Matrix Inverses

Consider the ordinary algebraic equation $ax = b$ and its solution shown below:

$$\begin{aligned} ax &= b \\ a^{-1}(ax) &= a^{-1}b \\ (a^{-1}a)\,x &= a^{-1}b \\ 1\,x &= a^{-1}b \\ x &= a^{-1}b \end{aligned}$$

Since the linear system

$$\begin{array}{cccccc} a_{11}\,x_1 & +\,a_{12}\,x_2 & +\,a_{13}\,x_2 & +\,\ldots & +\,a_{1n}\,x_n & =\,b_1 \\ a_{21}\,x_1 & +\,a_{22}\,x_2 & +\,a_{23}\,x_2 & +\,\ldots & +\,a_{2n}\,x_n & =\,b_1 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ a_{n1}\,x_1 & +\,a_{n2}\,x_2 & +\,a_{n3}\,x_2 & +\,\ldots & +\,a_{nn}\,x_n & =\,b_1 \end{array}$$

can be written as

$$\boldsymbol{A}\,\boldsymbol{x} = \boldsymbol{b}$$

where

$$A = \begin{pmatrix} a_{11} & a_{12} & \ldots & a_{1n} \\ a_{21} & a_{22} & \ldots & a_{2n} \\ \vdots & \vdots & \vdots & \vdots \\ a_{n1} & a_{n2} & \ldots & a_{nn} \end{pmatrix}, \quad \boldsymbol{x} = \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix}, \quad \text{and} \quad \boldsymbol{b} = \begin{pmatrix} b_1 \\ b_2 \\ \vdots \\ b_n \end{pmatrix},$$

($\boldsymbol{A}$ = coefficient matrix, $\boldsymbol{x}$ = variable vector, $\boldsymbol{b}$ = constant vector) it is reasonable to ask if the matrix equation $\boldsymbol{A}\,\boldsymbol{x} = \boldsymbol{b}$ corresponding to above system of n linear equation in n variables can be solved for $\boldsymbol{x}$ in a manner similar to the way the ordinary algebraic equation $ax = b$ is solved for x.

That is, is it possible to solve $\boldsymbol{A}\ \boldsymbol{x} = \boldsymbol{b}$ for the vector $\boldsymbol{x}$ as follows:

$$\begin{aligned} A\boldsymbol{x} &= \boldsymbol{b} \\ A^{-1}\left(A\boldsymbol{x}\right) &= A^{-1}\boldsymbol{b} \\ \left(A^{-1}A\right)\boldsymbol{x} &= A^{-1}\boldsymbol{b} \\ I_n\,\boldsymbol{x} &= A^{-1}\boldsymbol{b} \\ \boldsymbol{x} &= A^{-1}\boldsymbol{b} \end{aligned}$$

Recall that $\boldsymbol{I_n}$ is a $n \times n$ *identity matrix* with $I_{ij} = \begin{cases} 1 & \text{if } i = j \\ 0 & \text{if } i \neq j \end{cases}$. For all matrix products that are defined, $I_n\,\boldsymbol{B} = \boldsymbol{B}$ and $\boldsymbol{B}\,I_n = \boldsymbol{B}$.

The above scheme for solving the matrix equation $\boldsymbol{A}\ \boldsymbol{x} = \boldsymbol{b}$ for the vector $\boldsymbol{x}$ depends on our ability to find a matrix $\boldsymbol{B}$ so that the product of $\boldsymbol{B}$ with the coefficient matrix $\boldsymbol{A}$ is the identity matrix $\boldsymbol{I_n}$.

Example

1. Find a matrix $\boldsymbol{B} = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$, if possible, so that $B\begin{pmatrix} -1 & 2 \\ -3 & 5 \end{pmatrix} = I_2$.

Solution

The equation

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix}\begin{pmatrix} -1 & 2 \\ -3 & 5 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

after matrix multiplication on the left becomes

$$\begin{pmatrix} -a - 3b & 2a + 5b \\ -c - 3d & 2c + 5d \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}.$$

By matrix equality, we obtain the system

$$\begin{aligned} -a \quad - 3b \qquad\qquad\qquad &= 1 \\ 2a \quad + 5b \qquad\qquad\qquad &= 0 \\ -c \quad - 3d &= 0 \\ 2c \quad + 5d &= 1 \end{aligned}$$

which has solutions of $a = 5,\ b = -2,\ c = 3,\ d = -1$.

As a check, direct computation shows that

$$\begin{pmatrix} 5 & -2 \\ 3 & -1 \end{pmatrix} \begin{pmatrix} -1 & 2 \\ -3 & 5 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}.$$

We also observe that

$$\begin{pmatrix} -1 & 2 \\ -3 & 5 \end{pmatrix} \begin{pmatrix} 5 & -2 \\ 3 & -1 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}.$$

2. Find a matrix $\boldsymbol{B} = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$, if possible, so that $B \begin{pmatrix} -1 & 2 \\ -3 & 6 \end{pmatrix} = I_2$.

Solution

Here

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} -1 & 2 \\ -3 & 6 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

becomes

$$\begin{pmatrix} -a - 3b & 2a + 6b \\ -c - 3d & 2c + 6d \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

and the resulting system of linear equations is given by

$$\begin{aligned} -a \quad - 3b \qquad\qquad\qquad &= 1 \\ 2a \quad + 6b \qquad\qquad\qquad &= 0 \\ -c \quad - 3d &= 0 \\ 2c \quad + 6d &= 1 \end{aligned}$$

We quickly see that this particular system is dependent.

The point: Even for 2×2 *matrices it is not always possible to solve the matrix equation* $\boldsymbol{A}\,\boldsymbol{x} = \boldsymbol{b}$ *for the solution vector* $\boldsymbol{x}$ *as it is not always possible to find matrices* $\boldsymbol{B}$ *so that* $\boldsymbol{B}\,\boldsymbol{A} = I_n$.

Definition

An $n \times n$ square matrix $\boldsymbol{A}$ is ***invertible*** (or ***nonsingular***) if there exists an $n \times n$ matrix $\boldsymbol{B}$ such that

$$\boldsymbol{A}\,\boldsymbol{B} = \boldsymbol{B}\,\boldsymbol{A} = I_n.$$

In this case, we call the matrix $\boldsymbol{B}$ an ***inverse*** for $\boldsymbol{A}$. A matrix that does not have an inverse is said to be ***noninvertible*** or ***singular***.

Problem

What is the problem with the equation $\boldsymbol{A}\,\boldsymbol{B} = \boldsymbol{B}\,\boldsymbol{A}$ if $\boldsymbol{A}$ is an $m \times n$ matrix with $m \neq n$? Can a nonsquare matrix have an inverse in the sense of the above definition?

Examples

1. As seen above, the matrices $\begin{pmatrix} 5 & -2 \\ 3 & -1 \end{pmatrix}$ & $\begin{pmatrix} -1 & 2 \\ -3 & 5 \end{pmatrix}$ are inverses.

2. Also as seen above, the matrix $\begin{pmatrix} -1 & 2 \\ -3 & 6 \end{pmatrix}$ is singular.

3. The matrix $\begin{pmatrix} 4 & 8 & -7 & 14 \\ 2 & 5 & -4 & 6 \\ 0 & 2 & 1 & -7 \\ 3 & 6 & -5 & 10 \end{pmatrix}$ is invertible or nonsingular. An inverse can be shown to be

the 4×4 matrix $\begin{pmatrix} 27 & -10 & 4 & -29 \\ -16 & 5 & -2 & 18 \\ -17 & 4 & -2 & 20 \\ -7 & 2 & -1 & 8 \end{pmatrix}$. As a cheap check on this statement we note that in the product of the above two matrices that

$$a_{11} = \begin{pmatrix} 4 & 8 & -7 & 14 \end{pmatrix} \cdot \begin{pmatrix} 27 \\ -16 \\ -17 \\ -7 \end{pmatrix} = 1$$

and

$$a_{31} = \begin{pmatrix} 0 & 2 & 1 & -7 \end{pmatrix} \cdot \begin{pmatrix} 27 \\ -16 \\ -17 \\ -7 \end{pmatrix} = 0.$$

Example

Show that if $ad - bc \neq 0$, then $\begin{pmatrix} a & b \\ c & d \end{pmatrix}^{-1} = \dfrac{1}{ad - bc}\begin{pmatrix} d & -b \\ -c & a \end{pmatrix}$.

Solution

$$\frac{1}{ad - bc}\begin{pmatrix} d & -b \\ -c & a \end{pmatrix}\begin{pmatrix} a & b \\ c & d \end{pmatrix} = \frac{1}{ad - bc}\begin{pmatrix} ad - bc & bd - bd \\ -ac + ac & -bc + ad \end{pmatrix} = I_2$$

(The other product also yields the 2×2 identity matrix I_2.) The quantity $ad - bc$ is called the ***determinant*** of the matrix $\boldsymbol{A} = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ and is denoted by $Det(\boldsymbol{A}) = |\boldsymbol{A}|$.

We note that $\begin{pmatrix} -1 & 2 \\ -3 & 5 \end{pmatrix}^{-1} = \dfrac{1}{-1(5) - 2(-3)}\begin{pmatrix} 5 & -2 \\ 3 & -1 \end{pmatrix} = \begin{pmatrix} 5 & -2 \\ 3 & -1 \end{pmatrix}$.

<u>Theorem</u>

If $\boldsymbol{A}$ is an $\boldsymbol{n} \times \boldsymbol{n}$ nonsingular matrix, then $\boldsymbol{A}$ has exactly one inverse.

<u>Proof</u>

Suppose that $\boldsymbol{B}$ and $\boldsymbol{C}$ are matrix inverses of $\boldsymbol{A}$. Then, by definition,

$$\boldsymbol{A}\,\boldsymbol{B} = \boldsymbol{B}\,\boldsymbol{A} = I_n$$

and

$$\boldsymbol{A}\,\boldsymbol{C} = \boldsymbol{C}\,\boldsymbol{A} = I_n.$$

Now,

$$\begin{aligned} \boldsymbol{A}\,\boldsymbol{C} = I_n \quad &\Rightarrow \quad \boldsymbol{B}\,(\boldsymbol{A}\,\boldsymbol{C}) = \boldsymbol{B}\,I_n = \boldsymbol{B} \\ &\Rightarrow \quad (\boldsymbol{B}\,\boldsymbol{A})\,\boldsymbol{C} = \boldsymbol{B} \\ &\Rightarrow \quad I_n\,\boldsymbol{C} = \boldsymbol{B} \\ &\Rightarrow \quad \boldsymbol{C} = \boldsymbol{B}. \end{aligned}$$ >

Since an invertible square matrix $\boldsymbol{A}$ has a unique inverse, we will denote it by A^{-1}.

<u>Theorem</u>

If $\boldsymbol{A}$ is a nonsingular matrix, then the unique solution to the equation $\boldsymbol{A}\,\boldsymbol{x} = \boldsymbol{b}$ is given by $\boldsymbol{x} = A^{-1}\boldsymbol{b}$.

<u>Proof</u>

We first show that $A^{-1}\boldsymbol{b}$ solves the equation $\boldsymbol{A}\,\boldsymbol{x} = \boldsymbol{b}$. To this end we observe that

$$\boldsymbol{b} = \boldsymbol{b} \quad \Rightarrow \quad I_n\,\boldsymbol{b} = \boldsymbol{b} \quad \Rightarrow \quad (\boldsymbol{A}\,\boldsymbol{A}^{-1})\,\boldsymbol{b} = \boldsymbol{b} \quad \Rightarrow \quad \boldsymbol{A}\,(\boldsymbol{A}^{-1}\,\boldsymbol{b}) = \boldsymbol{b}$$

Thus, $\boldsymbol{x} = A^{-1}\boldsymbol{b}$ is a solution of $\boldsymbol{A}\,\boldsymbol{x} = \boldsymbol{b}$.

Now, we suppose that the vector $\boldsymbol{y}$ also solves $\boldsymbol{A}\,\boldsymbol{x} = \boldsymbol{b}$. Then

$$\begin{aligned} \boldsymbol{A}\,\boldsymbol{y} = \boldsymbol{b} \quad &\Rightarrow \quad \boldsymbol{A}^{-1}\,(\boldsymbol{A}\,\boldsymbol{y}) = \boldsymbol{A}^{-1}\,\boldsymbol{b} \\ &\Rightarrow \quad (\boldsymbol{A}^{-1}\,\boldsymbol{A})\,\boldsymbol{y} = \boldsymbol{A}^{-1}\,\boldsymbol{b} \end{aligned}$$

Y $\quad I_n\, \boldsymbol{y} = \boldsymbol{A}^{-1}\, \boldsymbol{b}$

Y $\quad \boldsymbol{y} = A^{-1} \boldsymbol{b}$. >

Theorem

Suppose that $\boldsymbol{A}$ and $\boldsymbol{B}$ are nonsingular matrices. Then $\boldsymbol{AB}$ is nonsingular and $(\boldsymbol{A}\boldsymbol{B})^{-1} = \boldsymbol{B}^{-1}\, \boldsymbol{A}^{-1}$.

To see that the above is true one only need simplify the two products

$$(\boldsymbol{A}\ \boldsymbol{B})\,(\boldsymbol{B}^{-1}\, \boldsymbol{A}^{-1}) \ \ \& \ \ (\boldsymbol{B}^{-1}\, \boldsymbol{A}^{-1})\,(\boldsymbol{A}\ \boldsymbol{B})$$

obtaining the identity matrix in each case. The above may be generalized via mathematical induction in a natural manner.

Example

1. The matrix equation $\begin{pmatrix} 5 & -2 \\ 3 & -1 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} b_1 \\ b_2 \end{pmatrix}$ has a unique solution for any scalars

$b_1,\ b_2 \in \mathbb{R}$ given by

$$\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 5 & -2 \\ 3 & -1 \end{pmatrix}^{-1} \begin{pmatrix} b_1 \\ b_2 \end{pmatrix} = \begin{pmatrix} -1 & 2 \\ -3 & 5 \end{pmatrix} \begin{pmatrix} b_1 \\ b_2 \end{pmatrix}.$$

2. The linear system

$$\begin{aligned} 4\,x_1 &+ 8\,x_2 - 7\,x_3 + 14\,x_4 = b_1 \\ 2\,x_1 &+ 5\,x_2 - 4\,x_3 + 6\,x_4 = b_2 \\ &+ 2\,x_2 + x_3 + -7\,x_4 = b_3 \\ 3\,x_1 &+ 6\,x_2 - 5\,x_3 + 10\,x_4 = b_4 \end{aligned}$$

has a unique solution for all scalars $b_1,\ b_2,\ b_3,\ b_4 \in \mathbb{R}$ given by

$$\begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix} = \begin{pmatrix} 4 & 8 & -7 & 14 \\ 2 & 5 & -4 & 6 \\ 0 & 2 & 1 & -7 \\ 3 & 6 & -5 & 10 \end{pmatrix}^{-1} \begin{pmatrix} b_1 \\ b_2 \\ b_3 \\ b_4 \end{pmatrix}.$$

$$= \begin{pmatrix} 27 & -10 & 4 & -29 \\ -16 & 5 & -2 & 18 \\ -17 & 4 & -2 & 20 \\ -7 & 2 & -1 & 8 \end{pmatrix} \begin{pmatrix} b_1 \\ b_2 \\ b_3 \\ b_4 \end{pmatrix}$$

In practice, one typically doesn't solve the matrix equation $\boldsymbol{A}\,\boldsymbol{x} = \boldsymbol{b}$ (or, equivalently, linear systems) by computing the inverse matrix of $\boldsymbol{A}$ and forming the unique solution $\boldsymbol{x} = A^{-1}\boldsymbol{b}$. The existence of the inverse matrix $\boldsymbol{A}^{-1}$ for a square matrix $\boldsymbol{A}$ does have theoretical value as shown by the next theorem.

<u>Theorem</u>

Suppose $\boldsymbol{A}$ is an $\boldsymbol{n} \times \boldsymbol{n}$ matrix. Then the following are equivalent:

1. $\boldsymbol{A}$ is invertible.
2. $\boldsymbol{A}$ is *row-equivalent* to the $\boldsymbol{n} \times \boldsymbol{n}$ identity matrix $I_{\boldsymbol{n}}$. (That is, it is possible to transform the matrix $\boldsymbol{A}$ into $I_{\boldsymbol{n}}$ using the elementary row operations of (i) multiply (or divide) one row by a nonzero number, (ii) add a multiple of one row to another row, and (iii) interchange two rows.)
3. $\boldsymbol{A}$ is the product of elementary matrices.
4. The homogeneous system $\boldsymbol{A}\,\boldsymbol{x} = \boldsymbol{0}_{\boldsymbol{n}\times 1}$ has only the trivial solution $\boldsymbol{x} = \boldsymbol{0}_{\boldsymbol{n}\times 1}$.
5. The linear system $\boldsymbol{A}\,\boldsymbol{x} = \boldsymbol{b}$ has a solution for each $\boldsymbol{n} \times 1$ column vector $\boldsymbol{b}$.

Gauss-Jordan Method for Computing $\boldsymbol{A}^{-1}$:

Assume $\boldsymbol{A}$ is an invertible square matrix of size n.

1. Form the augmented matrix $\left(A \mid I_n \right)$.

2. Use the elementary row operations to reduce the augmented matrix to the form $\left(I_n \mid \boldsymbol{B} \right)$.

3. Deduce that $\boldsymbol{A}^{-1} = \boldsymbol{B}$.

<u>Examples</u>

1. Consider the matrix $\begin{pmatrix} 2 & 1 \\ 3 & 2 \end{pmatrix}$. Using the above formula for computing the inverse of a 2×2

matrix

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix}^{-1} = \frac{1}{ad - bc} \begin{pmatrix} d & -b \\ -c & a \end{pmatrix}$$

we find that

$$\begin{pmatrix} 2 & 1 \\ 3 & 2 \end{pmatrix}^{-1} = \frac{1}{2(2) - 1(3)} \begin{pmatrix} 2 & -1 \\ -3 & 2 \end{pmatrix} = \begin{pmatrix} 2 & -1 \\ -3 & 2 \end{pmatrix}.$$

Now, by the *Gauss-Jordan Method*

$$\left(\begin{array}{cc|cc} 2 & 1 & 1 & 0 \\ 3 & 2 & 0 & 1 \end{array}\right) \sim \left(\begin{array}{cc|cc} 2 & 1 & 1 & 0 \\ 0 & 1/2 & -3/2 & 1 \end{array}\right)$$

$$\sim \left(\begin{array}{cc|cc} 2 & 1 & 1 & 0 \\ 0 & 1 & -3 & 2 \end{array}\right)$$

$$\sim \left(\begin{array}{cc|cc} 2 & 0 & 4 & -2 \\ 0 & 1 & -3 & 2 \end{array}\right)$$

$$\sim \left(\begin{array}{cc|cc} 1 & 0 & 2 & -1 \\ 0 & 1 & -3 & 2 \end{array}\right)$$

and, again, we see that

$$\begin{pmatrix} 2 & 1 \\ 3 & 2 \end{pmatrix}^{-1} = \begin{pmatrix} 2 & -1 \\ -3 & 2 \end{pmatrix}.$$

2. Consider

$$\left(\begin{array}{ccc|ccc} 2 & 4 & 3 & 1 & 0 & 0 \\ 0 & 1 & -1 & 0 & 1 & 0 \\ 3 & 5 & 7 & 0 & 0 & 1 \end{array}\right) \sim \left(\begin{array}{ccc|ccc} 1 & 2 & 3/2 & 1/2 & 0 & 0 \\ 0 & 1 & -1 & 0 & 1 & 0 \\ 3 & 5 & 7 & 0 & 0 & 1 \end{array}\right)$$

$$\sim \left(\begin{array}{ccc|ccc} 1 & 2 & 3/2 & 1/2 & 0 & 0 \\ 0 & 1 & -1 & 0 & 1 & 0 \\ 0 & -1 & 5/2 & -3/2 & 0 & 1 \end{array}\right)$$

$$\sim \left(\begin{array}{ccc|ccc} 1 & 2 & 3/2 & 1/2 & 0 & 0 \\ 0 & 1 & -1 & 0 & 1 & 0 \\ 0 & 0 & 3/2 & -3/2 & 1 & 1 \end{array}\right)$$

$$\sim \left(\begin{array}{ccc|ccc} 1 & 0 & 7/2 & 1/2 & -2 & 0 \\ 0 & 1 & -1 & 0 & 1 & 0 \\ 0 & 0 & 3/2 & -3/2 & 1 & 1 \end{array}\right)$$

$$\sim \left(\begin{array}{ccc|ccc} 1 & 0 & 7/2 & 1/2 & -2 & 0 \\ 0 & 1 & -1 & 0 & 1 & 0 \\ 0 & 0 & 1 & -1 & 2/3 & 2/3 \end{array}\right)$$

$$\sim \left(\begin{array}{ccc|ccc} 1 & 0 & 0 & 4 & -13/3 & -7/3 \\ 0 & 1 & -1 & 0 & 1 & 0 \\ 0 & 0 & 1 & -1 & 2/3 & 2/3 \end{array}\right)$$

$$\sim \left(\begin{array}{ccc|ccc} 1 & 0 & 0 & 4 & -13/3 & -7/3 \\ 0 & 1 & 0 & -1 & 5/3 & 2/3 \\ 0 & 0 & 1 & -1 & 2/3 & 2/3 \end{array}\right)$$

We conclude via the Gauss-Jordan method that $\begin{pmatrix} 2 & 4 & 3 \\ 0 & 1 & -1 \\ 3 & 5 & 7 \end{pmatrix}^{-1} = \begin{pmatrix} 4 & -13/3 & -7/3 \\ -1 & 5/3 & 2/3 \\ -1 & 2/3 & 2/3 \end{pmatrix}$.

Example

What if we apply the Gauss-Jordan method to a square matrix without knowing whether the matrix is in fact invertible? Well, consider the following sequence of matrices.

$$\begin{pmatrix} 1 & 2 & 0 & 1 & 0 & 0 \\ 3 & -1 & 2 & 0 & 1 & 0 \\ -2 & 3 & -2 & 0 & 0 & 1 \end{pmatrix} \sim \begin{pmatrix} 1 & 2 & 0 & 1 & 0 & 0 \\ 0 & -7 & 2 & -3 & 1 & 0 \\ -2 & 3 & -2 & 0 & 0 & 1 \end{pmatrix}$$

$$\sim \begin{pmatrix} 1 & 2 & 0 & 1 & 0 & 0 \\ 0 & -7 & 2 & -3 & 1 & 0 \\ 0 & 7 & -2 & 2 & 0 & 1 \end{pmatrix}$$

$$\sim \begin{pmatrix} 1 & 2 & 0 & 1 & 0 & 0 \\ 0 & -7 & 2 & -3 & 1 & 0 \\ 0 & 0 & 0 & -1 & 1 & 1 \end{pmatrix}$$

The above shows that the matrix $\begin{pmatrix} 1 & 2 & 0 \\ 3 & -1 & 2 \\ -2 & 3 & -2 \end{pmatrix}$ is *not* row-equivalent to the identity matrix

I_3 (why?) and so, by a theorem above, we conclude that the given matrix is not invertible.

Problem

1. True or false: If the matrix A has a row (or column) of zeros, then A is not invertible.
2. True or false: If the matrix A has a has two rows (or columns) that are proportional, then A is not invertible.

Problem

Let $S = \begin{pmatrix} -1 & 4 & 6 \\ 0 & 1 & 3 \\ 0 & 0 & -1 \end{pmatrix}$. Show that $S^2 = I_3$ and $S = S^{-1}$. (So what?)