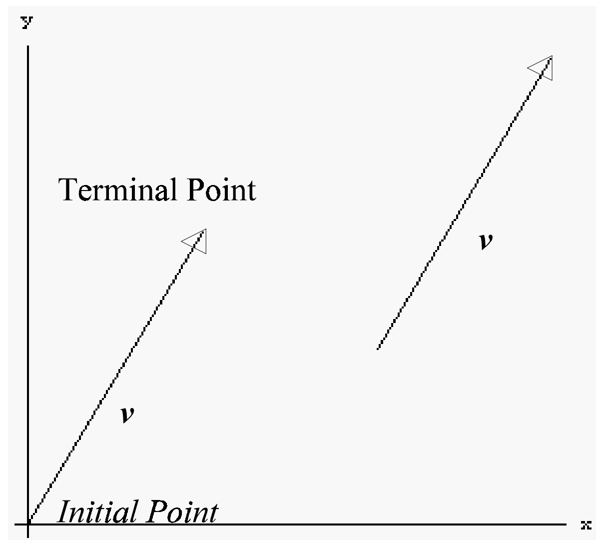
1 - Introduction to Vectors

Definition

A ***vector v*** in the plane $\mathbb{R}^2$ is an ordered pair of real numbers. We denote ***v*** by $\begin{pmatrix} v_1 \\ v_2 \end{pmatrix}$ or $\begin{pmatrix} v_1 & v_2 \end{pmatrix}$.



The term vector comes from the Latin word *vectus*, meaning "to carry." This leads nicely to the *geometric representation* of a vector in $\mathbb{R}^2$ as a directed line segment from the origin to the point (v_1, v_2). That is, one might envision an object being carried from the origin to the terminal point located at (v_1, v_2). We regard any directed line segment from initial point (a, b) to the terminal point $(a + v_1, b + v_2)$ as equivalent to the directed line segment from the origin to (v_1, v_2). So, just as the rational number ½ has many different equivalent representatives $\left(e.g., \frac{1}{2}, \frac{5}{10}, \frac{-12}{-24}, \ldots \right)$, a given vector ***v*** also has many equivalent directed line segments which may be used to stand for the given vector.

Problem

Suppose $\boldsymbol{v} = \begin{pmatrix} 3 \\ 7 \end{pmatrix}$. Find the terminal point (x, y) for the directed line segment representing ***v*** if the initial point is $(0, 0)$. Repeat for initial points of $(5, 1)$, $(-2, 4)$, and $(-3, -7)$.

Basic Vector Algebra in $\mathbb{R}^2$

1. *Vector Equality*: Two vectors $\boldsymbol{v} = \begin{pmatrix} v_1 \\ v_2 \end{pmatrix}$ and $\boldsymbol{u} = \begin{pmatrix} u_1 \\ u_2 \end{pmatrix}$ are ***equal*** if and only if $v_1 = u_1$ and $v_2 = u_2$.

2. *Vector Addition*: The ***sum*** of the vectors $\boldsymbol{v} = \begin{pmatrix} v_1 \\ v_2 \end{pmatrix}$ and $\boldsymbol{u} = \begin{pmatrix} u_1 \\ u_2 \end{pmatrix}$ is defined by

$$\boldsymbol{v} + \boldsymbol{u} \equiv \begin{pmatrix} v_1 + u_1 \\ v_2 + u_2 \end{pmatrix}.$$

3. *Scalar Multiplication:* Suppose $\boldsymbol{v} = \begin{pmatrix} v_1 \\ v_2 \end{pmatrix}$ is a vector and $\alpha \in \mathbb{R}$. Then the scalar product of $\alpha\, \boldsymbol{v}$ is defined by

$$\alpha\, \boldsymbol{v} \equiv \begin{pmatrix} \alpha\, v_1 \\ \alpha\, v_2 \end{pmatrix}.$$

Example

Find the sum of the following vectors.

1. $\boldsymbol{v} = \begin{pmatrix} 1 \\ 4 \end{pmatrix}, \ \boldsymbol{u} = \begin{pmatrix} 5 \\ 2 \end{pmatrix}$

2. $\boldsymbol{v} = \begin{pmatrix} 2 \\ -3 \end{pmatrix}, \ \boldsymbol{u} = \begin{pmatrix} -2 \\ 3 \end{pmatrix}$

3. $\boldsymbol{v} = \begin{pmatrix} 6 \\ 2 \end{pmatrix}, \ \boldsymbol{v} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$

Solution

1. $\boldsymbol{v} + \boldsymbol{u} = \begin{pmatrix} 1 \\ 4 \end{pmatrix} + \begin{pmatrix} 5 \\ 2 \end{pmatrix} = \begin{pmatrix} 1 + 5 \\ 4 + 2 \end{pmatrix} = \begin{pmatrix} 6 \\ 6 \end{pmatrix}$

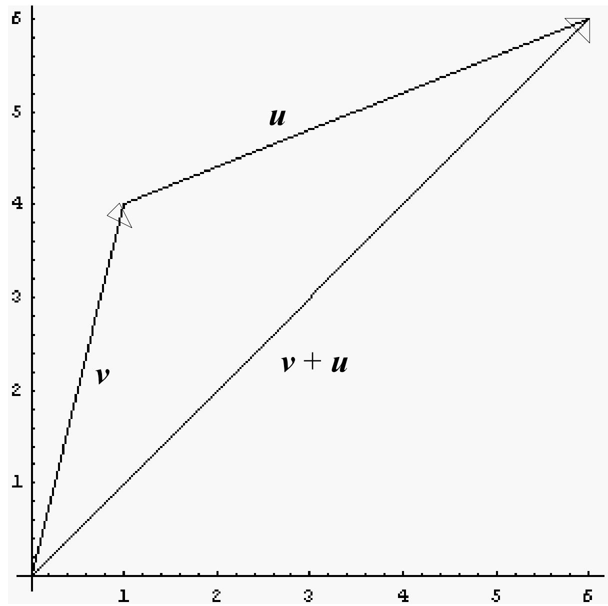
2. $\boldsymbol{v} + \boldsymbol{u} = \begin{pmatrix} 2 \\ -3 \end{pmatrix} + \begin{pmatrix} -2 \\ 3 \end{pmatrix} = \begin{pmatrix} 2 + (-2) \\ (-3) + 3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$

3. $\boldsymbol{v} + \boldsymbol{u} = \begin{pmatrix} 6 \\ 2 \end{pmatrix} + \begin{pmatrix} 0 \\ 0 \end{pmatrix} = \begin{pmatrix} 6 + 0 \\ 2 + 0 \end{pmatrix} = \begin{pmatrix} 6 \\ 2 \end{pmatrix}$

We illustrate

$$\boldsymbol{v} + \boldsymbol{u} = \begin{pmatrix} 1 \\ 4 \end{pmatrix} + \begin{pmatrix} 5 \\ 2 \end{pmatrix} = \begin{pmatrix} 1 + 5 \\ 4 + 2 \end{pmatrix} = \begin{pmatrix} 6 \\ 6 \end{pmatrix}$$

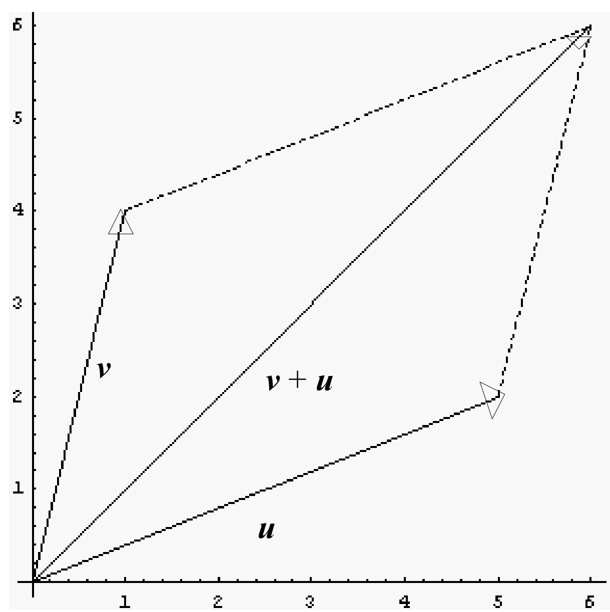
in the graphic at the right. As suggested by the graphic, vector addition may be regarded geometrically as head-to-tail addition of directed line segments.



We may also illustrate the vector sum

$$\boldsymbol{v} + \boldsymbol{u} = \begin{pmatrix} 1 \\ 4 \end{pmatrix} + \begin{pmatrix} 5 \\ 2 \end{pmatrix} = \begin{pmatrix} 1 + 5 \\ 4 + 2 \end{pmatrix} = \begin{pmatrix} 6 \\ 6 \end{pmatrix}$$

with $\boldsymbol{v} + \boldsymbol{u}$ as the diagonal of a parallelogram with sides determined by the vectors $\boldsymbol{v}$ and $\boldsymbol{u}$.



Problem

1. Find the sum of the following vectors:

(a) $\boldsymbol{v} = \begin{pmatrix} 3 \\ 5 \end{pmatrix}, \quad \boldsymbol{u} = \begin{pmatrix} 2 \\ 1 \end{pmatrix}$

(b) $\boldsymbol{v} = \begin{pmatrix} -5 \\ 6 \end{pmatrix}, \quad \boldsymbol{u} = \begin{pmatrix} 4 \\ 3 \end{pmatrix}$

2. Illustrate the above sums geometrically.

We note that vectors in $\mathbb{R}^3$ are simply ordered triples of real numbers of the form

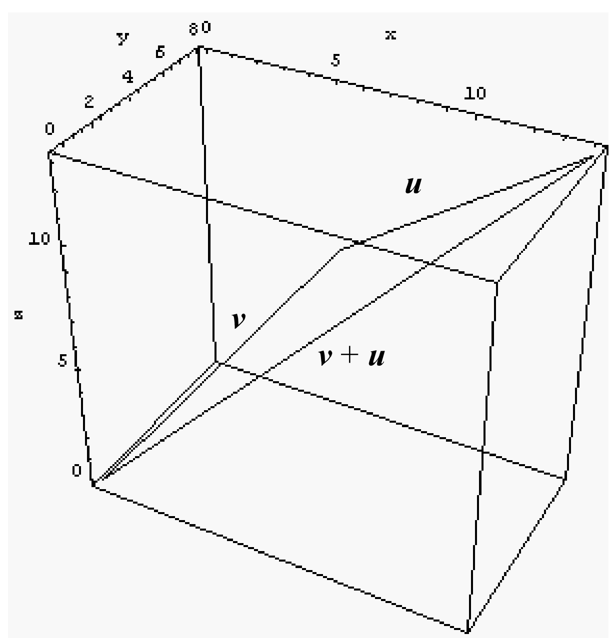
$$(v_1, v_2, v_3) \qquad \text{or} \qquad \begin{pmatrix} v_1 \\ v_2 \\ v_3 \end{pmatrix} \qquad \text{or} \qquad (v_1\ v_2\ v_3).$$

Vector addition in $\mathbb{R}^3$, like $\mathbb{R}^2$, is *componentwise* and is defined by

$$\boldsymbol{v} + \boldsymbol{u} = \begin{pmatrix} v_1 \\ v_2 \\ v_3 \end{pmatrix} + \begin{pmatrix} u_1 \\ u_2 \\ u_3 \end{pmatrix} \equiv \begin{pmatrix} v_1 + u_1 \\ v_2 + u_2 \\ v_3 + u_3 \end{pmatrix}.$$

Example

In $\mathbb{R}^3$, the sum of $\begin{pmatrix} v_1 \\ v_2 \\ v_3 \end{pmatrix} = \begin{pmatrix} 9 \\ 1 \\ 12 \end{pmatrix}$ and $\begin{pmatrix} u_1 \\ u_2 \\ u_3 \end{pmatrix} = \begin{pmatrix} 5 \\ 7 \\ 1 \end{pmatrix}$ is the ordered triple or column vector given by $\boldsymbol{v} + \boldsymbol{u} = \begin{pmatrix} 14 \\ 8 \\ 13 \end{pmatrix}$.



Example

Compute the following scalar products:

1. $\alpha = 3;\ \boldsymbol{v} = \begin{pmatrix} 4 \\ 2 \end{pmatrix}$

2. $\alpha = -2;\ \boldsymbol{v} = \begin{pmatrix} 1 \\ 3 \end{pmatrix}$

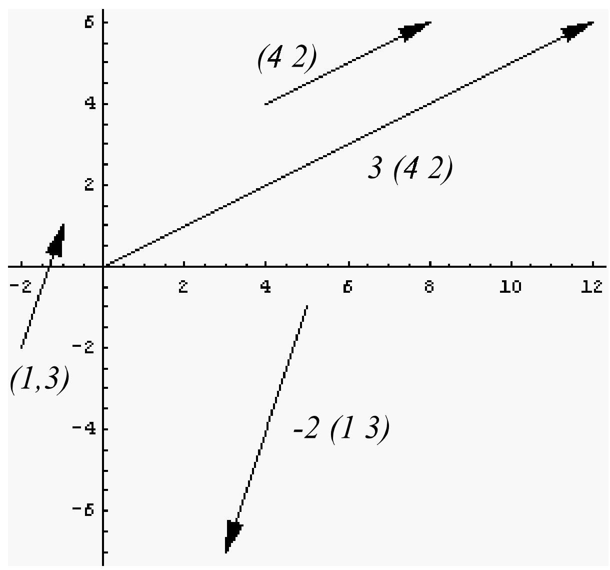
3. $\alpha = 4;\ \boldsymbol{v} = \begin{pmatrix} 5 \\ -2 \\ 6 \end{pmatrix}$

Solution

1. $3\begin{pmatrix} 4 \\ 2 \end{pmatrix} = \begin{pmatrix} 12 \\ 6 \end{pmatrix}$

2. $-2\begin{pmatrix} 1 \\ 3 \end{pmatrix} = \begin{pmatrix} -2 \\ -6 \end{pmatrix}$

3. $4\begin{pmatrix} 5 \\ -2 \\ 6 \end{pmatrix} = \begin{pmatrix} 20 \\ -8 \\ 24 \end{pmatrix}$



Observe that as directed line segments the illustration above suggests that the vector $\alpha\ \boldsymbol{v}$ is $|\alpha|$ times the length of the vector $\boldsymbol{v}$ and $\alpha\ \boldsymbol{v}$ has the same direction as $\boldsymbol{v}$ if " is positive and the opposite direction is " is negative.

Definition

Let $\boldsymbol{v_1}, \boldsymbol{v_2}, \boldsymbol{v_3}, \ldots, \boldsymbol{v_n}$ be n vectors in $\mathbb{R}^2$ (or $\mathbb{R}^3$). Then any vector of the form

$$\alpha_1 \boldsymbol{v_1} + \alpha_2 \boldsymbol{v_2} + \alpha_3 \boldsymbol{v_3} + \ldots + \alpha_n \boldsymbol{v_n}$$

where $\alpha_1, \alpha_2, \alpha_3, \ldots, \alpha_n$ are scalars is called a ***linear combination*** of $\boldsymbol{v_1}, \boldsymbol{v_2}, \boldsymbol{v_3}, \ldots, \boldsymbol{v_n}$.

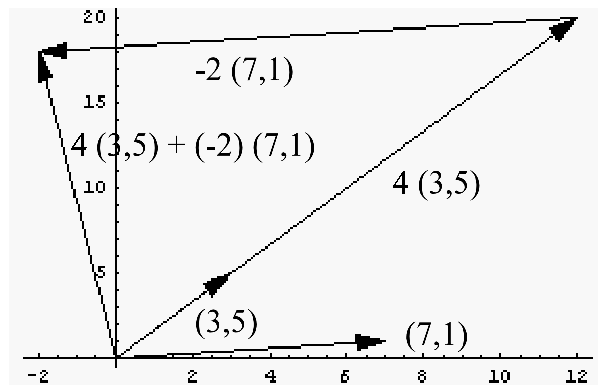
Examples

1. $$4\begin{pmatrix} 3 \\ 5 \end{pmatrix} + (-2)\begin{pmatrix} 7 \\ 1 \end{pmatrix}$$

$$= \begin{pmatrix} 4(3) \\ 4(5) \end{pmatrix} + \begin{pmatrix} (-2)(7) \\ (-2)(1) \end{pmatrix}$$

$$= \begin{pmatrix} 12 \\ 20 \end{pmatrix} + \begin{pmatrix} -14 \\ -2 \end{pmatrix}$$

$$= \begin{pmatrix} -2 \\ 18 \end{pmatrix}$$



2. $$5\begin{pmatrix} -2 \\ 10 \end{pmatrix} + (2)\begin{pmatrix} 8 \\ -1 \end{pmatrix} = \begin{pmatrix} 6 \\ 48 \end{pmatrix}$$

Example

Given $\boldsymbol{x} = \begin{pmatrix} -1 \\ -2 \\ -2 \end{pmatrix}$, $\boldsymbol{u} = \begin{pmatrix} 0 \\ 1 \\ 4 \end{pmatrix}$, $\boldsymbol{v} = \begin{pmatrix} -1 \\ 1 \\ 2 \end{pmatrix}$, and $\boldsymbol{w} = \begin{pmatrix} 3 \\ 1 \\ 2 \end{pmatrix}$ in $\mathbb{R}^3$, find scalars $a, b, c \in \mathbb{R}$,

if possible, so that $\boldsymbol{x} = a\,\boldsymbol{u} + b\,\boldsymbol{v} + c\,\boldsymbol{w}$.

Solution

$a\,\boldsymbol{u} + b\,\boldsymbol{v} + c\,\boldsymbol{w} = \boldsymbol{x}$

$$\Rightarrow \quad a\begin{pmatrix} 0 \\ 1 \\ 4 \end{pmatrix} + b\begin{pmatrix} -1 \\ 1 \\ 2 \end{pmatrix} + c\begin{pmatrix} 3 \\ 1 \\ 2 \end{pmatrix} = \begin{pmatrix} -1 \\ -2 \\ -2 \end{pmatrix}$$

$$\Rightarrow \quad \begin{pmatrix} 0 \\ a \\ 4a \end{pmatrix} + \begin{pmatrix} -b \\ b \\ 2b \end{pmatrix} + \begin{pmatrix} 3c \\ 1c \\ 2c \end{pmatrix} = \begin{pmatrix} -1 \\ -2 \\ -2 \end{pmatrix}$$

$$\Rightarrow \quad \begin{pmatrix} -b + 3c \\ a + b + c \\ 4a + 2b + 2c \end{pmatrix} = \begin{pmatrix} -1 \\ -2 \\ -2 \end{pmatrix}$$

Since two vectors are equal precisely when corresponding components are equal, the above yields the following system of three equations in three unknowns:

$$\begin{aligned} -b + 3c &= -1 \\ a + b + c &= -2 \\ 4a + 2b + 2c &= -2 \end{aligned}$$

or, equivalently,

$$\begin{aligned} a + \frac{1}{2}b + \frac{1}{2}c &= -\frac{1}{2} \\ b - 3c &= 1 \\ c &= -1 \end{aligned}$$

Solving the above system yields a unique solution of $a = 1,\ b = -2,\ c = -1$. That is,

$$1\begin{pmatrix} 0 \\ 1 \\ 4 \end{pmatrix} + (-2)\begin{pmatrix} -1 \\ 1 \\ 2 \end{pmatrix} + (-1)\begin{pmatrix} 3 \\ 1 \\ 2 \end{pmatrix} = \begin{pmatrix} 0 \\ 1 \\ 4 \end{pmatrix} + \begin{pmatrix} 2 \\ -2 \\ -4 \end{pmatrix} + \begin{pmatrix} -3 \\ -1 \\ -2 \end{pmatrix}$$

$$\Rightarrow \quad = \begin{pmatrix} 0 + 2 + (-3) \\ 1 + (-2) + (-1) \\ 4 + (-4) + (-2) \end{pmatrix}$$

$$= \begin{pmatrix} -1 \\ -2 \\ -2 \end{pmatrix}$$

Problem

Let $\boldsymbol{v_1} = \begin{pmatrix} 2 \\ 1 \end{pmatrix}$ and $\boldsymbol{v_2} = \begin{pmatrix} -4 \\ -2 \end{pmatrix}$. Determine if (a) $\boldsymbol{w} = \begin{pmatrix} 6 \\ 3 \end{pmatrix}$ or (b) $\boldsymbol{u} = \begin{pmatrix} 3 \\ 5 \end{pmatrix}$ is a linear combination of the two-vectors $\boldsymbol{v_1}$ & $\boldsymbol{v_2}$.

Question: When is a given vector a linear combination of a particular set of vectors?

Problem

Let $\boldsymbol{v}$, $\boldsymbol{u}$, & $\boldsymbol{w}$ be vectors in $\mathbb{R}^3$ and $a, b, c \in \mathbb{R}$.

1. What can be said about the set of all vectors of the form $a\,\boldsymbol{v}$?
2. What can be said about the set of all vectors of the form $a\,\boldsymbol{v} + b\,\boldsymbol{u}$?
3. What can be said about the set of all vectors of the form $a\,\boldsymbol{v} + \boldsymbol{u} + c\,\boldsymbol{w}$?