Rank, Row-Reduced Form, and Solutions to $\boldsymbol{A}\boldsymbol{x} = \boldsymbol{b}$

Example

1. Consider the 3×4 matrix $\boldsymbol{A}$ given by

$$\boldsymbol{A} = \begin{pmatrix} 1 & -2 & 3 & 9 \\ -1 & 3 & 0 & -4 \\ 2 & -5 & 5 & 17 \end{pmatrix}.$$

Using the three elementary row operations we may rewrite $\boldsymbol{A}$ in an *echelon form* as

$$\boldsymbol{U} = \begin{pmatrix} 1 & -2 & 3 & 9 \\ 0 & 1 & 3 & 5 \\ 0 & 0 & 2 & 4 \end{pmatrix}$$

or, continuing with additional row operations, in the *reduced row-echelon form*

$$\boldsymbol{R} = \begin{pmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & 1 & 2 \end{pmatrix}.$$

From the above, the homogeneous system $\boldsymbol{A}\boldsymbol{x} = \boldsymbol{0}$ has a solution that can be read as

$$\begin{aligned} x \qquad\qquad + w &= 0 \\ y \qquad - w &= 0 \\ z + 2w &= 0 \end{aligned}$$

or in vector form as

$$\begin{pmatrix} x \\ y \\ z \\ w \end{pmatrix} = \begin{pmatrix} -w \\ w \\ -2w \\ w \end{pmatrix} = w \begin{pmatrix} -1 \\ 1 \\ -2 \\ 1 \end{pmatrix}.$$

In the above, recall that w is a *free variable* while x, y, and z are the three *pivot variables*. The solution of the homogeneous system $\boldsymbol{A}\boldsymbol{x} = \boldsymbol{0}$ (i.e., the *null space* of $\boldsymbol{A}$) consists of all scalar multiples of the vector

$$\begin{pmatrix} -1 \\ 1 \\ -2 \\ 1 \end{pmatrix}$$

and, hence, has *dimension* 1 (the number of free variables).

2. Consider the 4×5 matrix given by

$$\begin{pmatrix} 1 & 0 & -2 & 1 & 0 \\ 0 & -1 & -3 & 1 & 3 \\ -2 & -1 & 1 & -1 & 3 \\ 0 & 3 & 9 & 0 & -12 \end{pmatrix}.$$

The associated *row reduced echelon form* is given by

$$\begin{pmatrix} 1 & 0 & -2 & 0 & 1 \\ 0 & 1 & 3 & 0 & -4 \\ 0 & 0 & 0 & 1 & -1 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}.$$

For the homogeneous system

$$\begin{pmatrix} 1 & 0 & -2 & 1 & 0 \\ 0 & -1 & -3 & 1 & 3 \\ -2 & -1 & 1 & -1 & 3 \\ 0 & 3 & 9 & 0 & -12 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{pmatrix} = \mathbf{0}_{5\times 1}$$

or, equivalently,

$$\begin{pmatrix} 1 & 0 & -2 & 0 & 1 \\ 0 & 1 & 3 & 0 & -4 \\ 0 & 0 & 0 & 1 & -1 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{pmatrix} = \mathbf{0}_{5\times 1}$$

we see that $x_1,\ x_2,\ \&\ x_4$ are the three *pivot variables* while $x_3\ \&\ x_5$ are the two *free variables*.

Here the *null space* of the given coefficient matrix is

$$\left\langle \begin{pmatrix} 2 \\ -3 \\ 1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} -1 \\ 4 \\ 0 \\ 1 \\ 1 \end{pmatrix} \right\rangle$$

and has *dimension* 2 (the number of free variables).

Definition

Suppose $\boldsymbol{A}$ is an $\boldsymbol{m} \times \boldsymbol{n}$ matrix.

1. We call the number of free variables of $\boldsymbol{A}\,\mathbf{x} = \boldsymbol{b}$ the ***nullity of A*** and we denote it by $Nullity(A)$.

2. We call the number of pivots of $\boldsymbol{A}$ the ***rank of A*** and we denoted it by $Rank(A)$.

Procedure for computing the rank of a matrix $\boldsymbol{A}$:

1. Use elementary row operations to transform $\boldsymbol{A}$ to a matrix $\boldsymbol{R}$ in reduced row echelon form.
2. $Rank(A)$ is the number of nonzero rows in $\boldsymbol{R}$. (Why?)

Example

1. For $\boldsymbol{A} = \begin{pmatrix} 1 & -2 & 3 & 9 \\ -1 & 3 & 0 & -4 \\ 2 & -5 & 5 & 17 \end{pmatrix}$ we see that $Rank(A) = 3$ and $Nullity(A) = 1$.

2. The *nullity* of $\begin{pmatrix} 1 & 0 & -2 & 1 & 0 \\ 0 & -1 & -3 & 1 & 3 \\ -2 & -1 & 1 & -1 & 3 \\ 0 & 3 & 9 & 0 & -12 \end{pmatrix}$ is 2 while its *rank* is 3.

Observe that in the above two examples that

$$Rank(A) + Nullity(A) = (\textit{number of columns in the coefficient matrix } \boldsymbol{A}).$$

This is no accident as the $Rank(A)$ counts the pivot variables, the $Nullity(A)$ counts the free variables, and the number of columns corresponds to the total number of variables for the coefficient matrix $\boldsymbol{A}$.

Theorem

Suppose $\boldsymbol{A}$ is an $\boldsymbol{m} \times \boldsymbol{n}$ matrix. Then

$$\boldsymbol{n} = Rank(A) + Nullity(A).$$

Example

Consider the matrix

$$\boldsymbol{B} = \begin{pmatrix} 1 & 1 & 4 & 1 & 2 \\ 0 & 1 & 2 & 1 & 1 \\ 0 & 0 & 0 & 1 & 2 \\ 1 & -1 & 0 & 0 & 2 \\ 2 & 1 & 6 & 0 & 1 \end{pmatrix}.$$

The *row reduced echelon form* of the above is

$$\boldsymbol{B} = \begin{pmatrix} 1 & 0 & 2 & 0 & 1 \\ 0 & 1 & 2 & 0 & -1 \\ 0 & 0 & 0 & 1 & 2 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}.$$

Since the number of nonzero rows is 3 (that is, there are three pivots), we see that

$$Rank(B) = 3$$

and that the

$$Nullity(B) = 5 - 3 = 2.$$

<u>Problem</u>

Is the above 5×5 matrix $\boldsymbol{B}$ invertible (a.k.a. nonsingular)? So?!!?

<u>Theorem</u>

Suppose that $\boldsymbol{A}$ is an $\boldsymbol{n} \times \boldsymbol{n}$ matrix. $\boldsymbol{A}$ is an invertible matrix if and only if $Rank(A) = \boldsymbol{n}$.

Why?

<u>Problem</u>

Find replacements for a, b, c, d so that the 3×3 matrix

$$\begin{pmatrix} 1 & 4 & 3 \\ 5 & a & b \\ c & d & 6 \end{pmatrix}$$

has rank of 1.

We now consider the nonhomogenous linear system $\boldsymbol{A}\,\boldsymbol{x} = \boldsymbol{b}$. We first observe that the solutions to the associated homogenous system $\boldsymbol{A}\,\boldsymbol{x} = \mathbf{0}$ form a vector space (called the ***null space***).

Problem

Can the same be said for the set of solutions to $\boldsymbol{A}\,\boldsymbol{x} = \boldsymbol{b}$?

Fortunately, there is a relationship between the solution sets to $\boldsymbol{A}\,\boldsymbol{x} = \mathbf{0}$ and $\boldsymbol{A}\,\boldsymbol{x} = \boldsymbol{b}$.

Theorem

If $\boldsymbol{x}_{\boldsymbol{p}}$ is a particular solution of $\boldsymbol{A}\,\boldsymbol{x} = \boldsymbol{b}$ (that is, $\boldsymbol{A}\,\boldsymbol{x}_{\boldsymbol{p}} = \boldsymbol{b}$), then every solution of $\boldsymbol{A}\,\boldsymbol{x} = \boldsymbol{b}$ can be written as

$$\boldsymbol{x} = \boldsymbol{x}_{\boldsymbol{h}} + \boldsymbol{x}_{\boldsymbol{p}}$$

where $\boldsymbol{x}_{\boldsymbol{h}}$ represents a solution of $\boldsymbol{A}\,\boldsymbol{x} = \mathbf{0}$.

Proof

Let $\boldsymbol{x}$ be any solution of $\boldsymbol{A}\,\boldsymbol{x} = \boldsymbol{b}$. Then

$$\boldsymbol{A}\left(\boldsymbol{x} - \boldsymbol{x}_{\boldsymbol{p}}\right) = \boldsymbol{A}\,\boldsymbol{x} - \boldsymbol{A}\,\boldsymbol{x}_{\boldsymbol{p}} - \boldsymbol{b} - \boldsymbol{b} = \mathbf{0}.$$

Thus, $\boldsymbol{x} - \boldsymbol{x}_{\boldsymbol{p}}$ is a solution of $\boldsymbol{A}\,\boldsymbol{x} = \mathbf{0}$, denote this solution by $\boldsymbol{x}_{\boldsymbol{h}}$. Then

$$\boldsymbol{x} = \boldsymbol{x}_{\boldsymbol{h}} + \boldsymbol{x}_{\boldsymbol{p}}$$

where $\boldsymbol{x}_{\boldsymbol{p}}$ is a particular solution of $\boldsymbol{A}\,\boldsymbol{x} = \boldsymbol{b}$ and $\boldsymbol{x}_{\boldsymbol{h}}$ represents a solution of $\boldsymbol{A}\,\boldsymbol{x} = \mathbf{0}$. >

Example

Solve $\boldsymbol{A}\ \boldsymbol{x} = \boldsymbol{b}$ where

$$\boldsymbol{A} = \begin{pmatrix} 1 & 2 & 0 & -1 & 1 \\ 1 & 3 & 1 & 1 & -1 \\ 2 & 5 & 1 & 0 & 0 \\ 3 & 6 & 0 & 0 & -6 \\ 1 & 5 & 3 & 5 & -5 \end{pmatrix}$$

and

$$\boldsymbol{b} = \begin{pmatrix} -10 \\ -9 \\ -19 \\ -27 \\ -7 \end{pmatrix}.$$

(Illustrate the above theorem in a particular setting.)

Solution

Using the elementary row operations (via technology!) on the augmented matrix $(\ \boldsymbol{A} \,|\, \boldsymbol{b}\ )$ we find that

$$(\ \boldsymbol{A} \,|\, \boldsymbol{b}\ ) = \begin{pmatrix} 1 & 2 & 0 & -1 & 1 & -10 \\ 1 & 3 & 1 & 1 & -1 & -9 \\ 2 & 5 & 1 & 0 & 0 & -19 \\ 3 & 6 & 0 & 0 & -6 & -27 \\ 1 & 5 & 3 & 5 & -5 & -7 \end{pmatrix}$$

is row equivalent to the *reduced row echelon form matrix*

$$\begin{pmatrix} 1 & 0 & -2 & 0 & -10 & -7 \\ 0 & 1 & 1 & 0 & 4 & -1 \\ 0 & 0 & 0 & 1 & -3 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}.$$

We see that $Rank(A) = 3$ (with free variables of x_3 & x_5). In vector form we may write the solution to $\boldsymbol{A}\,\boldsymbol{x} = \boldsymbol{b}$ as

$$\boldsymbol{x} = \begin{pmatrix} 2x_3 + 10x_5 - 7 \\ -x_3 - 4x_5 - 1 \\ x_3 \\ 3x_5 + 1 \\ x_5 \end{pmatrix} = x_3 \begin{pmatrix} 2 \\ -1 \\ 1 \\ 0 \\ 0 \end{pmatrix} + x_5 \begin{pmatrix} 10 \\ -4 \\ 0 \\ 3 \\ 1 \end{pmatrix} + \begin{pmatrix} -7 \\ -1 \\ 0 \\ 1 \\ 0 \end{pmatrix}.$$

The *null space for* $\boldsymbol{A}$ is given by $\left\langle \begin{pmatrix} 2 \\ -1 \\ 1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 10 \\ -4 \\ 0 \\ 3 \\ 1 \end{pmatrix} \right\rangle$ and $\boldsymbol{x_p} = \begin{pmatrix} -7 \\ -1 \\ 0 \\ 1 \\ 0 \end{pmatrix}$.

Definition

Suppose that $\boldsymbol{A}$ is an $m \times n$ matrix. Then we define the ***row space of A*** as

$$R_A = \langle r_1, r_2, r_3, \dots, r_m \rangle$$

where r_k is the k^{th} row of $\boldsymbol{A}$ and the ***column space of A*** as

$$C_A = \langle c_1, c_2, c_3, \dots, c_n \rangle$$

where c_k is the k^{th} column of $\boldsymbol{A}$.

Theorem

If an $m \times n$ matrix $\boldsymbol{A}$ is row equivalent to an $m \times n$ matrix $\boldsymbol{B}$, the row space of $\boldsymbol{A}$ is equal to the row space of $\boldsymbol{B}$.

Note: The above says that the elementary row operations do not change the row space of a matrix. However, the elementary row operations may change the column space.

Example

Since

$$\boldsymbol{A} = \begin{pmatrix} 1 & 2 & 0 & -1 & 1 \\ 1 & 3 & 1 & 1 & -1 \\ 2 & 5 & 1 & 0 & 0 \\ 3 & 6 & 0 & 0 & -6 \\ 1 & 5 & 3 & 5 & -5 \end{pmatrix} \quad \sim \quad \boldsymbol{B} = \begin{pmatrix} 1 & 0 & -2 & 0 & -10 \\ 0 & 1 & 1 & 0 & 4 \\ 0 & 0 & 0 & 1 & -3 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix},$$

the above theorem implies that $R_A = R_B$. In particular,

$$R_A = \langle \, (\ 1\ 0\ -2\ 0\ 10\ ), (\ 0\ 1\ 1\ 0\ 4\ ), (\ 0\ 0\ 0\ 1\ -3\ ) \, \rangle = R_B.$$

We note that only three row vectors (and not five!) were required to span R_A. Now, by forming

$$C = Transpose(rref(Tranpose(A))) = \left(rref(A^T)\right)^T$$

we find that

$$\boldsymbol{C} = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ -2 & 3 & 0 & 0 & 0 \end{pmatrix}$$

and so it follows by the above theorem that

$$C_A = \left\langle \begin{pmatrix} 1 \\ 0 \\ 1 \\ 0 \\ -2 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 1 \\ 0 \\ 3 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \\ 0 \end{pmatrix} \right\rangle$$

As with R_A, only three (column) vectors are required to span C_A. We observe that

$$(-10)\begin{pmatrix} 1 \\ 0 \\ 1 \\ 0 \\ -2 \end{pmatrix} + (-9)\begin{pmatrix} 0 \\ 1 \\ 1 \\ 0 \\ 3 \end{pmatrix} + (-27)\begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \\ 0 \end{pmatrix} = \begin{pmatrix} -10 \\ -9 \\ -19 \\ -27 \\ -7 \end{pmatrix}$$

and so $\boldsymbol{b} = \begin{pmatrix} -10 \\ -9 \\ -19 \\ -27 \\ -7 \end{pmatrix} \in C_A$.

Problem

Explain why in the above example that $C_A \neq C_B$. (Hint: Consider the rows of zeros in ***B***.)

The last example makes at least one direction of the following theorem plausible.

Theorem

The linear system $\boldsymbol{A}\,\boldsymbol{x} = \boldsymbol{b}$ is consistent if and only if $\boldsymbol{b} \in C_A$.

We close this unit by relating the rank of a coefficient matrix to the existence of at least one solution to $\boldsymbol{A}\ \boldsymbol{x} = \boldsymbol{b}$.

Theorem

The linear system $\boldsymbol{A}\ \boldsymbol{x} = \boldsymbol{b}$ is consistent if and only if $Rank(A) = Rank((A \mid b))$.