Bases and Dimension

Matrices are important in the study of vector spaces. They provide a wealth of examples of spaces (the spaces whose vectors are the matrices themselves as well as null spaces, column spaces, and row spaces as we have seen) and they also provide basic mappings between vector spaces as we shall see in a later chapter. Fundamental notions for all vector spaces are the concepts that we now consider: *linear independence* and *spanning*.

Definition

Suppose that $\mathbf{v_1}, \mathbf{v_2}, \ldots, \mathbf{v_n}$ are vectors in some vector space V. We say that they are **linearly independent** if $c_1 \mathbf{v_1} + c_2 \mathbf{v_2} + \ldots + c_n \mathbf{v_n} = \mathbf{0}$ implies that $c_1 = c_2 = \ldots = c_n = 0$, i.e., the only linear combination of the vectors that gives the zero vector is the one in which the coefficient of each vector is zero. The sequence of vectors is **linearly dependent** if there exist scalars $c_1, c_2, \ldots, c_n$ that are not all zero with $c_1 \mathbf{v_1} + c_2 \mathbf{v_2} + \ldots + c_n \mathbf{v_n} = \mathbf{0}$.

Examples

1. The vectors (8, -1, 9), (4, 3, -7), (22, 6, -4) in $\mathbf{R}^3$ are *linearly dependent* since $3(8, -1, 9) + 5 (4, 3, -7) + (-2) (22, 6, -4) = (0, 0, 0) = \mathbf{0}$.

2. The vectors $\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$, and $\begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}$ in $M_{2\times 2}$ are clearly *linearly independent*

   since if $a\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} + b\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} + c\begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} = \mathbf{0} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$, then from the 1, 1 position

   we must have a = 0, from the 1, 2 position b = 0, and from the 2, 1 position c = 0.

3. The vectors $x^2 + 5x - 2$ and $3x^2 + 10$ in P_3, the set of polynomials of degree 3 or less with real coefficients, are *linearly independent* since if $a(x^2 + 5x - 2) + b(3x^2 + 10) = \mathbf{0} = 0$, then the coefficient of x in the sum must be 0, so a = 0, and then, e.g., the coefficient of x^2, that is, a + 3b, must be 0, so b = 0.

4. Suppose that the sequence of vectors $\mathbf{v_1, v_2, ..., v_n}$ is linearly dependent and that $\mathbf{w}$ is any vector. Then $\mathbf{v_1, v_2, ..., v_n, w}$ is a linearly dependent set of vectors. Why?
5. Any set of vectors which includes the zero vector is linearly dependent.

<u>Theorem</u>

Suppose that $\mathbf{v_1, v_2, ..., v_n}$ is a linearly dependent set of at least two vectors ($n > 1$). Then there is some vector in the set which is a linear combination of the others.

<u>Proof</u>

Since the set is linearly dependent, there are scalars $c_1, c_2, ..., c_n$, not all 0 such that $c_1 \mathbf{v_1} + c_2 \mathbf{v_2} + \mathbf{...} + c_n \mathbf{v_n} = \mathbf{0}$. Choose one of these scalars which is non-zero. By renumbering the vectors and scalars, if necessary, we may assume that c_n ... 0. Solving for $\mathbf{v_n}$ we get $\mathbf{v_n} = -c_1/c_n \mathbf{v_1} - c_2/c_n \mathbf{v_2} - \mathbf{...} - c_{n-1}/c_n \mathbf{v_{n-1}}$. Thus, $\mathbf{v_n}$ is a linear combination of the remaining vectors in the set.

It should be clear that the theorem above can be stated as an if and only if result; if some vector in a set can be written as a linear combination of the others, then the set is linearly dependent. Thus linear dependence of a set of vectors can be characterized in terms of writing some vector in the set as a linear combination of the others.

<u>Example</u>

We have defined $<\mathbf{v_1, v_2, ..., v_n}>$ to be the subspace generated by the vectors $\mathbf{v_1, v_2, ..., v_n}$, Suppose that the $\mathbf{v_i}$ are linearly dependent. By the theorem above one of the vectors can be written as a linear combination of the others, say $\mathbf{v_n}$ is a linear combination of $\mathbf{v_1, v_2, ..., v_{n-1}}$. It is then easy to see that $<\mathbf{v_1, v_2, ..., v_n}> = <\mathbf{v_1, v_2, ..., v_{n-1}}>$, i.e. this subspace is also generated by a smaller subset of the $\mathbf{v_i}$.

How does one decide whether or not a sequence of vectors is linear independent? Write the definition - simply set up a linear combination of the vectors using variables for the scalars and equate this to $\mathbf{0}$. Then solve this equation to see if there is a solution other than the trivial solution in which each scalar is 0.

Example

Determine whether the following set of vectors is linearly independent.

(4, -3, 9, 5), (0, 7, 1, -2), (-5, 2, 0, 6), (1, 6, -8, 0)

Using the definition the question is exactly can we express the zero vector, **0**, as a nontrivial (not all the scalars being zero) linear combination of these vectors? Thus we proceed to solve the vector equation below for the scalars a, b, c, and d.

a(4, -3, 9, 5) + b (0, 7, 1, -2) + c (-5, 2, 0, 6) + d (1, 6, -8, 0) = (0, 0, 0, 0)

This becomes

(4a - 5c + d, -3a + 7b + 2c + 6d, 9a + b - 8d, 5a - 2b + 6c) = (0, 0, 0, 0)

which, equating the components of the vectors on each side of the equation, amounts to the homogeneous system

$$\begin{aligned} 4a \qquad - 5c + d &= 0 \\ -3a + 7b + 2c + 6d &= 0 \\ 9a + b \qquad - 8d &= 0 \\ 5a - 2b + 6c \qquad &= 0 \end{aligned}$$

Therefore we row reduce the coefficient matrix

$$\begin{pmatrix} 4 & 0 & -5 & 1 \\ -3 & 7 & 2 & 6 \\ 9 & 1 & 0 & -8 \\ 5 & -2 & 6 & 0 \end{pmatrix} \quad \text{to the equivalent matrix} \quad \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}.$$

This tells us that the only solution to the system (and so for the scalars in the vector equation) is the trivial solution. Hence, the vectors are linearly independent. If we replace the last vector in the set by (9, 2, 10, -3), the coefficient matrix becomes

$$\begin{pmatrix} 4 & 0 & -5 & 9 \\ -3 & 7 & 2 & 2 \\ 9 & 1 & 0 & 10 \\ 5 & -2 & 6 & -3 \end{pmatrix}. \text{ Since this matrix reduces to } \begin{pmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & -1 \\ 0 & 0 & 0 & 0 \end{pmatrix} \text{ the corresponding}$$

system of linear equations has many solutions (take d to be any value and adjust a, b, and c accordingly) and the new set of vectors is linearly dependent.

<u>Definition</u>

Suppose that $\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n$ are vectors in the vector space V and that S is a subset of V. The vectors **span** S provided each element of S can be written as a linear combination of the vectors $\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n$.

<u>Examples</u>

1. Decide whether (676, -677, 363) and (95, 71, -68) are in the span of (41, -25, 16), (-38, 61, -29), and (53, 119, -36).

   Combining the augmented matrices resulting from the vector equations a (41, -25, 16) + b (-38, 61, -29) + c (53, 119, -36) = (676, -677, 363) and also this left hand side equal to (95, 71, -68) we get the matrix

$$\begin{pmatrix} 41 & -38 & 53 & 676 & 95 \\ -25 & 61 & 119 & -677 & 71 \\ 16 & -29 & -36 & 363 & -68 \end{pmatrix}$$

   The equivalent reduced row echelon form of this matrix is

$$\begin{pmatrix} 1 & 0 & 5 & 10 & 0 \\ 0 & 1 & 4 & -7 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix}$$

   Therefore, (676, -677, 363) is in the span of these vectors but (95, 71, - 68) is not.

2. Prove that the vectors $\begin{pmatrix} 5 & -2 \\ 0 & 3 \end{pmatrix}$, $\begin{pmatrix} 1 & 7 \\ -4 & 6 \end{pmatrix}$, $\begin{pmatrix} 0 & 8 \\ 5 & -1 \end{pmatrix}$, and $\begin{pmatrix} -7 & -3 \\ 4 & 9 \end{pmatrix}$ span the space

$M_{2\times2}$. We need to show that for an arbitrary element $\begin{pmatrix} w & x \\ y & z \end{pmatrix}$ in $M_{2\times2}$, there are scalars

a, b, c, d such that

$$a\begin{pmatrix} 5 & -2 \\ 0 & 3 \end{pmatrix} + b\begin{pmatrix} 1 & 7 \\ -4 & 6 \end{pmatrix} + c\begin{pmatrix} 0 & 8 \\ 5 & -1 \end{pmatrix} + d\begin{pmatrix} -7 & -3 \\ 4 & 9 \end{pmatrix} = \begin{pmatrix} w & x \\ y & z \end{pmatrix}.$$

Performing the scalar multiplications, adding the matrices and equating corresponding elements results in the system

$$\begin{aligned} 5a + b \phantom{{}+ 8c} - 7d &= w \\ -2a + 7b + 8c - 3d &= x \\ -4b + 5c + 4d &= y \\ 3a + 6b - c + 9d &= z \end{aligned}$$

The coefficient matrix $\begin{pmatrix} 5 & 1 & 0 & -7 \\ -2 & 7 & 8 & -3 \\ 0 & -4 & 5 & 4 \\ 3 & 6 & -1 & 9 \end{pmatrix}$ of the system row reduces to the identity

matrix $\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$ so $M_{2\times2}$ is spanned by these vectors.

3. Notice that any set that is spanned by the vectors $\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n$ is contained in $\langle \mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n \rangle$ and is also spanned by $\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_{n+1}$ for any vector $\mathbf{v}_{n+1}$.

A subspace is defined by its generators. As a vector space, the number of generators in a *minimal set of generators* for any subspace completely determines the space. This is actually true of any vector space which motivates the following definition.

Definition

The vectors $\mathbf{v_1}, \mathbf{v_2}, ..., \mathbf{v_n}$ form a **basis** for the vector space V if

1. They are linearly independent and
2. They span V.

Example

1. The *standard basis* for $\mathbf{R^n}$ is the set $\mathbf{e_1}, \mathbf{e_2}, ..., \mathbf{e_n}$ where each $\mathbf{e_i}$ has all zero components except for a 1 in its i^{th} component. In $\mathbf{R^3}$ we have the standard basis $\mathbf{e_1} = (1, 0, 0)$, $\mathbf{e_2} = (0, 1, 0)$, and $(0, 0, 1)$.
2. For the vector space of n by m matrices, $M_{m\times n}$, the standard basis consists of the matrices $\mathbf{e_{ij}}$ for i = 1, 2, ..., m and j = 1, 2, ..., n where $\mathbf{e_{ij}} = (a_{st})$ and $a_{st} = 0$ if $(s,t) \ldots (i, j)$ and $a_{ij} = 1$. Thus, e.g., $M_{2\times 3}$ has as its standard basis the vectors

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

3. The vector space consisting of all polynomials of degree n or less, P_n, has $\{1, x, x^2, ..., x^n\}$ as its standard basis.

Every spanning set of a vector space contains a basis for the space.

Theorem

Suppose that $S = \{\mathbf{v_1}, \mathbf{v_2}, ..., \mathbf{v_n}\}$ spans the vector space V. Then there is a basis of V consisting of a subset of S.

Proof

S is a linearly independent set, then S is a basis for V. So suppose that S is a linearly dependent set. Then there is some vector (say $\mathbf{v_n}$) in S which is a linear combination of the others. By a previous example we know $<\mathbf{v_1}, \mathbf{v_2}, ..., \mathbf{v_n}> = <\mathbf{v_1}, \mathbf{v_2}, ..., \mathbf{v_{n-1}}>$. Thus the set $\{\mathbf{v_1}, \mathbf{v_2}, ..., \mathbf{v_{n-1}}\}$ spans V. Repeating his argument we arrive at a subset of S which is linearly independent and also spans V, i.e. is a basis for V.

Any two bases for a fixed vector space have the same number of elements.

Theorem

Suppose that S = {$\mathbf{v_1}, \mathbf{v_2}, ..., \mathbf{v_n}$} and T = {$\mathbf{w_1}, \mathbf{w_2}, ..., \mathbf{w_m}$} are both bases for a vector space V. Then n = m.

Definition

The **dimension** of a vector space V, denoted dim(V), is the number os elements in any basis of V.

Examples

From the standard basis given earlier we observe that dim($\mathbf{R}^n$) = n, dim($M_{m\times n}$) = nm, and dim(P_n) = n + 1.

Theorem (Unique Representation)

Suppose that {$\mathbf{v_1}, \mathbf{v_2}, ..., \mathbf{v_n}$} is a basis for V and **w** is a vector in V. Then there exist *unique* scalars $c_1, c_2, ..., c_n$ such that $\mathbf{w} = c_1 \mathbf{v_1} + c_2 \mathbf{v_2} + ... + c_n \mathbf{v_n}$.

Proof

Since S is a basis, it spans V so **w** is a linear combination of the $\mathbf{v_i}$. This gives the existence. For the uniqueness, suppose that $\mathbf{w} = c_1 \mathbf{v_1} + c_2 \mathbf{v_2} + ... + c_n \mathbf{v_n}$ and also $\mathbf{w} = d_1 \mathbf{v_1} + d_2 \mathbf{v_2} + ... + d_n \mathbf{v_n}$. Then $\mathbf{0} = \mathbf{w} - \mathbf{w} = (c_1 - d_1) \mathbf{v_1} + (c_2 - d_2) \mathbf{v_2} + ... + (c_n - d_n) \mathbf{v_n}$. But since S is a linearly independent set, $c_i - d_i = 0$ for each i. Hence, $c_i = d_i$ for all i = 1, 2, ..., n which proves uniqueness.