5 - Determinants

Associated with any square matrix A there is a number called the **determinant** of A denoted |A| or det A . One way to define the determinant, this function from the set of all n×n matrices to the set of real numbers, is by the following three properties. All matrices below are square.

1. Det I_n = 1 for all n.
2. If matrix B is obtained from matrix A by interchanging two rows, then det B = - det A.
3. The determinant is *linear* with respect to operations on a single row. That is, suppose that $\mathbf{s}, \mathbf{r}_1, \mathbf{r}_2, \ldots, \mathbf{r}_n$ are vectors in $\mathbf{R}^n$ (the rows of matrices), that c is any scalar, and that i is some integer from 1 to n (the number of a row). We require that

$$\det\begin{pmatrix} r_1 \\ r_2 \\ \vdots \\ \\ \\ \\ r_n \end{pmatrix} + \det\begin{pmatrix} r_1 \\ r_2 \\ \vdots \\ r_{i-1} \\ s \\ r_{i+1} \\ \vdots \\ r_n \end{pmatrix} = \det\begin{pmatrix} r_1 \\ r_2 \\ \vdots \\ r_{i-1} \\ r_i + s \\ r_{i+1} \\ \vdots \\ r_n \end{pmatrix}$$

$$\text{and } \det\begin{pmatrix} r_1 \\ r_2 \\ \vdots \\ r_{i-1} \\ cr_i \\ r_{i+1} \\ \vdots \\ r_n \end{pmatrix} = c\det\begin{pmatrix} r_1 \\ r_2 \\ \vdots \\ r_{i-1} \\ r_i \\ r_{i+1} \\ \vdots \\ r_n \end{pmatrix}$$

These three properties uniquely define the determinant (though do not provide a traditional formula.) Notice that properties 2 and 3 permit us to track the effect that two of the elementary row operations have on the determinant of a matrix. How about the third row operation, adding a scalar multiple of one row to another? It does not change the determinant. Here is how this can be seen. Suppose that matrix A has two rows which are equal. Then interchanging those two rows yields the same matrix. From property 2 above, det A = - det A, hence, det A = 0. More generally, if A has some row which is a multiple of another of its rows, det A = 0. For suppose that in A row i is c times row j ($\mathbf{r_i} = c\mathbf{r_j}$). Let B be the matrix which is the same as A except in place of row i of A, B has row j of A (i.e. rows i and j of B are both $\mathbf{r_j}$). Then, from the second part of property 3, det A = c det B. But we have just seen that det B = 0, so det A = 0. Finally then using the first part of property 3, we can see that adding a scalar multiple of some row to another changes the determinant by addition of the determinant of a matrix of the type of B above (a matrix with some row being a multiple of another.) But that determinant is zero, so the operation does not change the determinant.

We illustrate using these properties to calculate a determinant below.

<u>Example</u>

Let A = $\begin{pmatrix} 5 & -3 & 7 \\ 10 & 8 & 20 \\ 0 & 1 & -6 \end{pmatrix}$ and write d for det A. We reduce A keeping track of the effect on the

determinant.

Matrix	Determinant
$\begin{pmatrix} 5 & -3 & 7 \\ 10 & 8 & 20 \\ 0 & 1 & -6 \end{pmatrix}$	d
$\begin{pmatrix} 5 & -3 & 7 \\ 0 & 14 & 6 \\ 0 & 1 & -6 \end{pmatrix}$	d
$\begin{pmatrix} 5 & -3 & 7 \\ 0 & 1 & -6 \\ 0 & 14 & 6 \end{pmatrix}$	-d
$\begin{pmatrix} 5 & 0 & -11 \\ 0 & 1 & -6 \\ 0 & 0 & 90 \end{pmatrix}$	-d

$$\begin{pmatrix} 5 & 0 & -11 \\ 0 & 1 & -6 \\ 0 & 0 & 1 \end{pmatrix} \qquad \frac{-1}{90}d$$

$$\begin{pmatrix} 5 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \qquad \frac{-1}{90}d$$

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \qquad \frac{-1}{450}d$$

Thus we have det $I_3 = 1 = \frac{-1}{450}d$, which yields det A = d = -450.

In considering solutions for systems of linear equations we found it convenient to write the systems as matrix equations such as A**x** = **b** where **x** and **b** are vectors, **x** collecting the unknowns and **b** the constants which comprise the right hand side of each equation. This approach provides a way to collect large amounts of data and gives an overview of the problem. Suppose that the coefficient matrix A is a square matrix, say A has size n by n so that the system consists of n equations in n unknowns. We have seen that the system has a *unique solution* exactly when A row reduces to the identity matrix I_n.

Considering the 2 × 2 case, let A = $\begin{pmatrix} a & c \\ b & d \end{pmatrix}$. Under what conditions does A reduce to I_2 so that

the system has a unique solution? Assuming a ... 0, the reduction yields the following matrices:

$$\begin{pmatrix} a & c \\ b & d \end{pmatrix} \quad \begin{pmatrix} 1 & \frac{c}{a} \\ b & d \end{pmatrix} \quad \begin{pmatrix} 1 & \frac{c}{a} \\ 0 & d-\frac{bc}{a} \end{pmatrix}$$

For this to reduce to I_2 then we must have $d-\frac{bc}{a}=\frac{ad-bc}{a}$ ...0, so ad - bc ... 0. If a = 0, then b must necessarily be nonzero and the reduction looks like

$$\begin{pmatrix} a & c \\ b & d \end{pmatrix}\begin{pmatrix} b & d \\ a & c \end{pmatrix}\begin{pmatrix} 1 & \frac{d}{b} \\ a & c \end{pmatrix}\begin{pmatrix} 1 & \frac{d}{b} \\ 0 & c-\frac{ad}{b} \end{pmatrix}$$

Here we must have $c-\frac{ad}{b}=\frac{bc-ad}{b}$ ...0 so bc - ad ... 0, or, as before, ad - bc ...0. The number ad - bc determines the form of the result. How about the 3 × 3 case? Set S = $\begin{pmatrix} a & b & e \\ c & d & f \\ g & h & i \end{pmatrix}$.

With appropriate quantities being nonzero the reduction proceeds as indicated below.

$$\begin{pmatrix} a & b & e \\ c & d & f \\ g & h & i \end{pmatrix} \quad \begin{pmatrix} 1 & \frac{b}{a} & \frac{e}{a} \\ 0 & \frac{ad-bc}{a} & f-\frac{ec}{a} \\ 0 & h-\frac{bg}{a} & i-\frac{eg}{a} \end{pmatrix} \quad \begin{pmatrix} 1 & * & * \\ 0 & 1 & \frac{a}{ad-bc}\frac{fa-ec}{a} \\ 0 & 0 & X \end{pmatrix}$$

Note that the 2 ×2 matrix in the upper lefthand corner is simply the 2 × 2 matrix we reduced above (with its 2,2 entry rewritten as a single fraction), that asterisks have been placed in entries that are of no consequence, and that reduction to I_3 is dependent on X being nonzero where

$$X = (i - \frac{eg}{a}) - \frac{fa - ec}{ad - bc} \cdot \left(h - \frac{bg}{a} \right) = \frac{ia - eg}{a} - \frac{fa - ec}{ad - bc} \cdot \frac{ha - bg}{a}$$

$$= \frac{(ia - eg)(ad - bc) + (ce - fa)(ha - bg)}{a(ad - bc)} = \frac{ia^2 d - iabc - egad + egbc + ceha - cebg - faha + fabg}{a(ad - bc)}$$

$$= \frac{a(iad - ibc + hec - hfa + gbf - gde)}{a(ad - bc)} = \frac{i(ad - bc) + h(ec - fa) + g(bf - de)}{ad - bc}$$

Hence, we get a unique solution exactly when the number i(ad - bc) + h(ec - fa) + g(bf - de) is not 0. All the other cases in this 3×3 problem, e.g. when a = 0 or ad - bc = 0, yield this same result.

Patterns that begin to emerge from these beginning cases lead to two different formulas for computing this number associated with the square matrix A, its *determinant*.

First note that each term in each sum, both the sum in the 2×2 problem and the sum from the 3×3 case, is the product of entries from A, exactly one entry from each row and one from each column in each product. Also the sum consists of exactly every possible product with exactly half of them also having a minus sign. Viewing the determinant this way leads to the *permutation formula* that we explain below.

From a slightly different angle, the formulas arrived at for the 2×2 and 3×3, namely ad - bc and i(ad - bc) + h(ec - fa) + g(bf - de), respectively, both consist of sums of products with the first factors being the entries from a particular row. In the 2×2 case the a and b make up the first row of A: for the 3×3 case it is the i, h, and g which is the third row of A. Examining the term

multiplying i, it is readily recognized as the determinant of the 2×2 upper lefthand corner of A. In fact, each of the quantities in the parentheses is (plus or minus) the determinant of the submatrix of A obtained by deleting the row and column of the entry which multiplies that group. This is also true of the 2×2 case if we interpret the determinant of a 1×1 matrix as being its lone entry. The *recursive formula* follows from this viewpoint.

Recursive Formula for the Determinant

If A is a 1×1 matrix, then set det A = a_{11}, the only element of A. Suppose that A is an n×n matrix with n > 1. Let A = (a_{ij}) and, for the i,jth entry of A, define the *minor* of a_{ij}, denoted M_{ij}, to be the square matrix formed by deleting the ith row and the jth column of A. The *cofactor* of a_{ij}, C_{ij}, is the determinant of M_{ij}. Then det A = $a_{11}C_{11} - a_{12}C_{12} + a_{13}C_{13} - ...(-1)^{1+n}a_{1n}C_{1n}$. This is called the *cofactor expansion* of A about its first row.

<u>Example</u>

Consider the matrix A = $\begin{pmatrix} 5 & -3 & -1 & 7 \\ 10 & 8 & -6 & 20 \\ 0 & 0 & 2 & 0 \\ 0 & 1 & 0 & -6 \end{pmatrix}$. The cofactor expansion of A about the first row yields the determinant of A is

$$5\det\begin{pmatrix} 8 & -6 & 20 \\ 0 & 2 & 0 \\ 1 & 0 & -6 \end{pmatrix} - (-3)\det\begin{pmatrix} 10 & -6 & 20 \\ 0 & 2 & 0 \\ 0 & 0 & -6 \end{pmatrix} + (-1)\det\begin{pmatrix} 10 & 8 & 20 \\ 0 & 0 & 0 \\ 0 & 1 & -6 \end{pmatrix} - 7\det\begin{pmatrix} 10 & 8 & -6 \\ 0 & 0 & 2 \\ 0 & 1 & 0 \end{pmatrix}$$

Demonstrating the recursion in the formula we compute (again using expansion about the first row) the cofactor associated with the 1,1 entry 5.

$$\det\begin{pmatrix} 8 & -6 & 20 \\ 0 & 2 & 0 \\ 1 & 0 & -6 \end{pmatrix} = 8 \det\begin{pmatrix} 2 & 0 \\ 0 & -6 \end{pmatrix} - (-6) \det\begin{pmatrix} 0 & 0 \\ 1 & -6 \end{pmatrix} + 20 \det\begin{pmatrix} 0 & 2 \\ 1 & 0 \end{pmatrix}$$

$$= 8(-12) + 6(0) + 20(-2) = -96 + 0 - 40 = -136$$

Filling in the values for all these cofactors we get

det A = 5(-136) + 3(-120) - 1(0) - 7(-20) = -680 - 360 + 140 = -900

Note that the cofactor of the 1,3 entry (-1) is zero since we could expand about the second row using the fact stated below. In fact, our task would have been simpler had we expanded the original matrix A about row 3. With the remaining entries all bing zeros, we would have just gotten $(-1)^{3+3}$ 2 det $\begin{pmatrix} 5 & -3 & 7 \\ 10 & 8 & 20 \\ 0 & 1 & -6 \end{pmatrix}$ = 2(-450) since this is the matrix whose determinant we computed in the previous example.

Fact: The determinant can be computed by expanding about any row or column, i.e.

Det A = $\sum_{j=1}^{n}(-1)^{i+j} a_{ij} C_{ij} = \sum_{i=1}^{n}(-1)^{i+j} a_{ij} C_{ij}$ the first sum is the expansion about row i,

the second the expansion about column j.

Permutation Formula for the Determinant

Again suppose that A = (a_{ij}) is an n×n matrix. Let σ be a permutation of the set {1, 2, ...,n}, i.e. {1, 2, ...,n} = {σ(1), σ(2), ...σ(n)}. (What we have is σ is as bijection, a one-to-one, onto function from {1, 2, ...,n} to itself.) For any permutation σ, we define the *sign* of σ, sg σ, by counting the number of moves required to rearrange the sequence σ(1)σ(2)...σ(n) in order to form the sequence 123...n. If this number is even then sg σ = 1, while if this number is odd set sg σ = -1.

Finally an alternate, equivalent definition of determinant is

det A = $\sum sg(\sigma)a_{1\sigma(1)}a_{2\sigma(2)}\ldots a_{n\sigma(n)}$ where the sum is taken over all the permutations of {1, 2, ...,n}. Since there are n! (n factorial) permutation or rearrangements of {1, 2, ...,n}, Strang refers to this as the *Big Formula*.

<u>Example</u>

Let A = $\begin{pmatrix} 2 & 3 & 4 \\ 5 & 6 & 7 \\ 8 & 9 & 0 \end{pmatrix}$. The permutations of {1, 2, 3} are 123, 231, 312,132, 213, and 321. The

first three of these are even, the last three odd. Thus, using the permutation formula,

det A = $a_{11}a_{22}a_{33} + a_{12}a_{23}a_{31} + a_{13}a_{21}a_{32} - a_{11}a_{23}a_{32} - a_{12}a_{21}a_{33} - a_{13}a_{22}a_{31}$

= 2@6@0 + 3@7@8 + 4@5@9 - 2@7@9 - 3@5@0 - 4@6@8

= 30

Applications

The fact that det A … 0 was a necessary and sufficient condition for 2×2 and 3×3 systems with coefficient matrix A to have a unique solution holds for any n which we record in the following **<u>Theorem</u>**.

Let A be an n×n matrix.

1. The equation A**x** = **b** has a unique solution (for any **b** in $\mathbf{R}^n$) if and only if det A … 0.
2. A is invertible if and only if det A … 0.

Besides signaling the existence of an inverse for A, the determinant is used in a formula for A^{-1}.

$A^{-1} = \dfrac{1}{\det A} C^T$ where C = (C_{ij}) is the n×n matrix of cofactors of A (the i, j^{th} entry of C, C_{ij} , is the i,j^{th} cofactor of A.) So A^{-1} consists of the transpose of the matrix of cofactors of A

times the scalar $\frac{1}{\det A}$.

Collecting some properties of the determinant we note that for any n×n matrices A and B and any scalar c:

1. det (AB) = det A det B,
2. det cA = c^n det A,
3. det A^T = det A, and
4. det A^{-1} = det A.

Incorporating definitions and results from previous sections, we gather a number of conditions that are equivalent for square matrices in the theorem below.

<u>Theorem</u>

Suppose that A is an n×n matrix. The following are equivalent.

1. The determinant of A is not 0.
2. A is invertible.
3. A reduces to the identity matrix, I_n.
4. The equation A**x** = **b** has a unique solution for each **b** in $\mathbf{R}^n$.
5. The equation A**x** = **0** has only the trivial solution.
6. The rank of A is n.
7. The dimension of the row space of A is n.
8. The dimension of the column space of A is n.
9. The dimension of the null space of A is 0.
10. The columns of A are linearly independent.
11. The columns of A span $\mathbf{R}^n$.
12. The rows of A are linearly independent.
13. The rows of S span $\mathbf{R}^n$.