Bases and Dimensions of C_A, R_a, and N(A)

A close look at our previous work with the vector spaces associated with a matrix readily yields additional information about the spaces in light of the definitions made in the last section. We illustrate with an example. Consider the matrix A below

$$\begin{pmatrix} 3 & 15 & -1 & -5 & -2 \\ 36 & 180 & 8 & -20 & 3 \\ 8 & 40 & 2 & -4 & 1 \\ -22 & -110 & -7 & 8 & -3 \end{pmatrix}$$

. The reduced row echelon form of A is rref(A) =

$$\begin{pmatrix} 1 & 5 & 0 & -1 & 0 \\ 0 & 0 & 1 & 2 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}.$$

Recall that the nullity of A is the number of free variables of A and the rank of A is the number of pivots of A. For our example Nullity(A) = 2 and Rank(A) = 3. Take a look at the null space of A, N(A), which is the same as the null space of rref(A). From rref(A) we know that the

solutions to A**x** = **0**, the elements of these null spaces, are all of the form a $\begin{pmatrix} -5 \\ 1 \\ 0 \\ 0 \\ 0 \end{pmatrix}$ + b

$\begin{pmatrix} 1 \\ 0 \\ -2 \\ 1 \\ 0 \end{pmatrix}$ for arbitrary scalars a and b. Therefore. N(A) = $\left\langle \begin{pmatrix} -5 \\ 1 \\ 0 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 \\ 0 \\ -2 \\ 1 \\ 0 \end{pmatrix} \right\rangle$. These vectors then

span this subspace of $\mathbf{R}^5$ and are clearly linearly independent. Thus, these vectors are a basis for the null space and dim(N(A)) = nullity(A).

We have seen that the row spaces of A and rref(A) are equal, $R_A = R_{rref(A)}$. But it is easy to see that $R_{rref(A)}$ = <(1, 5, 0, -1, 0), (0, 0, 1, 2, 0), (0, 0, 0, 0, 1)> and that these vectors are linearly independent. Hence, these vectors are a basis for R_A (as are the pivot rows from which these rows came) and dim(R_A) = 3 = rank(A).

The number of pivots also gives the dimension of the column space of A, dim(C_A) = rank(A). As with the pivot rows for R_A, the pivot columns of A give a basis for C_A. A notable difference is that for C_A the pivot columns of rref(A) do <u>not</u> give a basis. So in our example, C_A has a basis given by $\begin{pmatrix} 3 \\ 36 \\ 8 \\ -22 \end{pmatrix}$, $\begin{pmatrix} -1 \\ 8 \\ 2 \\ -7 \end{pmatrix}$, and $\begin{pmatrix} -2 \\ 3 \\ 1 \\ -3 \end{pmatrix}$ but <u>not</u> by $\begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix}$, $\begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \end{pmatrix}$, and $\begin{pmatrix} 0 \\ 0 \\ 1 \\ 0 \end{pmatrix}$ from rref(A). Here this

is clearly the case since, e.g., the space spanned by the latter set contains only vectors with 0's in the fourth component whereas that is not the case for C_A. Why the difference? The row operations used to create rref(A) from A traded rows for *linear combinations* of rows. Columns, on the other hand, had their elements jumbled internally.

We then can find bases and dimensions for N(A), C_A, and R_A by row reducing A. These results on dimension are summarized in the following theorem.

Fundamental Theorem of Linear Algebra

Let A be am $m \times n$ matrix and $r = \text{rank}(A)$. Then $\dim(R_A) = r = \dim(C_A)$, $\dim(N(A)) = n - r$, and $\dim(N(A^T)) = m - r$.