Definition

The ***dot product*** of two vectors $\boldsymbol{v}$ and $\boldsymbol{u}$, denoted by $\boldsymbol{v} \cdot \boldsymbol{u}$, is the scalar given by

$$\boldsymbol{v} \cdot \boldsymbol{u} = \begin{pmatrix} v_1 \\ v_2 \end{pmatrix} \cdot \begin{pmatrix} u_1 \\ u_2 \end{pmatrix} \equiv v_1 u_1 + v_2 u_2 \text{ in } \mathbb{R}^2$$

or

$$\boldsymbol{v} \cdot \boldsymbol{u} = \begin{pmatrix} v_1 \\ v_2 \\ v_3 \end{pmatrix} \cdot \begin{pmatrix} u_1 \\ u_2 \\ u_3 \end{pmatrix} \equiv v_1 u_1 + v_2 u_2 + v_3 u_3 .$$

The dot product is also called the ***inner product*** or the ***scalar product*** of the vectors $\boldsymbol{v}$ and $\boldsymbol{u}$. *We note that the dot product of two vectors is a scalar and not another vector.*

Example

Given $\boldsymbol{u} = \begin{pmatrix} 4 \\ -4 \end{pmatrix}$, $\boldsymbol{v} = (5,8)$, & $\boldsymbol{w} = \begin{pmatrix} 8 \\ -6 \end{pmatrix}$, find the following.

(1) $\boldsymbol{u} \cdot \boldsymbol{v}$ (2) $(\boldsymbol{u} \cdot \boldsymbol{v})\, \boldsymbol{w}$ (3) $\boldsymbol{u} \cdot (4\, \boldsymbol{v})$ (4) $\boldsymbol{u} \cdot (3\, \boldsymbol{v} - \boldsymbol{w})$

Solution

(1) $\boldsymbol{u} \cdot \boldsymbol{v} = \begin{pmatrix} 4 \\ -4 \end{pmatrix} \cdot \begin{pmatrix} 5 \\ 8 \end{pmatrix} = 4\,(5) + (-4)\,(8) = -12$

(2) $(\boldsymbol{u} \cdot \boldsymbol{v})\, \boldsymbol{w} = (-12)\, \boldsymbol{w} = -12 \begin{pmatrix} 8 \\ -6 \end{pmatrix} = \begin{pmatrix} (-12)\,8 \\ (-12)\,(-6) \end{pmatrix} = \begin{pmatrix} -96 \\ 72 \end{pmatrix}$

(3) $\boldsymbol{u} \cdot (4\, \boldsymbol{v}) = \begin{pmatrix} 4 \\ -4 \end{pmatrix} \cdot \left(4 \begin{pmatrix} 5 \\ 8 \end{pmatrix} \right) = \begin{pmatrix} 4 \\ -4 \end{pmatrix} \cdot \begin{pmatrix} 4\,(5) \\ 4\,(8) \end{pmatrix}$

$$= \begin{pmatrix} 4 \\ -4 \end{pmatrix} \cdot \begin{pmatrix} 20 \\ 32 \end{pmatrix} = 4\,(20) + (-4)\,(32) = -48$$

(4) $\quad \boldsymbol{u} \cdot (3\boldsymbol{v} - \boldsymbol{w}) = \begin{pmatrix} 4 \\ -4 \end{pmatrix} \cdot \left(3 \begin{pmatrix} 5 \\ 8 \end{pmatrix} - \begin{pmatrix} 8 \\ -6 \end{pmatrix} \right) = \begin{pmatrix} 4 \\ -4 \end{pmatrix} \cdot \left(\begin{pmatrix} 3\,(5) \\ 3\,(8) \end{pmatrix} - \begin{pmatrix} 8 \\ -6 \end{pmatrix} \right)$

$$= \begin{pmatrix} 4 \\ -4 \end{pmatrix} \cdot \left(\begin{pmatrix} 15 \\ 24 \end{pmatrix} - \begin{pmatrix} 8 \\ -6 \end{pmatrix} \right) = \begin{pmatrix} 4 \\ -4 \end{pmatrix} \cdot \begin{pmatrix} 15 - 8 \\ 24 - (-6) \end{pmatrix}$$

$$= \begin{pmatrix} 4 \\ -4 \end{pmatrix} \cdot \begin{pmatrix} 7 \\ 30 \end{pmatrix} = 4\,(7) + (-4)\,(30) = -92$$

Example

Suppose a manufacturer produces three items. The demand for these items is given by the *demand vector* $\boldsymbol{d} = \begin{pmatrix} 1100 \\ 550 \\ 18400 \end{pmatrix}$. The price per unit that is received for the items is given by the *price vector* $\boldsymbol{p} = \begin{pmatrix} \$47.50 \\ \$38.25 \\ \$9.75 \end{pmatrix}$. If the manufacturer meets demand this period, how much money will the company gross?

Solution

The follow shows that the dot product of $\boldsymbol{d}$ and $\boldsymbol{p}$ yields the gross income:

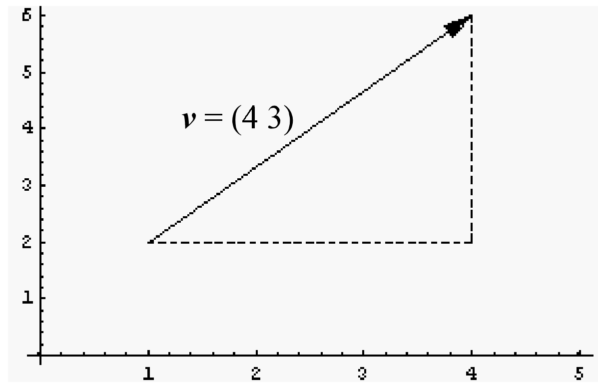
$$\boldsymbol{d} \cdot \boldsymbol{p} = \begin{pmatrix} 1100 \\ 550 \\ 18400 \end{pmatrix} \cdot \begin{pmatrix} \$47.50 \\ \$38.25 \\ \$9.75 \end{pmatrix}$$

$$= 1100\,(47.50) + 550\,(38.25) + 18400\,(9.75)$$

$$= 252{,}687.50$$

Example

Let $\boldsymbol{v} = \begin{pmatrix} 4 \\ 3 \end{pmatrix}$.



(1) Compute the length of the vector $\boldsymbol{v}$ regarding $\boldsymbol{v}$ as a directed line segment in the plane $\mathbb{R}^2$.

(2) Compute the inner product $\boldsymbol{v} \cdot \boldsymbol{v}$.

(3) Relate your answers to parts (1) and (2).

Solution

(1) By the Pythagorean Theorem, the length of "the" direct line segment associated with $\boldsymbol{v}$ is given by

$$\sqrt{4^2 + 3^2} = \sqrt{25} = 5.$$

(2) $\boldsymbol{v} \cdot \boldsymbol{v} = \begin{pmatrix} 4 \\ 3 \end{pmatrix} \cdot \begin{pmatrix} 4 \\ 3 \end{pmatrix} = 4(4) + 3(3) = 4^2 + 3^2 = 25$

(3) The square of the length of a directed line segment associated with $\boldsymbol{v}$ equals $\boldsymbol{v} \cdot \boldsymbol{v}$.

Definition

The ***norm*** of a vector $\boldsymbol{v}$, denoted by $\| \boldsymbol{v} \|$, in either $\mathbb{R}^2$ or $\mathbb{R}^3$ is given by

$$\| \boldsymbol{v} \| = \sqrt{\boldsymbol{v} \cdot \boldsymbol{v}}.$$

In view of the above example, we may regard the quantity $\| \boldsymbol{v} \|$ as the length of $\boldsymbol{v}$. We also refer to the quantity $\| \boldsymbol{v} \| = \sqrt{\boldsymbol{v} \cdot \boldsymbol{v}}$ as the ***magnitude*** of $\boldsymbol{v}$.

Example

Find $\| \boldsymbol{v} \|$ if (1) $\boldsymbol{v} = \begin{pmatrix} 5 \\ 12 \end{pmatrix}$, (2) $\boldsymbol{v} = \begin{pmatrix} 2 \\ -6 \\ 4 \end{pmatrix}$, (3) $\boldsymbol{v} = \begin{pmatrix} 2/\sqrt{30} \\ 5/\sqrt{30} \\ 1/\sqrt{30} \end{pmatrix}$.

Solution

(1) $\qquad \|\boldsymbol{v}\| = \left\| \begin{pmatrix} 5 \\ 12 \end{pmatrix} \right\| = \sqrt{5^2 + 12^2} = \sqrt{169} = 13$

(2) $\qquad \|\boldsymbol{v}\| = \left\| \begin{pmatrix} 2 \\ -6 \\ 4 \end{pmatrix} \right\| = \sqrt{2^2 + (-6)^2 + 4^2} = \sqrt{56} = 2\sqrt{14}$

(3) $\qquad \|\boldsymbol{v}\| = \left\| \begin{pmatrix} 2/\sqrt{30} \\ 5/\sqrt{30} \\ 1/\sqrt{30} \end{pmatrix} \right\| = \sqrt{\left(\frac{2}{\sqrt{30}} \right)^2 + \left(\frac{5}{\sqrt{30}} \right)^2 + \left(\frac{1}{\sqrt{30}} \right)^2}$

$$= \sqrt{\frac{4}{30} + \frac{25}{30} + \frac{1}{30}} = 1$$

Definition

Any vector with norm of 1 is called a ***unit vector***.

Example

The following are all unit vectors: $\begin{pmatrix} 2/\sqrt{30} \\ 5/\sqrt{30} \\ 1/\sqrt{30} \end{pmatrix}$, $\begin{pmatrix} 1 \\ 0 \end{pmatrix}$, $\begin{pmatrix} 0 \\ 1 \end{pmatrix}$, $\begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}$

Example

Suppose that $\boldsymbol{v} = \begin{pmatrix} v_1 \\ v_2 \\ v_3 \end{pmatrix} \neq \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$. Show that $\dfrac{1}{\| \boldsymbol{v} \|} \boldsymbol{v}$ is a unit vector.

Solution

Since $\| \boldsymbol{v} \| = \left\| \begin{pmatrix} v_1 \\ v_2 \\ v_3 \end{pmatrix} \right\| = \sqrt{v_1^2 + v_2^2 + v_3^2}$,

$$\frac{1}{\| \boldsymbol{v} \|} \boldsymbol{v} = \frac{1}{\sqrt{v_1^2 + v_2^2 + v_3^2}} \begin{pmatrix} v_1 \\ v_2 \\ v_3 \end{pmatrix} = \begin{pmatrix} v_1/\sqrt{v_1^2 + v_2^2 + v_3^2} \\ v_2/\sqrt{v_1^2 + v_2^2 + v_3^2} \\ v_3/\sqrt{v_1^2 + v_2^2 + v_3^2} \end{pmatrix}$$

and so

$$\left\| \frac{1}{\| \boldsymbol{v} \|} \boldsymbol{v} \right\| = \sqrt{\left(\frac{v_1}{\sqrt{v_1^2 + v_2^2 + v_3^2}} \right)^2 + \left(\frac{v_2}{\sqrt{v_1^2 + v_2^2 + v_3^2}} \right)^2 + \left(\frac{v_3}{\sqrt{v_1^2 + v_2^2 + v_3^2}} \right)^2}$$

$$= \sqrt{\frac{v_1^2}{v_1^2 + v_2^2 + v_3^2} + \frac{v_2^2}{v_1^2 + v_2^2 + v_3^2} + \frac{v_3^2}{v_1^2 + v_2^2 + v_3^2}}$$

$$= 1$$

Problem

Formulate a conjecture about the relationship between $\boldsymbol{v} \cdot \boldsymbol{u}$ and $\| \boldsymbol{v} \| \; \| \boldsymbol{u} \|$ by considering the following pairs of vectors. Hint: Consider $\dfrac{\boldsymbol{v} \cdot \boldsymbol{u}}{\| \boldsymbol{v} \| \; \| \boldsymbol{u} \|}$ and sketch each vector pair in the plane.

1. $\boldsymbol{v} = \begin{pmatrix} 5 \\ 0 \end{pmatrix}; \; \boldsymbol{u} = \begin{pmatrix} 3 \\ 3 \end{pmatrix}$

2. $\boldsymbol{v} = \begin{pmatrix} 3 \\ -4 \end{pmatrix}; \; \boldsymbol{u} = \begin{pmatrix} 4 \\ 3 \end{pmatrix}$

3. $\boldsymbol{v} = \begin{pmatrix} \sqrt{2} \\ \sqrt{2} \end{pmatrix}; \; \boldsymbol{u} = \begin{pmatrix} \dfrac{\sqrt{3}+1}{2\sqrt{2}} \\ \dfrac{\sqrt{3}-1}{2\sqrt{2}} \end{pmatrix}$

Cauchy-Schwarz Inequality

If $\boldsymbol{u}$ and $\boldsymbol{v}$ are vectors in $\mathbb{R}^2$ or $\mathbb{R}^3$, then

$$\left| \boldsymbol{v} \cdot \boldsymbol{u} \right| \le \| \boldsymbol{v} \| \; \| \boldsymbol{u} \| .$$

Proof

If $\boldsymbol{u} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$ or $\boldsymbol{u} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$, then the result is clearly true. So, suppose that $\boldsymbol{u}$ is not the zero vector. Then we consider the vector $t\,\boldsymbol{u} + \boldsymbol{v}$ where t is any real number. Since

$$(t\,\boldsymbol{u} + \boldsymbol{v}) \cdot (t\,\boldsymbol{u} + \boldsymbol{v}) \ge 0 \qquad \text{(Why?)}$$

it follows by a direct computation that

$$(\boldsymbol{u} \cdot \boldsymbol{u})\, t^2 + 2(\boldsymbol{u} \cdot \boldsymbol{v})\, t + \boldsymbol{v} \cdot \boldsymbol{v} \ge 0.$$

The above is a quadratic in the real variable t with $a = \boldsymbol{u} \cdot \boldsymbol{u}$, $b = 2(\boldsymbol{u} \cdot \boldsymbol{v})$, and $c = \boldsymbol{v} \cdot \boldsymbol{v}$.

Because

$$(\boldsymbol{u}\cdot\boldsymbol{u})\,t^2 + 2(\boldsymbol{u}\cdot\boldsymbol{v})\,t + \boldsymbol{v}\cdot\boldsymbol{v} \geq 0,$$

we have that

$$b^2 - 4ac \leq 0. \qquad\qquad \text{(Why?)}$$

Hence,

$$\begin{aligned} b^2 - 4ac &\leq 0 \\ b^2 &\leq 4ac \\ 4(\boldsymbol{u}\cdot\boldsymbol{v})^2 &\leq 4\,(\boldsymbol{u}\cdot\boldsymbol{u})\,(\boldsymbol{v}\cdot\boldsymbol{v}) \\ (\boldsymbol{u}\cdot\boldsymbol{v})^2 &\leq (\boldsymbol{u}\cdot\boldsymbol{u})\,(\boldsymbol{v}\cdot\boldsymbol{v}) \end{aligned}$$

The desired result follows by taking the square root of both sides of the last inequality above.

Question: In the above proof, why is it important that $\boldsymbol{u} \neq \mathbf{0}$?

So, by the Cauchy-Schwarz Inequality, it follows that for nonzero vectors $\boldsymbol{u}$ and $\boldsymbol{v}$ that

$$\frac{|\boldsymbol{v}\cdot\boldsymbol{u}|}{\|\boldsymbol{v}\|\ \|\boldsymbol{u}\|} \leq 1.$$

Definition

The ***angle*** 2 between two nonzero vectors $\boldsymbol{u}$ and $\boldsymbol{v}$ is given by

$$\cos\theta = \frac{\boldsymbol{u}\cdot\boldsymbol{v}}{\|\boldsymbol{u}\|\ \|\boldsymbol{v}\|}, \quad 0 \leq \theta \leq \pi.$$

Example

1. Find the angle 2 between the vectors $\boldsymbol{u} = \begin{pmatrix} 2 \\ 3 \end{pmatrix}$ & $\boldsymbol{v} = \begin{pmatrix} -7 \\ 1 \end{pmatrix}$.

2. Find the angle 2 between the vectors $\boldsymbol{u} = \begin{pmatrix} 3 \\ -7 \\ 2 \end{pmatrix}$ & $\boldsymbol{v} = \begin{pmatrix} 10 \\ 4 \\ -1 \end{pmatrix}$.

Solution

1. By the above definition,

$$\cos\theta = \frac{\boldsymbol{u} \cdot \boldsymbol{v}}{\|\boldsymbol{u}\| \, \|\boldsymbol{v}\|}$$

$$= \frac{(2)(-7) + (3)(1)}{\sqrt{2^2 + 3^2}\sqrt{(-7)^2 + 1^2}}$$

$$= \frac{-11}{\sqrt{650}}$$

and so

$$\theta = \arccos\frac{-11}{\sqrt{650}} \approx 2.0169018757 \; rad \; (115.55996517°).$$

2. Again, applying the above definition, we find that

$$\cos\theta = \frac{\boldsymbol{u} \cdot \boldsymbol{v}}{\|\boldsymbol{u}\| \, \|\boldsymbol{v}\|}$$

$$= \frac{(3)(10) + (-7)(4) + (2)(-1)}{\sqrt{3^2 + (-7)^2 + 2^2}\sqrt{10^2 + 4^2 + (-1)^2}}$$

$$= 0$$

and so

$$\theta = \arccos 0 = \frac{\pi}{2} \; rad \; (90°).$$

Definition

Suppose that $\boldsymbol{u}$ and $\boldsymbol{v}$ are two nonzero vectors.

1. We say that $\boldsymbol{u}$ and $\boldsymbol{v}$ are ***parallel*** if the angle between them is either 0 or B.
2. We say that $\boldsymbol{u}$ and $\boldsymbol{v}$ are ***orthogonal*** (or perpendicular) if the angle between them is $\frac{\pi}{2}$ *rad*.

Problem

Explain why two nonzero vectors are orthogonal precisely when their dot product is zero.

Problem

True or false.

1. If $\boldsymbol{u}$ is orthogonal to both $\boldsymbol{v}$ and $\boldsymbol{w}$, then $\boldsymbol{v}$ and $\boldsymbol{w}$ are parallel.

   (Does your answer depend on whether the vectors are in $\mathbb{R}^2$ or $\mathbb{R}^3$?)
2. If $\boldsymbol{u}$ is orthogonal to both $\boldsymbol{v}$ and $\boldsymbol{w}$, then $\boldsymbol{u}$ is orthogonal to $\boldsymbol{v} + 2\,\boldsymbol{w}$.