Seminar 2: Equation-solving continued

A+S 101-003: High school mathematics from a more advanced point of view

Goals of today's discussion:

1. Briefly discuss the following. Yea or Nay?: Each seminar two students (rotating basis) take notes during the seminar and during the week summarize the discussion.
2. Continue discussion concerning an algorithm for solving linear equations in one variable: why is that standard algorithm valid?
3. Which operations on an (arbitrary) equation lead to an equivalent equation? Discussion.
4. Solving linear equations in one variable in other algebraic systems: linear congruences. Compare with linear equations in the real numbers.
5. Discuss quadratic equations in the real numbers: develop an algorithm(s) to solve quadratic equations.
6. What higher mathematics (studied in Abstract Algebra, for example) is relevant to these discussions of equation-solving?
7. Discuss systems of (linear) equations and algorithms to solve such systems.
8. Time permitting, connect linear systems and matrix equations.

At the end of the seminar last week, students were asked to think about a couple of questions. Summarized, these were:

Given an equation E, of the form $A(t) = B(t)$, involving one variable t, does adding an algebraic expression involving the variable t to both sides result in an equivalent equation? Same question can be asked with "multiplying" in place of "adding".

A challenge put forth by Dr. Jones: how can the problem above be re-stated so that it could be presented to a middle-school or high-school student?

We'll return to the questions after we go through a standard algorithm for solving a linear equation in one unknown (in the real numbers). Let's do the algorithm on this equation, pausing to reflect on our goals and justification for the steps in the procedure. Can we identify natural "sub-algorithms" in the procedure?

$2x + 1 = 5x - 4$

What about $.2(3x - 4) = \frac{x}{2} + 1$? How would have a student begin?

We sadly (?) leave algebra in the real numbers and take up the equation $2x + 1 = 5x - 4$ in $\mathbf{Z_{12}}$, the integers mod 12. Let's recall some of the basics there.

1. $\mathbf{Z_{12}}$ has twelve elements $\{0, 1, \ldots, 10, 11\}$.
2. We add and multiply "mod 12". All questions below refer to $\mathbf{Z_{12}}$: What is 3×5? What is $9 + 6$? What is the additive identity? What is -5, the additive inverse of 5? What is the multiplicative identity?
3. Addition and multiplication $\mathbf{Z_{12}}$ are associative.

Let's see what happens if we run on our algorithm on $2x + 1 = 5x - 4$ but in $\mathbf{Z_{12}}$.

Let's reflect on what happened.

Write a story problem whose solution is modelled by a linear equation in $\mathbf{Z_{12}}$ Suggestion: something involving clocks and hours.

Equivalent *systems* of linear equations in the real numbers

Consider the following system of two equations in two unknowns:
$2x - 5y = 4$
$4x + y = 2$
What do you we mean by a *solution* to the above system?

Assuming we are going to solve the system algebraically, say with pencil and paper, how do you proceed?

What does it mean it that two linear systems *equivalent system of linear equations*?

Quadratic equations in the real numbers

The algorithm for solving a linear equation with one variable calls for *isolating the variable*. Consider the equation

$x^2 = -2x + 5.$

Is isolating the variable helpful here? Explain.

Consider the equation $(x + 1)^2 = 4$. How might we solve it?

Now solve $x^2 = -2x + 5$.

We continue the discussion (assuming time permits)– our goal is an algorithm for quadratic equations in the real numbers.