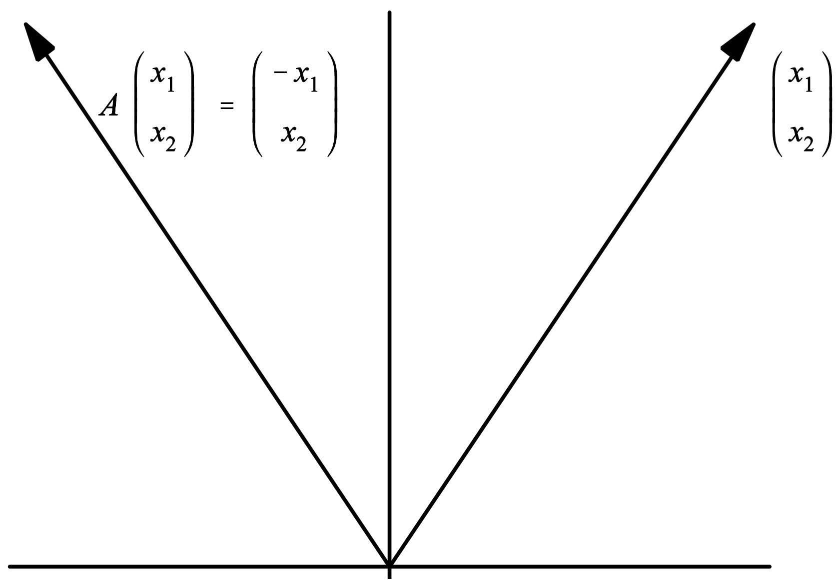
Eigenvalues & Eigenvectors

Example

Suppose $A = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}$. Then $A\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} -x_1 \\ x_2 \end{pmatrix}$. So, geometrically,

multiplying a vector in $\mathbb{R}^2$ by the matrix $\boldsymbol{A}$ results in a vector which is a reflection of the given vector about the *y*-axis.



We observe that

$$A\begin{pmatrix} x_1 \\ 0 \end{pmatrix} = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} x_1 \\ 0 \end{pmatrix} = \begin{pmatrix} -x_1 \\ 0 \end{pmatrix} = -1\begin{pmatrix} x_1 \\ 0 \end{pmatrix}$$

and

$$A\begin{pmatrix} 0 \\ x_2 \end{pmatrix} = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} 0 \\ x_2 \end{pmatrix} = \begin{pmatrix} 0 \\ x_2 \end{pmatrix}.$$

Thus, vectors on the coordinate axes get mapped to vectors on the same coordinate axis. That is, for vectors on the coordinate axes we see that $A \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$ and $\begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$ are parallel or, equivalently, for vectors on the coordinate axes there exists a scalar λ so that $A \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \lambda \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$. In particular, $\lambda = -1$ for vectors on the x-axis and $\lambda = 1$ for vectors on the y-axis. Given the geometric properties of $A = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}$ we see that $A \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \lambda \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$ has solutions only when $\begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$ is on one of the coordinate axes.

Definition

Let $\boldsymbol{A}$ be an $\boldsymbol{n} \times \boldsymbol{n}$ matrix. We call a scalar λ an ***eigenvalue*** of $\boldsymbol{A}$ provided there exists a nonzero n-vector $\boldsymbol{x}$ so that $\boldsymbol{A}\,\boldsymbol{x} = \lambda\,\boldsymbol{x}$. In this case, we call the n-vector $\boldsymbol{x}$ an ***eigenvector*** of $\boldsymbol{A}$ corresponding to λ.

We note that $\boldsymbol{A}\,\boldsymbol{x} = \lambda\,\boldsymbol{x}$ is true for all λ in the case that $\boldsymbol{x} = 0_{n \times 1}$ and, hence, is not particularly interesting. We do allow for the possibility that $\lambda = 0$.

Eigenvalues are also called ***proper values*** ("eigen" is German for the word "own" or "proper") or ***characteristic values*** or ***latent values***. Eigenvalues were initial used by Leonhard Euler in 1743 in connection with the solution to an n^{th} order linear differential equation with constant coefficients.

Geometrically, the equation $\boldsymbol{A}\,\boldsymbol{x} = \lambda\,\boldsymbol{x}$ implies that the n-vectors $\boldsymbol{A}\,\boldsymbol{x}$ & $\boldsymbol{x}$ are parallel.

Example

Suppose $A = \begin{pmatrix} 1 & 1 \\ -2 & 4 \end{pmatrix}$. Then $\begin{pmatrix} 6 \\ 6 \end{pmatrix}$ is an *eigenvector* for $\boldsymbol{A}$ corresponding to the *eigenvalue* of $\boldsymbol{\lambda} = 2$ as

$$A \begin{pmatrix} 6 \\ 6 \end{pmatrix} = \begin{pmatrix} 1 & 1 \\ -2 & 4 \end{pmatrix} \begin{pmatrix} 6 \\ 6 \end{pmatrix} = \begin{pmatrix} 12 \\ 12 \end{pmatrix} = 2 \begin{pmatrix} 6 \\ 6 \end{pmatrix}.$$

In fact, by direct computation, any vector of the form $\begin{pmatrix} k \\ k \end{pmatrix}$ is an eigenvector for $\boldsymbol{A}$ corresponding to $\boldsymbol{\lambda} = 2$. We also see that $\begin{pmatrix} 2 \\ 4 \end{pmatrix}$ is an *eigenvector* for $\boldsymbol{A}$ corresponding to the *eigenvalue* $\boldsymbol{\lambda} = 3$ since

$$A \begin{pmatrix} 2 \\ 4 \end{pmatrix} = \begin{pmatrix} 1 & 1 \\ -2 & 4 \end{pmatrix} \begin{pmatrix} 2 \\ 4 \end{pmatrix} = \begin{pmatrix} 6 \\ 12 \end{pmatrix} = 3 \begin{pmatrix} 2 \\ 4 \end{pmatrix}.$$

Suppose $\boldsymbol{A}$ is an $\boldsymbol{n} \times \boldsymbol{n}$ matrix and $\boldsymbol{\lambda}$ is a eigenvalue of $\boldsymbol{A}$. If $\boldsymbol{x}$ is an eigenvector of $\boldsymbol{A}$ corresponding to $\boldsymbol{\lambda}$ and k is any scalar, then

$$\boldsymbol{A} \left(k\boldsymbol{x} \right) = k \left(\boldsymbol{A} \ \boldsymbol{x} \right) = k \left(\lambda \ \boldsymbol{x} \right) = \lambda \left(k\boldsymbol{x} \right).$$

So, any scalar multiple of an eigenvector is also an eigenvector for the given eigenvalue $\boldsymbol{\lambda}$.

Now, if $\boldsymbol{x}_1$ & $\boldsymbol{x}_2$ are both eigenvectors of $\boldsymbol{A}$ corresponding to $\boldsymbol{\lambda}$, then

$$\boldsymbol{A} \left(\boldsymbol{x}_1 + \boldsymbol{x}_2 \right) = \boldsymbol{A} \ \boldsymbol{x}_1 + \boldsymbol{A} \ \boldsymbol{x}_2 = \lambda \ \boldsymbol{x}_1 + \lambda \ \boldsymbol{x}_2 = \lambda \left(\boldsymbol{x}_1 + \boldsymbol{x}_2 \right).$$

Thus, the set of all eigenvectors of $\boldsymbol{A}$ corresponding to given eigenvalue $\boldsymbol{\lambda}$ is closed under scalar multiplication and vector addition. This proves the following result:

Theorem

If $\boldsymbol{A}$ is an $\boldsymbol{n} \times \boldsymbol{n}$ matrix and λ is a eigenvalue of $\boldsymbol{A}$, then the set of all eigenvectors of λ, together with the zero vector, forms a subspace of $\mathbb{R}^n$. We call this subspace the ***eigenspace*** of λ.

Example

Find the eigenvalues and the corresponding eigenspaces for the matrix $\boldsymbol{A} = \begin{pmatrix} 10 & -18 \\ 6 & -11 \end{pmatrix}$.

Solution

We first seek all scalars λ so that $\boldsymbol{A}\,\boldsymbol{x} = \lambda\,\boldsymbol{x}$:

$$\begin{pmatrix} 10 & -18 \\ 6 & -11 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \lambda \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$$

$$\begin{pmatrix} 10 & -18 \\ 6 & -11 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \lambda \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$$

$$\begin{pmatrix} 10 & -18 \\ 6 & -11 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} - \lambda \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

$$\begin{pmatrix} 10 - \lambda & -18 \\ 6 & -11 - \lambda \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}.$$

The above has nontrivial solutions precisely when $\begin{pmatrix} 10 - \lambda & -18 \\ 6 & -11 - \lambda \end{pmatrix}$ is singular. That is,

the above matrix equation has nontrivial solutions when

$$Det\begin{pmatrix} 10 - \lambda & -18 \\ 6 & -11 - \lambda \end{pmatrix} = \begin{vmatrix} 10 - \lambda & -18 \\ 6 & -11 - \lambda \end{vmatrix}$$

$$= (10 - \lambda)(-11 - \lambda) - (-18)6$$

$$= \lambda^2 + \lambda - 2$$

$$= (\lambda + 2)(\lambda - 1)$$

$$= 0.$$

Thus, the eigenvalues for $\boldsymbol{A} = \begin{pmatrix} 10 & -18 \\ 6 & -11 \end{pmatrix}$ are $\lambda = -2$ & $\lambda = 1$. Since

$$\begin{pmatrix} 10 & -18 \\ 6 & -11 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \lambda\begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$$

implies

$$\begin{pmatrix} 10 - \lambda & -18 \\ 6 & -11 - \lambda \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix},$$

the eigenspace of $\boldsymbol{A} = \begin{pmatrix} 10 & -18 \\ 6 & -11 \end{pmatrix}$ corresponding to $\lambda = -2$ is the null space of

$$\begin{pmatrix} 10 - (-2) & -18 \\ 6 & -11 - (-2) \end{pmatrix} = \begin{pmatrix} 12 & -18 \\ 6 & -9 \end{pmatrix}.$$

Because

$$\begin{pmatrix} 12 & -18 \\ 6 & -9 \end{pmatrix} \sim \begin{pmatrix} 2 & -3 \\ 0 & 0 \end{pmatrix}$$

we see that the null space of $\begin{pmatrix} 12 & -18 \\ 6 & -9 \end{pmatrix}$ is given by $\left\langle \begin{pmatrix} 3 \\ 2 \end{pmatrix} \right\rangle$. In a similar manner, the

eigenspace for $\lambda = 1$ is the null space of $\begin{pmatrix} 10 - (1) & -18 \\ 6 & -11 - (1) \end{pmatrix} = \begin{pmatrix} 9 & -18 \\ 6 & -12 \end{pmatrix}$ which is given by $\left\langle \begin{pmatrix} 2 \\ 1 \end{pmatrix} \right\rangle$.

Finding Eigenvalues:

Let $\boldsymbol{A}$ be an $\boldsymbol{n} \times \boldsymbol{n}$ matrix. Then

$$\boldsymbol{A}\,\boldsymbol{x} = \lambda\,\boldsymbol{x}$$

$$\Rightarrow \quad \boldsymbol{A}\,\boldsymbol{x} = \lambda\,\boldsymbol{I_n}\,\boldsymbol{x}$$

$$\Rightarrow \quad \boldsymbol{A}\,\boldsymbol{x} - \lambda\,\boldsymbol{I_n}\,\boldsymbol{x} = 0_{n \times 1}$$

$$\Rightarrow \quad \left(\boldsymbol{A} - \lambda\,\boldsymbol{I_n}\right)\boldsymbol{x} = 0_{n \times 1}.$$

It follows that

λ is an eigenvalue of $\boldsymbol{A}$ if and only if $\left(\boldsymbol{A} - \lambda\,\boldsymbol{I_n}\right)\boldsymbol{x} = 0_{n \times 1}$

if and only if $Det\left(\boldsymbol{A} - \lambda\,\boldsymbol{I_n}\right) = 0$.

<u>**Theorem**</u>

Let $\boldsymbol{A}$ be an $\boldsymbol{n} \times \boldsymbol{n}$ matrix. Then λ is an eigenvalue of $\boldsymbol{A}$ if and only if $Det\left(\boldsymbol{A} - \lambda\,\boldsymbol{I_n}\right) = 0$. Further, $Det\left(\boldsymbol{A} - \lambda\,\boldsymbol{I_n}\right) = 0$ is a polynomial in λ of degree n called the ***characteristic polynomial*** of $\boldsymbol{A}$. We call $Det\left(\boldsymbol{A} - \lambda\,\boldsymbol{I_n}\right) = 0$ the ***characteristic equation*** of $\boldsymbol{A}$.

Examples

1. Find the eigenvalues and the corresponding eigenspaces of the 2×2 matrix $\begin{pmatrix} 2 & -1 \\ -4 & 2 \end{pmatrix}$.

Solution

Here

$$\begin{aligned} Det\left(A - \lambda I_n\right) &= Det\begin{pmatrix} 2 - \lambda & -1 \\ -4 & 2 - \lambda \end{pmatrix} \\ &= (2 - \lambda)(2 - \lambda) - (-1)(-4) \\ &= \lambda^2 - 4\lambda \\ &= \lambda(\lambda - 4) \end{aligned}$$

and so the eigenvalues are $\lambda = 0$ & $\lambda = 4$. The eigenspace corresponding to $\lambda = 0$ is just the null space of the given matrix $\begin{pmatrix} 2 & -1 \\ -4 & 2 \end{pmatrix}$ which is $\left\langle \begin{pmatrix} 1 \\ 2 \end{pmatrix} \right\rangle$. The eigenspace corresponding to $\lambda = 4$ is the null space of $\begin{pmatrix} -2 & -1 \\ -4 & -2 \end{pmatrix}$ which is $\left\langle \begin{pmatrix} 1 \\ -2 \end{pmatrix} \right\rangle$.

Note: Here we have two distinct eigenvalues and two linearly independent eigenvectors (as $\begin{pmatrix} 1 \\ 2 \end{pmatrix}$ is not a multiple of $\begin{pmatrix} 1 \\ -2 \end{pmatrix}$). We also see that $\begin{vmatrix} 2 & -1 \\ -4 & 2 \end{vmatrix} = 0\ (4) = 0$.

2. Find the eigenvalues and the corresponding eigenspaces of the 2×2 matrix $\begin{pmatrix} 2 & -1 \\ 5 & -2 \end{pmatrix}$.

Solution

Here

$$Det\left(\boldsymbol{A} - \lambda\, \boldsymbol{I}_n\right) = Det\begin{pmatrix} 2 - \lambda & -1 \\ 5 & -2 - \lambda \end{pmatrix}$$

$$= (2 - \lambda)(-2 - \lambda) - (-1)(5)$$

$$= \lambda^2 + 1$$

and so the eigenvalues are $\lambda = -i$ & $\lambda = i$. (This example illustrates that a matrix with real entries may have complex eigenvalues.) To find the eigenspace corresponding to $\lambda = -i$ we must solve

$$\begin{pmatrix} 2 - (-i) & -1 \\ 5 & -2 - (-i) \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}.$$

As always, we set up an appropriate augmented matrix and row reduce:

$$\left(\begin{array}{cc|c} 2 + i & -1 & 0 \\ 5 & -2 + i & 0 \end{array}\right) \sim \left(\begin{array}{cc|c} 1 & -\frac{2}{5} + \frac{i}{5} & 0 \\ 5 & -2 + i & 0 \end{array}\right)$$

$$\text{Recall: } \frac{-1}{2 + i} = \frac{(-1)(2 - i)}{(2 + i)(2 - i)} = \frac{-2 + i}{2^2 - (-1)} = -\frac{2}{5} + \frac{i}{5}$$

$$\sim \left(\begin{array}{cc|c} 1 & -\frac{2}{5} + \frac{i}{5} & 0 \\ 0 & 0 & 0 \end{array}\right)$$

Hence, $x_1 = \left(\frac{2}{5} - \frac{i}{5}\right) x_2$ and so $\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = t\begin{pmatrix} 2 - i \\ 5 \end{pmatrix}$ for all scalars t.

To find the eigenspace corresponding to $\boldsymbol{\lambda} = i$ we must solve

$$\begin{pmatrix} 2 - (i) & -1 \\ 5 & -2 - (i) \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}.$$

We again set up an appropriate augmented matrix and row reduce:

$$\left(\begin{array}{cc|c} 2 - i & -1 & 0 \\ 5 & -2 - i & 0 \end{array}\right) \sim \left(\begin{array}{cc|c} 1 & -\frac{2}{5} - \frac{i}{5} & 0 \\ 5 & -2 - i & 0 \end{array}\right)$$

$$\sim \left(\begin{array}{cc|c} 1 & -\frac{2}{5} - \frac{i}{5} & 0 \\ 0 & 0 & 0 \end{array}\right)$$

Hence, $x_1 = \left(\frac{2}{5} + \frac{i}{5}\right) x_2$ and so $\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = t \begin{pmatrix} 2 + i \\ 5 \end{pmatrix}$ for all scalars t.

Note: Again, we have two distinct eigenvalues with linearly independent eigenvectors. We also see that $\begin{vmatrix} 2 & -1 \\ 5 & -2 \end{vmatrix} = (-i)(i) = 1$

Fact: Let $\boldsymbol{A}$ be an $\boldsymbol{n} \times \boldsymbol{n}$ matrix with real entries. If $\boldsymbol{\lambda}$ is an eigenvalue of $\boldsymbol{A}$ with associated eigenvector $\boldsymbol{v}$, then $\overline{\boldsymbol{\lambda}}$ is also an eigenvalue of $\boldsymbol{A}$ with associated eigenvector $\overline{\boldsymbol{v}}$.

3. Find the eigenvalues and the corresponding eigenspaces of the 3×3 matrix $\begin{pmatrix} 5 & 4 & 2 \\ 4 & 5 & 2 \\ 2 & 2 & 2 \end{pmatrix}$.

Solution

Here

$$Det\left(A - \lambda I_n\right) = Det\begin{pmatrix} 5-\lambda & 4 & 2 \\ 4 & 5-\lambda & 2 \\ 2 & 2 & 2-\lambda \end{pmatrix}$$

$$= (5-\lambda)\begin{vmatrix} 5-\lambda & 2 \\ 2 & 2-\lambda \end{vmatrix}$$

$$- 4\begin{vmatrix} 4 & 2 \\ 2 & 2-\lambda \end{vmatrix}$$

$$+ 2\begin{vmatrix} 4 & 5-\lambda \\ 2 & 2 \end{vmatrix}$$

$$= (5-\lambda)\left((5-\lambda)(2-\lambda) - (2)(2) \right)$$

$$- 4\left(4(2-\lambda) - (2)(2) \right)$$

$$+ 2\left((4)(2) - (5-\lambda)(2) \right)$$

$$= -\lambda^3 + 12\lambda^2 - 21\lambda + 10$$

$$= -\left(\lambda - 1\right)^2 \left(\lambda - 10\right)$$

Recall: **Rational Root Theorem**

Let $p(x) = a_n x^n + a_{n-1} x^{n-1} + \ldots + a_2 x^2 + a_1 x + a_0 \ \left(a_n \neq 0\right)$ be a polynomial of degree n with integer coefficients. If r and s are relatively prime and $p\left(\frac{r}{s}\right) = 0$, then $r \mid a_0$ and $s \mid a_n$.

For $\boldsymbol{\lambda} = 1$, we obtain

$$\left(\boldsymbol{A} - (1)\,\boldsymbol{I_3}\right)\boldsymbol{x} = 0_{3\times 1}$$

or

$$\begin{pmatrix} 5-1 & 4 & 2 \\ 4 & 5-1 & 2 \\ 2 & 2 & 2-1 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = 0_{3\times 1}$$

or

$$\begin{pmatrix} 4 & 4 & 2 \\ 4 & 4 & 2 \\ 2 & 2 & 1 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = 0_{3\times 1}$$

or

$$\begin{pmatrix} 2 & 2 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = 0_{3\times 1}.$$

So, $\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} -s - \frac{1}{2}t \\ s \\ t \end{pmatrix} = s\begin{pmatrix} -1 \\ 1 \\ 0 \end{pmatrix} + t\begin{pmatrix} -\frac{1}{2} \\ 0 \\ 1 \end{pmatrix}$ and the eigenspace corresponding to

$\boldsymbol{\lambda} = 1$ is given by $\left\langle \begin{pmatrix} -1 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} -1 \\ 0 \\ 2 \end{pmatrix} \right\rangle$.

For $\boldsymbol{\lambda} = 10$, we obtain

$$\left(\boldsymbol{A} - (10)\,\boldsymbol{I_3}\right)\boldsymbol{x} = 0_{3 \times 1}$$

or

$$\begin{pmatrix} 5 - 10 & 4 & 2 \\ 4 & 5 - 10 & 2 \\ 2 & 2 & 2 - 10 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = 0_{3 \times 1}$$

or

$$\begin{pmatrix} -5 & 4 & 2 \\ 4 & -5 & 2 \\ 2 & 2 & -8 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = 0_{3 \times 1}.$$

Hence,

$$\left(\begin{array}{ccc|c} -5 & 4 & 2 & 0 \\ 4 & -5 & 2 & 0 \\ 2 & 2 & -8 & 0 \end{array}\right) \sim \quad \ldots$$

$$\sim \left(\begin{array}{ccc|c} 1 & 0 & -2 & 0 \\ 0 & 1 & -2 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right)$$

and so

$\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 2s \\ 2s \\ s \end{pmatrix} = s \begin{pmatrix} 2 \\ 2 \\ 1 \end{pmatrix}$. The eigenspace corresponding to $\boldsymbol{\lambda} = 10$ is given by $\left\langle \begin{pmatrix} 2 \\ 2 \\ 1 \end{pmatrix} \right\rangle$.

Note: Here we have two distinct eigenvalues with three linearly independent eigenvectors. We see that $\begin{vmatrix} 5 & 4 & 2 \\ 4 & 5 & 2 \\ 2 & 2 & 2 \end{vmatrix} = (1)(1)(10) = 10.$

Examples (details left to the student)

1. Find the eigenvalues and corresponding eigenspaces for $\begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & -3 & 3 \end{pmatrix}$.

Solution

Here

$$\begin{vmatrix} -\lambda & 1 & 0 \\ 0 & -\lambda & 1 \\ 1 & -3 & 3 - \lambda \end{vmatrix} = -\lambda^3 + 3\lambda^2 - 3\lambda + 1 = -(\lambda - 1)^3.$$

The eigenspace corresponding to the lone eigenvalue $\lambda = 1$ is given by $\left\langle \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} \right\rangle$.

Note: Here we have one eigenvalue and one eigenvector. Once again

$$\begin{vmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & -3 & 3 \end{vmatrix} = (1)(1)(1) = 1.$$

2. Find the eigenvalues and the corresponding eigenspaces for $\begin{pmatrix} 2 & 0 & 1 & -3 \\ 0 & 2 & 10 & 4 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 3 \end{pmatrix}$.

Solution

Here

$$\begin{vmatrix} 2-\lambda & 0 & 1 & -3 \\ 0 & 2-\lambda & 10 & 4 \\ 0 & 0 & 2-\lambda & 0 \\ 0 & 0 & 0 & 3-\lambda \end{vmatrix} = \lambda^4 - 9\lambda^3 + 30\lambda^2 - 44\lambda + 24$$

$$= (\lambda - 2)^3 (\lambda - 3)$$

The eigenspace corresponding to $\lambda = 2$ is given by $\left\langle \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \end{pmatrix} \right\rangle$ and the eigenspace

corresponding to $\lambda = 3$ is given by $\left\langle \begin{pmatrix} -3 \\ 4 \\ 0 \\ 1 \end{pmatrix} \right\rangle$.

Note: Here we have two distinct eigenvalues and three linearly independent eigenvectors. Yet again $\begin{vmatrix} 2 & 0 & 1 & -3 \\ 0 & 2 & 10 & 4 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 3 \end{vmatrix} = 2^3 \; (3) = 24.$

Theorem

If $\boldsymbol{A}$ is an $\boldsymbol{n} \times \boldsymbol{n}$ matrix with

$$Det\left(\boldsymbol{A} - \lambda\, \boldsymbol{I_n}\right) = \left(\lambda - r_1\right)^{n_1} \left(\lambda - r_2\right)^{n_2} \left(\lambda - r_3\right)^{n_3} \ldots \left(\lambda - r_k\right)^{n_k},$$

then

$$Det\ A = r_1^{n_1}\, r_2^{n_2}\, r_3^{n_3} \ldots r_k^{n_k}.$$

We note that in the above example the eigenvalues for the matrix $\begin{pmatrix} 2 & 0 & 1 & -3 \\ 0 & 2 & 10 & 4 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 3 \end{pmatrix}$ are

(formally) 2, 2, 2, and 3, the elements along the main diagonal. This is no accident.

Theorem

If $\boldsymbol{A}$ is an $\boldsymbol{n} \times \boldsymbol{n}$ upper (or lower) triangular matrix, the eigenvalues are the entries on its main diagonal.

Definition

Let $\boldsymbol{A}$ be an $\boldsymbol{n} \times \boldsymbol{n}$ matrix and let

$$Det\left(\boldsymbol{A} - \lambda\, \boldsymbol{I_n}\right) = \left(\lambda - r_1\right)^{n_1} \left(\lambda - r_2\right)^{n_2} \left(\lambda - r_3\right)^{n_3} \ldots \left(\lambda - r_k\right)^{n_k}.$$

(1) The numbers $n_1, n_2, n_3, \ldots, n_k$ are the ***algebraic multiplicities*** of the eigenvalues $r_1, r_2, r_3, \ldots, r_k$, respectively.

(2) The ***geometric multiplicity*** of the eigenvalue r_j $(j = 1, 2, 3, \ldots, k)$ is the dimension of the null space $A - r_j I_n$ $(j = 1, 2, 3, \ldots, k)$.

Example

1. The table below gives the algebraic and geometric multiplicity for each eigenvalue of the 2×2 matrix $\begin{pmatrix} 2 & -1 \\ -4 & 2 \end{pmatrix}$:

Eigenvalue	*Algebraic Multiplicity*	*Geometric Multiplicity*
0	1	1
4	1	1

2. The table below gives the algebraic and geometric multiplicity for each eigenvalue of the 3×3 matrix $\begin{pmatrix} 5 & 4 & 2 \\ 4 & 5 & 2 \\ 2 & 2 & 2 \end{pmatrix}$:

Eigenvalue	*Algebraic Multiplicity*	*Geometric Multiplicity*
1	2	2
10	1	1

3. The table below gives the algebraic and geometric multiplicity for each eigenvalue of the 3×3 matrix $\begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & -3 & 3 \end{pmatrix}$:

Eigenvalue	*Algebraic Multiplicity*	*Geometric Multiplicity*
1	3	1

4. The table below gives the algebraic and geometric multiplicity for each eigenvalue of the 4×4 matrix $\begin{pmatrix} 2 & 0 & 1 & -3 \\ 0 & 2 & 10 & 4 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 3 \end{pmatrix}$:

Eigenvalue	*Algebraic Multiplicity*	*Geometric Multiplicity*
2	3	2
3	1	1

The above examples suggest the following theorem:

Theorem

Let $\boldsymbol{A}$ be an $\boldsymbol{n} \times \boldsymbol{n}$ matrix with eigenvalue $\boldsymbol{\lambda}$. Then the geometric multiplicity of $\boldsymbol{\lambda}$ is less than or equal to the algebra multiplicity of $\boldsymbol{\lambda}$.