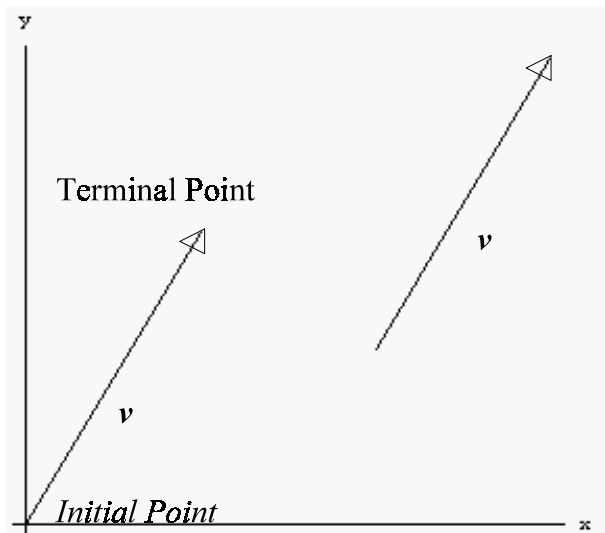


1 - Introduction to Vectors

Definition

A **vector** \mathbf{v} in the plane \mathbb{R}^2 is an ordered pair of real numbers. We denote \mathbf{v} by $\begin{pmatrix} v_1 \\ v_2 \end{pmatrix}$ or $(v_1 \ v_2)$.

The term vector comes from the Latin word *vectus*, meaning “to carry.” This leads nicely to the *geometric representation* of a vector in \mathbb{R}^2 as a directed line segment from the origin to the point (v_1, v_2) . That is, one might envision an object being carried from the origin to the terminal point located at (v_1, v_2) . We regard any directed line segment from initial point (a, b) to the terminal point $(a + v_1, b + v_2)$ as equivalent



to the directed line segment from the origin to (v_1, v_2) . So, just as the rational number $\frac{1}{2}$ has many different equivalent representatives $\left(\text{e.g., } \frac{1}{2}, \frac{5}{10}, \frac{-12}{-24}, \dots \right)$, a given vector \mathbf{v} also has many equivalent directed line segments which may be used to stand for the given vector.

Problem

Suppose $\mathbf{v} = \begin{pmatrix} 3 \\ 7 \end{pmatrix}$. Find the terminal point (x, y) for the directed line segment representing \mathbf{v} if the initial point is $(0, 0)$. Repeat for initial points of $(5, 1)$, $(-2, 4)$, and $(-3, -7)$.

Basic Vector Algebra in \mathbb{R}^2

1. *Vector Equality*: Two vectors $\mathbf{v} = \begin{pmatrix} v_1 \\ v_2 \end{pmatrix}$ and $\mathbf{u} = \begin{pmatrix} u_1 \\ u_2 \end{pmatrix}$ are *equal* if and only if $v_1 = u_1$ and $v_2 = u_2$.

2. *Vector Addition*: The *sum* of the vectors $\mathbf{v} = \begin{pmatrix} v_1 \\ v_2 \end{pmatrix}$ and $\mathbf{u} = \begin{pmatrix} u_1 \\ u_2 \end{pmatrix}$ is defined by

$$\mathbf{v} + \mathbf{u} \equiv \begin{pmatrix} v_1 + u_1 \\ v_2 + u_2 \end{pmatrix}.$$

3. *Scalar Multiplication*: Suppose $\mathbf{v} = \begin{pmatrix} v_1 \\ v_2 \end{pmatrix}$ is a vector and $\alpha \in \mathbb{R}$. Then the scalar product of $\alpha \mathbf{v}$ is defined by

$$\alpha \mathbf{v} \equiv \begin{pmatrix} \alpha v_1 \\ \alpha v_2 \end{pmatrix}.$$

Example

Find the sum of the following vectors.

1. $\mathbf{v} = \begin{pmatrix} 1 \\ 4 \end{pmatrix}$, $\mathbf{u} = \begin{pmatrix} 5 \\ 2 \end{pmatrix}$
2. $\mathbf{v} = \begin{pmatrix} 2 \\ -3 \end{pmatrix}$, $\mathbf{u} = \begin{pmatrix} -2 \\ 3 \end{pmatrix}$
3. $\mathbf{v} = \begin{pmatrix} 6 \\ 2 \end{pmatrix}$, $\mathbf{v} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$

Solution

$$1. \quad \mathbf{v} + \mathbf{u} = \begin{pmatrix} 1 \\ 4 \end{pmatrix} + \begin{pmatrix} 5 \\ 2 \end{pmatrix} = \begin{pmatrix} 1 + 5 \\ 4 + 2 \end{pmatrix} = \begin{pmatrix} 6 \\ 6 \end{pmatrix}$$

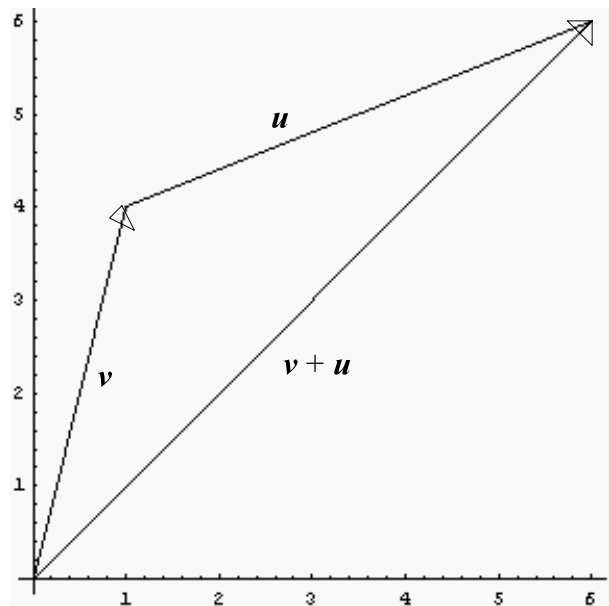
$$2. \quad \mathbf{v} + \mathbf{u} = \begin{pmatrix} 2 \\ -3 \end{pmatrix} + \begin{pmatrix} -2 \\ 3 \end{pmatrix} = \begin{pmatrix} 2 + (-2) \\ (-3) + 3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

$$3. \quad \mathbf{v} + \mathbf{u} = \begin{pmatrix} 6 \\ 2 \end{pmatrix} + \begin{pmatrix} 0 \\ 0 \end{pmatrix} = \begin{pmatrix} 6 + 0 \\ 2 + 0 \end{pmatrix} = \begin{pmatrix} 6 \\ 2 \end{pmatrix}$$

We illustrate

$$\mathbf{v} + \mathbf{u} = \begin{pmatrix} 1 \\ 4 \end{pmatrix} + \begin{pmatrix} 5 \\ 2 \end{pmatrix} = \begin{pmatrix} 1 + 5 \\ 4 + 2 \end{pmatrix} = \begin{pmatrix} 6 \\ 6 \end{pmatrix}$$

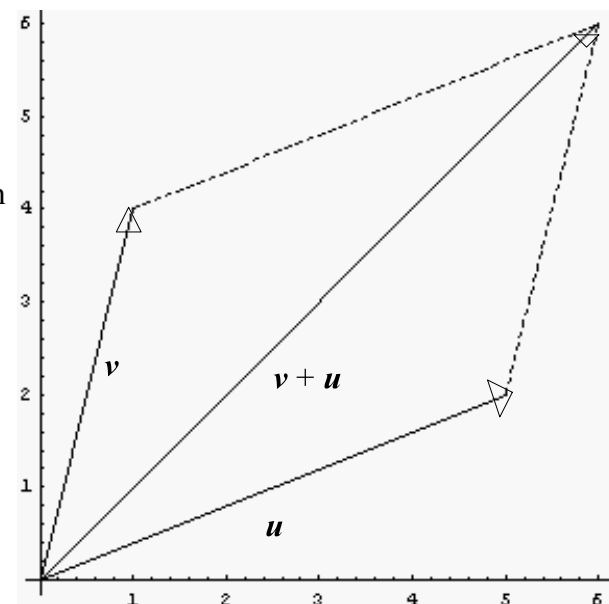
in the graphic at the right. As suggested by the graphic, vector addition may be regarded geometrically as head-to-tail addition of directed line segments.



We may also illustrate the vector sum

$$\mathbf{v} + \mathbf{u} = \begin{pmatrix} 1 \\ 4 \end{pmatrix} + \begin{pmatrix} 5 \\ 2 \end{pmatrix} = \begin{pmatrix} 1 + 5 \\ 4 + 2 \end{pmatrix} = \begin{pmatrix} 6 \\ 6 \end{pmatrix}$$

with $\mathbf{v} + \mathbf{u}$ as the diagonal of a parallelogram with sides determined by the vectors \mathbf{v} and \mathbf{u} .



Problem

1. Find the sum of the following vectors:

(a) $\mathbf{v} = \begin{pmatrix} 3 \\ 5 \end{pmatrix}$, $\mathbf{u} = \begin{pmatrix} 2 \\ 1 \end{pmatrix}$

(b) $\mathbf{v} = \begin{pmatrix} -5 \\ 6 \end{pmatrix}$, $\mathbf{u} = \begin{pmatrix} 4 \\ 3 \end{pmatrix}$

2. Illustrate the above sums geometrically.

We note that vectors in \mathbb{R}^3 are simply ordered triples of real numbers of the form

$$(\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3) \quad \text{or} \quad \begin{pmatrix} \mathbf{v}_1 \\ \mathbf{v}_2 \\ \mathbf{v}_3 \end{pmatrix} \quad \text{or} \quad (\mathbf{v}_1 \ \mathbf{v}_2 \ \mathbf{v}_3).$$

Vector addition in \mathbb{R}^3 , like \mathbb{R}^2 , is *componentwise* and is defined by

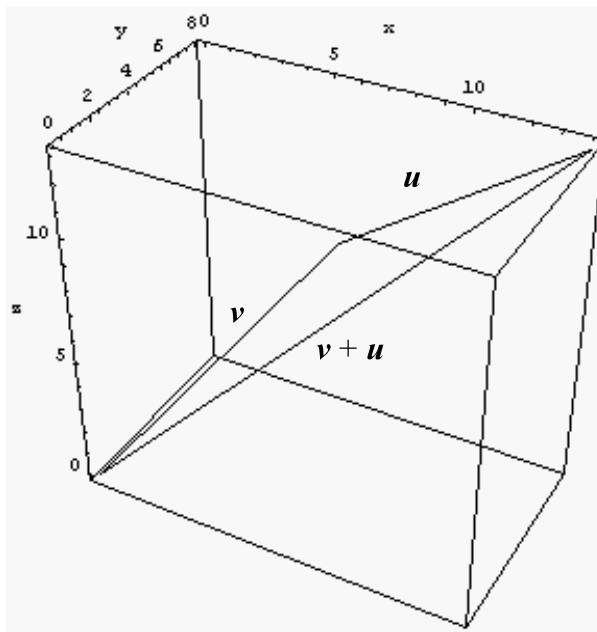
$$\mathbf{v} + \mathbf{u} = \begin{pmatrix} \mathbf{v}_1 \\ \mathbf{v}_2 \\ \mathbf{v}_3 \end{pmatrix} + \begin{pmatrix} \mathbf{u}_1 \\ \mathbf{u}_2 \\ \mathbf{u}_3 \end{pmatrix} \equiv \begin{pmatrix} \mathbf{v}_1 + \mathbf{u}_1 \\ \mathbf{v}_2 + \mathbf{u}_2 \\ \mathbf{v}_3 + \mathbf{u}_3 \end{pmatrix}.$$

Example

In \mathbb{R}^3 , the sum of $\begin{pmatrix} \mathbf{v}_1 \\ \mathbf{v}_2 \\ \mathbf{v}_3 \end{pmatrix} = \begin{pmatrix} 9 \\ 1 \\ 12 \end{pmatrix}$ and

$\begin{pmatrix} \mathbf{u}_1 \\ \mathbf{u}_2 \\ \mathbf{u}_3 \end{pmatrix} = \begin{pmatrix} 5 \\ 7 \\ 1 \end{pmatrix}$ is the ordered triple or column

vector given by $\mathbf{v} + \mathbf{u} = \begin{pmatrix} 14 \\ 8 \\ 13 \end{pmatrix}$.



Example

Compute the following scalar products:

1. $\alpha = 3; \mathbf{v} = \begin{pmatrix} 4 \\ 2 \end{pmatrix}$

2. $\alpha = -2; \mathbf{v} = \begin{pmatrix} 1 \\ 3 \end{pmatrix}$

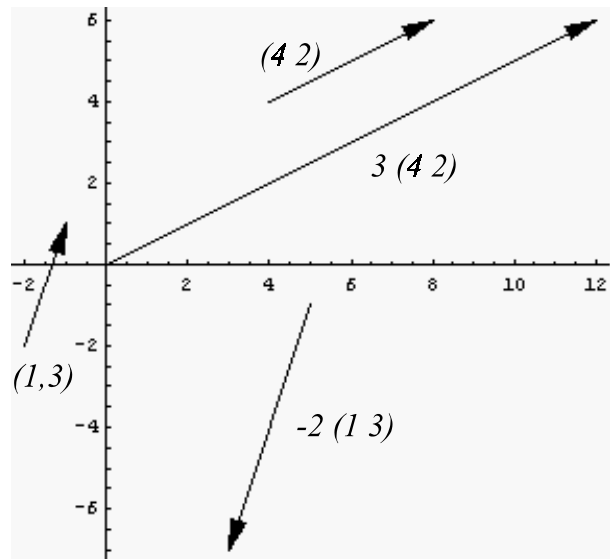
3. $\alpha = 4; \mathbf{v} = \begin{pmatrix} 5 \\ -2 \\ 6 \end{pmatrix}$

Solution

1. $3 \begin{pmatrix} 4 \\ 2 \end{pmatrix} = \begin{pmatrix} 12 \\ 6 \end{pmatrix}$

2. $-2 \begin{pmatrix} 1 \\ 3 \end{pmatrix} = \begin{pmatrix} -2 \\ -6 \end{pmatrix}$

3. $4 \begin{pmatrix} 5 \\ -2 \\ 6 \end{pmatrix} = \begin{pmatrix} 20 \\ -8 \\ 24 \end{pmatrix}$



Observe that as directed line segments the illustration above suggests that the vector $\alpha \mathbf{v}$ is $|\alpha|$ times the length of the vector \mathbf{v} and $\alpha \mathbf{v}$ has the same direction as \mathbf{v} if α is positive and the opposite direction if α is negative.

Definition

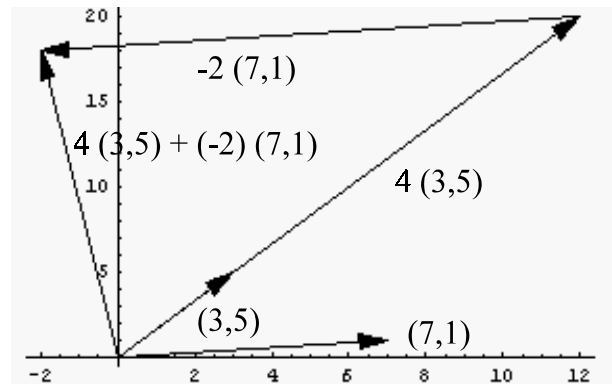
Let $\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3, \dots, \mathbf{v}_n$ be n vectors in \mathbb{R}^2 (or \mathbb{R}^3). Then any vector of the form

$$\alpha_1 \mathbf{v}_1 + \alpha_2 \mathbf{v}_2 + \alpha_3 \mathbf{v}_3 + \dots + \alpha_n \mathbf{v}_n$$

where $\alpha_1, \alpha_2, \alpha_3, \dots, \alpha_n$ are scalars is called a *linear combination* of $\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3, \dots, \mathbf{v}_n$.

Examples

$$\begin{aligned} 1. \quad & 4 \begin{pmatrix} 3 \\ 5 \end{pmatrix} + (-2) \begin{pmatrix} 7 \\ 1 \end{pmatrix} \\ &= \begin{pmatrix} 4(3) \\ 4(5) \end{pmatrix} + \begin{pmatrix} (-2)(7) \\ (-2)(1) \end{pmatrix} \\ &= \begin{pmatrix} 12 \\ 20 \end{pmatrix} + \begin{pmatrix} -14 \\ -2 \end{pmatrix} \\ &= \begin{pmatrix} -2 \\ 18 \end{pmatrix} \end{aligned}$$



$$2. \quad 5 \begin{pmatrix} -2 \\ 10 \end{pmatrix} + (2) \begin{pmatrix} 8 \\ -1 \end{pmatrix} = \begin{pmatrix} 6 \\ 48 \end{pmatrix}$$

Example

Given $\mathbf{x} = \begin{pmatrix} -1 \\ -2 \\ -2 \end{pmatrix}$, $\mathbf{u} = \begin{pmatrix} 0 \\ 1 \\ 4 \end{pmatrix}$, $\mathbf{v} = \begin{pmatrix} -1 \\ 1 \\ 2 \end{pmatrix}$, and $\mathbf{w} = \begin{pmatrix} 3 \\ 1 \\ 2 \end{pmatrix}$ in \mathbb{R}^3 , find scalars $a, b, c \in \mathbb{R}$,

if possible, so that $\mathbf{x} = a\mathbf{u} + b\mathbf{v} + c\mathbf{w}$.

Solution

$$a\mathbf{u} + b\mathbf{v} + c\mathbf{w} = \mathbf{x}$$

$$\Rightarrow a \begin{pmatrix} 0 \\ 1 \\ 4 \end{pmatrix} + b \begin{pmatrix} -1 \\ 1 \\ 2 \end{pmatrix} + c \begin{pmatrix} 3 \\ 1 \\ 2 \end{pmatrix} = \begin{pmatrix} -1 \\ -2 \\ -2 \end{pmatrix}$$

$$\Rightarrow \begin{pmatrix} 0 \\ a \\ 4a \end{pmatrix} + \begin{pmatrix} -b \\ b \\ 2b \end{pmatrix} + \begin{pmatrix} 3c \\ 1c \\ 2c \end{pmatrix} = \begin{pmatrix} -1 \\ -2 \\ -2 \end{pmatrix}$$

$$\Rightarrow \begin{pmatrix} -b + 3c \\ a + b + c \\ 4a + 2b + 2c \end{pmatrix} = \begin{pmatrix} -1 \\ -2 \\ -2 \end{pmatrix}$$

Since two vectors are equal precisely when corresponding components are equal, the above yields the following system of three equations in three unknowns:

$$-b + 3c = -1$$

$$a + b + c = -2$$

$$4a + 2b + 2c = -2$$

or, equivalently,

$$a + \frac{1}{2}b + \frac{1}{2}c = -\frac{1}{2}$$

$$b - 3c = 1$$

$$c = -1$$

Solving the above system yields a unique solution of $a = 1$, $b = -2$, $c = -1$. That is,

$$\begin{aligned} 1 \begin{pmatrix} 0 \\ 1 \\ 4 \end{pmatrix} + (-2) \begin{pmatrix} -1 \\ 1 \\ 2 \end{pmatrix} + (-1) \begin{pmatrix} 3 \\ 1 \\ 2 \end{pmatrix} &= \begin{pmatrix} 0 \\ 1 \\ 4 \end{pmatrix} + \begin{pmatrix} 2 \\ -2 \\ -4 \end{pmatrix} + \begin{pmatrix} -3 \\ -1 \\ -2 \end{pmatrix} \\ \Rightarrow &= \begin{pmatrix} 0 + 2 + (-3) \\ 1 + (-2) + (-1) \\ 4 + (-4) + (-2) \end{pmatrix} \\ &= \begin{pmatrix} -1 \\ -2 \\ -2 \end{pmatrix} \end{aligned}$$

Problem

Let $\mathbf{v}_1 = \begin{pmatrix} 2 \\ 1 \end{pmatrix}$ and $\mathbf{v}_2 = \begin{pmatrix} -4 \\ -2 \end{pmatrix}$. Determine if (a) $\mathbf{w} = \begin{pmatrix} 6 \\ 3 \end{pmatrix}$ or (b) $\mathbf{u} = \begin{pmatrix} 3 \\ 5 \end{pmatrix}$ is a linear

combination of the two-vectors \mathbf{v}_1 & \mathbf{v}_2 .

Question: When is a given vector a linear combination of a particular set of vectors?

Problem

Let \mathbf{v} , \mathbf{u} , & \mathbf{w} be vectors in \mathbb{R}^3 and $a, b, c \in \mathbb{R}$.

1. What can be said about the set of all vectors of the form $a\mathbf{v}$?
2. What can be said about the set of all vectors of the form $a\mathbf{v} + b\mathbf{u}$?
3. What can be said about the set of all vectors of the form $a\mathbf{v} + \mathbf{u} + c\mathbf{w}$?