* iff no row interchanges are required in the Gaussian reduction of $\boldsymbol{A}$ to upper triangular form. If $\boldsymbol{A}$ is nonsingular, the decomposition is unique.

We leave the above as an exercise.

Example

Find the *LU decomposition* for the matrix

$$C = \begin{pmatrix} 2 & 8 & 1 & 1 \\ 4 & 13 & 3 & -1 \\ -2 & -5 & -3 & 3 \\ -6 & -18 & -1 & 1 \end{pmatrix}.$$

Solution

We "build" the unit lower triangular matrix $\boldsymbol{L}$ as we use Gaussian elimination to transform $\boldsymbol{C}$ into an upper triangular matrix $\boldsymbol{U}$.

$$\begin{pmatrix} 2 & 8 & 1 & 1 \\ 4 & 13 & 3 & -1 \\ -2 & -5 & -3 & 3 \\ -6 & -18 & -1 & 1 \end{pmatrix} \to \begin{pmatrix} 2 & 8 & 1 & 1 \\ 0 & -3 & 1 & -3 \\ -2 & -5 & -3 & 3 \\ -6 & -18 & -1 & 1 \end{pmatrix}, \qquad L = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 2 & 1 & 0 & 0 \\ ? & ? & 1 & 0 \\ ? & ? & ? & 1 \end{pmatrix}$$

$$\to \begin{pmatrix} 2 & 8 & 1 & 1 \\ 0 & -3 & 1 & -3 \\ 0 & 3 & -2 & 4 \\ -6 & -18 & -1 & 1 \end{pmatrix}, \qquad L = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 2 & 1 & 0 & 0 \\ -1 & ? & 1 & 0 \\ ? & ? & ? & 1 \end{pmatrix}$$

$$\to \begin{pmatrix} 2 & 8 & 1 & 1 \\ 0 & -3 & 1 & -3 \\ 0 & 3 & -2 & 4 \\ 0 & 6 & 2 & 4 \end{pmatrix}, \qquad L = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 2 & 1 & 0 & 0 \\ -1 & ? & 1 & 0 \\ -3 & ? & ? & 1 \end{pmatrix}$$

$$\rightarrow \begin{pmatrix} 2 & 8 & 1 & 1 \\ 0 & -3 & 1 & -3 \\ 0 & 0 & -1 & 1 \\ 0 & 6 & 2 & 4 \end{pmatrix}, \qquad L = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 2 & 1 & 0 & 0 \\ -1 & -1 & 1 & 0 \\ -3 & ? & ? & 1 \end{pmatrix}$$

$$\rightarrow \begin{pmatrix} 2 & 8 & 1 & 1 \\ 0 & -3 & 1 & -3 \\ 0 & 0 & -1 & 1 \\ 0 & 0 & 4 & -2 \end{pmatrix}, \qquad L = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 2 & 1 & 0 & 0 \\ -1 & -1 & 1 & 0 \\ -3 & -2 & ? & 1 \end{pmatrix}$$

$$\rightarrow \begin{pmatrix} 2 & 8 & 1 & 1 \\ 0 & -3 & 1 & -3 \\ 0 & 0 & -1 & 1 \\ 0 & 0 & 0 & 2 \end{pmatrix} = U, \qquad L = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 2 & 1 & 0 & 0 \\ -1 & -1 & 1 & 0 \\ -3 & -2 & -4 & 1 \end{pmatrix}$$

So, we have that

$$\boldsymbol{C} = \begin{pmatrix} 2 & 8 & 1 & 1 \\ 4 & 13 & 3 & -1 \\ -2 & -5 & -3 & 3 \\ -6 & -18 & -1 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 2 & 1 & 0 & 0 \\ -1 & -1 & 1 & 0 \\ -3 & -2 & -4 & 1 \end{pmatrix} \begin{pmatrix} 2 & 8 & 1 & 1 \\ 0 & -3 & 1 & -3 \\ 0 & 0 & -1 & 1 \\ 0 & 0 & 0 & 2 \end{pmatrix} = \boldsymbol{L}\,\boldsymbol{U}$$

<u>Problem</u>

Find the *LU decomposition* for the following matrices.

1. $\begin{pmatrix} 2 & 4 \\ 6 & 3 \end{pmatrix}$

2. $\begin{pmatrix} 2 & 1 & 3 \\ -2 & 5 & 1 \\ 4 & 2 & 4 \end{pmatrix}$

Key Questions:

1. How does one use the *LU decomposition* to solve the system $\boldsymbol{A}\,\boldsymbol{x} = \boldsymbol{b}$?
2. Why does one care about this method of solution?

Consider the system $\boldsymbol{A}\,\boldsymbol{x} = \boldsymbol{b}$ and suppose $\boldsymbol{A}$ has an *LU decomposition* given by $A = L\,U$. Then

$$\boldsymbol{A}\,\boldsymbol{x} = \boldsymbol{b} \quad \text{Y} \quad \boldsymbol{L}\,\boldsymbol{U}\,\boldsymbol{x} = \boldsymbol{b}$$

$$\text{Y} \quad \boldsymbol{L}\,\boldsymbol{y} = \boldsymbol{b} \qquad \text{where } \boldsymbol{y} = \boldsymbol{U}\,\boldsymbol{x}$$

Now, first solve the lower triangular system $\boldsymbol{L}\,\boldsymbol{y} = \boldsymbol{b}$ for $\boldsymbol{y}$ using ***forward substitution*** and then solve the upper triangular system $\boldsymbol{U}\,\boldsymbol{x} = \boldsymbol{y}$ for $\boldsymbol{x}$ using ***backward substitution***. The solution vector $\boldsymbol{x}$ then solves the original system $\boldsymbol{A}\,\boldsymbol{x} = \boldsymbol{b}$.

<u>**Example**</u>

Solve the system $\boldsymbol{A}\,\boldsymbol{x} = \boldsymbol{b}$ using the *LU decompositions* where

$$\boldsymbol{A} = \begin{pmatrix} 2 & 8 & 1 & 1 \\ 4 & 13 & 3 & -1 \\ -2 & -5 & -3 & 3 \\ -6 & -18 & -1 & 1 \end{pmatrix} \text{ and } \boldsymbol{b} = \begin{pmatrix} 1 \\ 0 \\ -2 \\ 1 \end{pmatrix}.$$

<u>Solution</u>

From a previous example we know that

$$\begin{pmatrix} 2 & 8 & 1 & 1 \\ 4 & 13 & 3 & -1 \\ -2 & -5 & -3 & 3 \\ -6 & -18 & -1 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 2 & 1 & 0 & 0 \\ -1 & -1 & 1 & 0 \\ -3 & -2 & -4 & 1 \end{pmatrix} \begin{pmatrix} 2 & 8 & 1 & 1 \\ 0 & -3 & 1 & -3 \\ 0 & 0 & -1 & 1 \\ 0 & 0 & 0 & 2 \end{pmatrix}.$$

So, we first solve the lower triangular system $\boldsymbol{L}\ \boldsymbol{y} = \boldsymbol{b}$ using forward substitution:

$$\begin{pmatrix} 1 & 0 & 0 & 0 \\ 2 & 1 & 0 & 0 \\ -1 & -1 & 1 & 0 \\ -3 & -2 & -4 & 1 \end{pmatrix} \begin{pmatrix} y_1 \\ y_2 \\ y_3 \\ y_4 \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \\ -2 \\ 1 \end{pmatrix}$$

Here we have that

$$y_1 = 1 \qquad \text{Y} \qquad 2\,y_1 + y_2 = 0 \qquad \text{Y} \qquad y_2 = -2$$

$$y_1 = 1 \ \&\ y_2 = -2 \qquad \text{Y} \qquad -y_1 - y_2 + y_3 = -2 \qquad y_3 = -3$$

$$y_1 = 1 \,\&\, y_2 = -2 \,\&\, y_3 = -3 \quad \text{Y} \qquad -3\,y_1 - 2\,y_2 - 4\,y_3 + y_4 = 1 \qquad \text{Y} \qquad y_4 = -12$$

We now solve the upper triangular system

$$\begin{pmatrix} 2 & 8 & 1 & 1 \\ 0 & -3 & 1 & -3 \\ 0 & 0 & -1 & 1 \\ 0 & 0 & 0 & 2 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix} = \begin{pmatrix} 1 \\ -2 \\ -3 \\ -12 \end{pmatrix}$$

We now have that

$$2\,x_4 = -12 \qquad \text{Y} \qquad x_4 = -6$$

$$x_4 = -6 \qquad \text{Y} \qquad -x_3 + x_4 = -3 \qquad \text{Y} \qquad x_3 = -3$$

$x_3 = -3 \;\&\; x_4 = -6 \quad \Rightarrow \quad -3x_2 + x_3 - 3x_4 = -2 \quad \Rightarrow \quad x_2 = \dfrac{17}{3}$

$x_2 = \dfrac{17}{3} \;\&\; x_3 = -3 \;\&\; x_4 = -6 \quad \Rightarrow \quad 2x_1 + 8x_2 + x_3 + x_4 = 1 \quad \Rightarrow \quad x_1 = -\dfrac{53}{3}$

Hence, $\begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix} = \begin{pmatrix} -53/3 \\ 17/3 \\ -3 \\ -6 \end{pmatrix}$ solves the system

$$\begin{pmatrix} 2 & 8 & 1 & 1 \\ 4 & 13 & 3 & -1 \\ -2 & -5 & -3 & 3 \\ -6 & -18 & -1 & 1 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \\ -2 \\ 1 \end{pmatrix}.$$

Now for the second question, why does one care about using the *LU decomposition* method to solve the system $\boldsymbol{A}\,\boldsymbol{x} = \boldsymbol{b}$ where the coefficient matrix $\boldsymbol{A}$ is an $\boldsymbol{n} \times \boldsymbol{n}$ square matrix? Solving $\boldsymbol{A}\,\boldsymbol{x} = \boldsymbol{b}$ by Gaussian elimination requires roughly $\dfrac{\boldsymbol{n}^3}{3}$ operations and solving $\boldsymbol{A}\,\boldsymbol{x} = \boldsymbol{b}$ via *LU decompositions* requires roughly $\dfrac{\boldsymbol{n}^3}{3}$ operations. So, for a single system there is no compuational savings using the *LU decomposition*. However, if we must solve the systems $\boldsymbol{A}\,\boldsymbol{x} = \boldsymbol{b}_{\boldsymbol{k}}$ where $\boldsymbol{b}_{\boldsymbol{k}}$ is a constant vector for $\boldsymbol{k} = 1, 2, \ldots, \boldsymbol{n}$, then it requires roughly $\dfrac{\boldsymbol{n}^3}{3}$ operations to solve $\boldsymbol{A}\,\boldsymbol{x} = \boldsymbol{b}_{\boldsymbol{1}}$ via *LU decomposition* but roughly only $\boldsymbol{n}^2$ operations to solve the remaining $\boldsymbol{A}\,\boldsymbol{x} = \boldsymbol{b}_{\boldsymbol{k}}$

for $k = 2, 3, \ldots, n$ using *LU decomposition* since we have the unit lower triangular matrix $\boldsymbol{L}$ and the upper triangular matrix $\boldsymbol{U}$ from solving $\boldsymbol{A}\,\boldsymbol{x} = \boldsymbol{b}_1$.

The table below shows approximately the number of operations required to solve $\boldsymbol{A}\,\boldsymbol{x} = \boldsymbol{b}_1$ and $\boldsymbol{A}\,\boldsymbol{x} = \boldsymbol{b}_2$ via two Gaussian eliminations and via two *LU decompositions* where $\boldsymbol{A}$ is of size $n \times n$.

n	*Gaussian Elimination*	*LU Decompositions*
10	700	400
20	5300	3100
30	18000	9900
40	42700	22900
50	83300	44200