Having a glimpse of some of the possibilities for solutions of linear systems, we move to methods of finding these solutions. The basic idea we shall use is to try to simplify the system by eliminating some of the variables in some its equations. Thus, we refer to the method as **elimination**. The primary operation involved is the addition of a multiple of one equation to another.

Example

Consider the system

$$5x - 20y = 40$$
$$3x - 14y = -2.$$

Multiplying the first equation by -3/5 and adding that to the second eliminates the variable x in the second and the system becomes

$$5x - 20y = 40$$
$$-2y = -26$$

Then the second equation clearly requires that y = 13 and, substituting that value for y into the first equation yields 5x - 260 = 40 so x = 60.

Example

Next we examine the 3 by 3 system below.

$$2x + 18y - 10z = 6$$
$$x + 10y + 7z = -3$$
$$-4x - 33y + 5z = -13$$

The variable x is eliminated from the second and third equations by adding, respectively, - ½ and 4/2 = 2 times equation one to them resulting in the system below.

$$2x + 18y - 10z = 6$$
$$y + 12z = -6$$
$$3y - 15z = -1$$

Finally, multiplying equation two by - 3 and adding to equation three yields the following system.

$$2x + 18y - 10z = 6$$
$$y + 12z = -6$$
$$-51z = 17$$

From the third equation it is clear that z = -1/3. Substituting the value for z into equation two and solving we get y = -2. Plugging these values for y and z into the first equation gives

2x - 36 + 10/3 = 6, so x = 19 1/3.

Definitions

1. The first nonzero entry in the row that does the elimination is called a **pivot.**
2. The value of the entry eliminated divided by the pivot is the **multiplier**. It is multiplied by the row which contains the pivot which is then subtracted from the row containing the entry eliminated.
3. The form that the system is reduced to is **upper triangular.**
4. The process used to produce the solution from the upper triangular form is called **back substitution.**

A system obtained from another by the *addition of a multiple of one equation to another* has the same solution set as the original system. Each solution of the first is also a solution of the second and visa versa. We say these systems are **equivalent**. *Interchanging equations* naturally will not change the solution set of a system; so an equivalent system also results from this procedure. To achieve our intermediate goal of obtaining an equivalent upper triangular system often requires the use of equation interchanges as the examples below show.

Problem

Consider reducing the systems below to upper triangular form. Identify the next pivot and corresponding multiplier(s).

$$\begin{aligned} 8x - 4y &= 10 \\ 3x + 12y &= 20 \end{aligned}$$

$$\begin{aligned} 3y + z &= 26 \\ 5x + y - 15z &= 0 \\ x - 6y + 9z &= -8 \end{aligned}$$

$$\begin{aligned} 7x + y - 14z &= 2 \\ 2y + 18z &= -30 \\ 11y - 5z &= 19 \end{aligned}$$

Examples

Below are upper triangular systems. Discuss the <u>form</u> of the solution of each.

$$\begin{aligned} 6x + 52y - 37z &= 135 \\ 10y + 22z &= -42 \\ 60z &= 225 \end{aligned}$$

$$\begin{aligned} 3x - 78y + 34z &= 61 \\ 3y + 81z &= -49 \\ 0 &= 0 \end{aligned} \qquad \begin{aligned} 3x - 78y + 34z &= 61 \\ 3y + 81z &= -49 \\ 0 &= 6 \end{aligned}$$

$$\begin{aligned} 45x - 92y + 3z &= 100 \\ 53z &= 63 \end{aligned}$$

$$\begin{aligned} a + 5b - 8c + 16d &= 47 \\ 7c + 9d &= -2 \end{aligned}$$

Expressing the coefficient matrix, A, of the general system using its rows we wrote the system with dot products. We now take this a couple of steps further by first declaring this to be the product of a matrix and a vector, i.e. A**x** = $\begin{pmatrix} r_1 \cdot x \\ r_2 \cdot x \\ \vdots \\ r_m \cdot x \end{pmatrix}$. It is important that the dot products are defined, that the number of columns of A equals the number of elements (the number of rows) of **x**. Secondly, we extend this idea to the product of matrices. Let A = $\begin{pmatrix} r_1 \\ r_2 \\ \vdots \\ r_m \end{pmatrix}$ and B = $(\mathbf{c}_1\ \mathbf{c}_2 \ldots \mathbf{c}_p)$

where the rows of A, the $\mathbf{r}_k$, and the columns of B, the $\mathbf{c}_k$, are vectors in $\mathbf{R}^n$. Their **matrix**

product AB is defined by $AB = \begin{pmatrix} r_1 \cdot c_1 & r_1 \cdot c_2 & \cdots & r_1 \cdot c_p \\ r_2 \cdot c_1 & r_2 \cdot c_2 & \cdots & r_2 \cdot c_p \\ \vdots & \vdots & & \vdots \\ r_m \cdot c_1 & r_m \cdot c_2 & \cdots & r_m \cdot c_p \end{pmatrix} = (A\mathbf{c_1}\, A\mathbf{c_2} \ldots A\mathbf{c_p})$. An equivalent way of expressing this product is by setting AB = C = (c_{ij}) where C is the m×p matrix whose i,j[th] entry $c_{ij} = \mathbf{r_i} \cdot \mathbf{c_j}$, the dot product of the i[th] row of A with the j[th] column of B.

Examples

$$\begin{pmatrix} 6 & -2 \\ 4 & 7 \end{pmatrix}\begin{pmatrix} -3 & 5 & 0 \\ -1 & 8 & 3 \end{pmatrix} = \begin{pmatrix} 6(-3)+4(-1) & 6(5)+4(8) & 6(0)+4(3) \\ 4(-3)+7(-1) & 4(5)+7(8) & 4(0)+7(3) \end{pmatrix} = \begin{pmatrix} -22 & 62 & 12 \\ -19 & 76 & 21 \end{pmatrix}$$

$$\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} 34 & -73 \\ 67 & 189 \end{pmatrix} = \begin{pmatrix} 34 & -73 \\ 67 & 189 \end{pmatrix}$$

Definitions

1. A square matrix M = (m_{ij}) is a **diagonal matrix** if m_{ij} = 0 if i…j, i.e. its entries that are not on its diagonal are zero.
2. The **identity matrix** is an n by n diagonal matrix I_n = (a_{ij}) with ones along its diagonal, a_{ii} = 1 for all i.
3. A **permutation matrix $\mathbf{P_{ij}}$** is the identity matrix with rows i and j interchanged. Multiplying P_{ij} by a matrix A of the same size to form the product P_{ij} A interchanges the i[th] and j[th] rows of A.
4. An **elementary matrix** or **elimination matrix** E_{ij} is a matrix which differs from the identity matrix only in the i,j[th] entry. For any matrix A having the same size as E_{ij}, the product E_{ij}A is the matrix A except that row j has had m times row i added to it where the i,j[th] entry of E_{ij} is m.

Examples

$$P_{12}\begin{pmatrix} a & b & c \\ d & e & f \\ g & h & i \end{pmatrix} = \begin{pmatrix} d & e & f \\ a & b & c \\ g & h & i \end{pmatrix}$$

Letting $E_{12} = \begin{pmatrix} 1 & 3 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$ so that the multiplier in the 1,2 position is 3, we have

$$E_{12}\begin{pmatrix} a & b & c \\ d & e & f \\ g & h & i \end{pmatrix} = \begin{pmatrix} a+3d & b+3e & c+3f \\ d & e & f \\ g & h & i \end{pmatrix}$$

As we observed earlier, the augmented matrix, A, of a system contains all of the information in the system. Also the critical steps in solving the system involved only the two operations, row addition and interchange, that we can mow mimic on A by multiplications by E_{ij} and P_{ij}. We illustrate the procedure on the 3 by 3 system that we solved earlier.

Example

The augmented matrix is $A = \begin{pmatrix} 2 & 18 & -10 & 6 \\ 1 & 10 & 7 & -3 \\ -4 & -33 & 5 & -13 \end{pmatrix}$. For $E_{21} = \begin{pmatrix} 1 & 0 & 0 \\ -\frac{1}{2} & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$ and

$E_{31} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 2 & 0 & 1 \end{pmatrix}$ The product $E_{21}(E_{31}A) = C$ is the augmented matrix of the second system we obtained in reducing the system. Then letting $E_{32} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & -3 & 1 \end{pmatrix}$ $U = E_{32}C$ is the

augmented matrix of the upper triangular system. Since it turns out as we shall see that matrix multiplication is associative, $A(BC) = (AB)C$ whenever these products are defined, we get $LA = U$ where $L = E_{32}E_{21}E_{31}$. Care does need to be taken since matrix multiplication is not commutative. We have denoted the product of these elimination matrices L since it is lower triangular being the product of lower triangular matrices.