Diagonalizing a Matrix

Definition

1. We say that two square matrices ***A*** and ***B*** are ***similar*** provided there exists an invertible matrix ***P*** so that $B = P^{-1} A P$.
2. We say a matrix ***A*** is ***diagonalizable*** if it is similar to a diagonal matrix.

Example

1. The matrices $\begin{pmatrix} 1 & 1 \\ -2 & 4 \end{pmatrix}$ and $\begin{pmatrix} 2 & 0 \\ 0 & 3 \end{pmatrix}$ are similar matrices since

$$\begin{pmatrix} 2 & 0 \\ 0 & 3 \end{pmatrix} = \begin{pmatrix} 1 & 1 \\ 1 & 2 \end{pmatrix}^{-1} \begin{pmatrix} 1 & 1 \\ -2 & 4 \end{pmatrix} \begin{pmatrix} 1 & 1 \\ 1 & 2 \end{pmatrix}.$$

We conclude that $\begin{pmatrix} 1 & 1 \\ -2 & 4 \end{pmatrix}$ is diagonalizable.

2. The matrices $\begin{pmatrix} 2 & 1 & 0 \\ -1 & 0 & 1 \\ 1 & 3 & 1 \end{pmatrix}$ and $\begin{pmatrix} 2 & 1 & -1 \\ 10 & 4 & -7 \\ 6 & 2 & -3 \end{pmatrix}$ are similar matrices since

$$\begin{pmatrix} 2 & 1 & -1 \\ 10 & 4 & -7 \\ 6 & 2 & -3 \end{pmatrix} = \begin{pmatrix} -1 & -1 & 2 \\ 2 & 1 & -2 \\ 1 & 1 & -1 \end{pmatrix}^{-1} \begin{pmatrix} 2 & 1 & 0 \\ -1 & 0 & 1 \\ 1 & 3 & 1 \end{pmatrix} \begin{pmatrix} -1 & -1 & 2 \\ 2 & 1 & -2 \\ 1 & 1 & -1 \end{pmatrix}.$$

After we have developed some additional theory, we will be able to conclude that the matrices $\begin{pmatrix} 2 & 1 & 0 \\ -1 & 0 & 1 \\ 1 & 3 & 1 \end{pmatrix}$ and $\begin{pmatrix} 2 & 1 & -1 \\ 10 & 4 & -7 \\ 6 & 2 & -3 \end{pmatrix}$ are *not* diagonalizable.

Theorem

Suppose ***A***, ***B*** and ***C*** are square matrices.

(1) ***A*** is similar to ***A***.

(2) If ***A*** is similar to ***B***, then ***B*** is similar to ***A***.

(3) If ***A*** is similar to ***B*** and if ***B*** is similar to ***C***, then ***A*** is similar to ***C***.

Proof of (3)

Since ***A*** is similar to ***B***, there exists an invertible matrix ***P*** so that $B = P^{-1} A P$. Also, since ***B*** is similar to ***C***, there exists an invertible matrix ***R*** so that $C = R^{-1} B R$. Now,

$$C = R^{-1} B R = R^{-1} \left(P^{-1} A P\right) R = \left(R^{-1} P^{-1}\right) A \left(P R\right) = \left(P R\right)^{-1} A \left(P R\right)$$

and so ***A*** is similar to ***C***.

Thus, "***A*** is similar to ***B***" is an equivalence relation.

Theorem

If ***A*** is similar to ***B***, then ***A*** and ***B*** have the same eigenvalues.

Proof

Since ***A*** is similar to ***B***, there exists an invertible matrix ***P*** so that $B = P^{-1} A P$. Now,

$$\begin{aligned}
Det\left(B - \lambda I_n\right) &= Det\left(P^{-1}AP - \lambda I_n\right) \\
&= Det\left(P^{-1}AP - P^{-1}\left(\lambda I_n\right)P\right) \\
&= Det\left(P^{-1}\left(AP - \left(\lambda I_n\right)P\right)\right) \\
&= Det\left(P^{-1}\left(A - \lambda I_n\right) P\right) \\
&= Det\left(P^{-1}\right) Det\left(A - \lambda I_n\right) Det\left(P\right) \\
&= Det\left(P^{-1}\right) Det\left(P\right) Det\left(A - \lambda I_n\right) \\
&= Det\left(P^{-1}P\right) Det\left(A - \lambda I_n\right) \\
&= Det\left(I_n\right) Det\left(A - \lambda I_n\right) \\
&= Det\left(A - \lambda I_n\right).
\end{aligned}$$

Since $\boldsymbol{A}$ and $\boldsymbol{B}$ have the same characteristic equation, they have the same eigenvalues. >

Example

Find the eigenvalues for $\begin{pmatrix} 1 & 1 \\ -2 & 4 \end{pmatrix}$.

Solution

Since $\begin{pmatrix} 1 & 1 \\ -2 & 4 \end{pmatrix}$ is similar to the diagonal matrix $\begin{pmatrix} 2 & 0 \\ 0 & 3 \end{pmatrix}$, they have the same eigenvalues.

Because the eigenvalues of an upper (or lower) triangular matrix are the entries on the main diagonal, we see that the eigenvalues for $\begin{pmatrix} 2 & 0 \\ 0 & 3 \end{pmatrix}$, and, hence, $\begin{pmatrix} 1 & 1 \\ -2 & 4 \end{pmatrix}$ are

$\lambda = 2 \quad \& \quad \lambda = 3$. As a check we observe that

$$\begin{vmatrix} 1-\lambda & 1 \\ -2 & 4-\lambda \end{vmatrix} = (1-\lambda)(4-\lambda) - (1)(-2)$$

$$= \lambda^2 - 5\lambda + 6$$

$$= (\lambda - 2)(\lambda - 3).$$

Continuing with the above example, we can show that an eigenvector associated with $\lambda = 2$ is $\begin{pmatrix} 1 \\ 1 \end{pmatrix}$ and an eigenvector associated with $\lambda = 3$ is $\begin{pmatrix} 1 \\ 2 \end{pmatrix}$. Recall that

$$\begin{pmatrix} 2 & 0 \\ 0 & 3 \end{pmatrix} = P^{-1} \begin{pmatrix} 1 & 1 \\ -2 & 4 \end{pmatrix} P$$

where $P = \begin{pmatrix} 1 & 1 \\ 1 & 2 \end{pmatrix}$. Observe that the columns of $\boldsymbol{P}$ are the linearly independent eigenvectors for $\begin{pmatrix} 1 & 1 \\ -2 & 4 \end{pmatrix}$. We note that interchanging the two columns in ***P*** to obtain a new matrix ***P*** and computing $P^{-1} \begin{pmatrix} 1 & 1 \\ -2 & 4 \end{pmatrix} P$ yields

$$\begin{pmatrix} 1 & 1 \\ 2 & 1 \end{pmatrix}^{-1} \begin{pmatrix} 1 & 1 \\ -2 & 4 \end{pmatrix} \begin{pmatrix} 1 & 1 \\ 2 & 1 \end{pmatrix} = \begin{pmatrix} 3 & 0 \\ 0 & 2 \end{pmatrix}$$

which is another diagonal matrix with eigenvalues of $\lambda = 2 \;\; \& \;\; \lambda = 3$.

The next result characterizes matrices that are diagonalizable.

<u>Theorem</u>

An $n \times n$ matrix ***A*** is diagonalizable if and only if ***A*** has *n* linearly independent eigenvectors.

<u>Proof</u>

(Z) Assume ***A*** has *n* linearly independent eigenvectors. Assume these *n* eigenvectors are column vectors $v_1, v_2, v_3, \ldots, v_n$ with associated eigenvectors $\lambda_1, \lambda_2, \lambda_3, \ldots, \lambda_n$. Define $S = (v_1 \vdots v_2 \vdots v_3 \vdots \ldots \vdots v_n)$ and let Λ be the diagonal matrix with ii-entry equal to λ_i.

Since the columns of ***S*** are linearly independent, ***S*** is invertible. Now,

$$\begin{aligned} A\,S &= A\,(v_1 \vdots v_2 \vdots v_3 \vdots \ldots \vdots v_n) \\ &= (A\,v_1 \vdots A\,v_2 \vdots A\,v_3 \vdots \ldots \vdots A\,v_n) \\ &= (\lambda_1 v_1 \vdots \lambda_2 v_2 \vdots \lambda_3 v_3 \vdots \ldots \vdots \lambda_n v_n) \\ &= (v_1 \vdots v_2 \vdots v_3 \vdots \ldots \vdots v_n)\,\Lambda \\ &= S\,\Lambda. \end{aligned}$$

Since $A\,S = S\,\Lambda$, it follows that $S^{-1} A\,S = \Lambda$. Thus, ***A*** and Λ are similar and so A is diagonalizable. >

The next result gives us sufficient conditions for a matrix to be diagonalizable.

Theorem

Let $\boldsymbol{A}$ be an $\boldsymbol{n} \times \boldsymbol{n}$ matrix . If $\boldsymbol{A}$ has eigenvalues that are real and distinct, then $\boldsymbol{A}$ is diagonalizable.

Example

Determine if the following matrices are diagonalizable.

1. $\begin{pmatrix} 0 & -1 \\ 2 & 3 \end{pmatrix}$

Solution

Since $\begin{vmatrix} -\lambda & -1 \\ 2 & 3-\lambda \end{vmatrix} = (-\lambda)(3-\lambda) - (-1)(2) = \lambda^2 - 3\lambda + 2 = (\lambda - 2)(\lambda - 3)$, the given matrix $\begin{pmatrix} 0 & -1 \\ 2 & 3 \end{pmatrix}$ has distinct real eigenvalues of $\lambda = 2$ & $\lambda = 3$. Thus, $\begin{pmatrix} 0 & -1 \\ 2 & 3 \end{pmatrix}$ is diagonalizable.

2. $\begin{pmatrix} 1 & -3 & 3 \\ 0 & -5 & 6 \\ 0 & -3 & 4 \end{pmatrix}$

Solution

Here

$$\begin{vmatrix} 1-\lambda & -3 & 3 \\ 0 & -5-\lambda & 6 \\ 0 & -3 & 4-\lambda \end{vmatrix} = -\lambda^3 + 3\lambda - 2 = -(\lambda - 1)^2(\lambda + 2)$$

and so we have repeated eigenvalues. So, we must go ahead and find the associated eigenvectors for $\lambda = 1 \quad \& \quad \lambda = -2$ and determine if they are linearly independent. We seek the null space for

$$\begin{pmatrix} 0 & -3 & 3 \\ 0 & -6 & 6 \\ 0 & -3 & 3 \end{pmatrix}$$

corresponding to $\lambda = 1$. Since

$$\begin{pmatrix} 0 & -3 & 3 \\ 0 & -6 & 6 \\ 0 & -3 & 3 \end{pmatrix} \sim \begin{pmatrix} 0 & 1 & -1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix},$$

the null space has a basis of $\left\{ \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix} \right\}$.

The null space for

$$\begin{pmatrix} 3 & -3 & 3 \\ 0 & -3 & 6 \\ 0 & -3 & 6 \end{pmatrix}$$

corresponding to $\lambda = -2$ is obtained by

$$\begin{pmatrix} 3 & -3 & 3 \\ 0 & -3 & 6 \\ 0 & -3 & 6 \end{pmatrix} \sim \begin{pmatrix} 1 & 0 & -1 \\ 0 & 1 & -2 \\ 0 & 0 & 0 \end{pmatrix}.$$

From this row reduced matrix we see that a basis for the null space is $\left\langle \begin{pmatrix} 1 \\ 2 \\ 1 \end{pmatrix} \right\rangle$.

We close this section with a result that can simplify the computation of powers of a square matrix ***A***.

<u>Theorem</u>

Suppose that ***A*** is diagonalizable, say $P^{-1}A\,P = D$, a diagonal matrix. Then

$$A^{k} = PD^{k}P^{-1} \text{ for all } k \in \mathbb{N}.$$

<u>Proof</u>

For $k = 1$ we have that $A = PDP^{-1}$. Assume $A^{k} = PD^{k}P^{-1}$ holds and show that $A^{k+1} = PD^{k+1}P^{-1}$. Now,

$$A^{k+1} = A^{k}A = \left(PD^{k}P^{-1}\right)\left(PDP^{-1}\right) = PD^{k+1}P^{-1}.$$

So, by mathematical induction, $A^{k} = PD^{k}P^{-1}$ for all natural numbers *k*. >