Seminar 5: *Equation-solving V*
A+S 101-003: High school mathematics from a more advanced point of view

1. We’ll continue with quadratic equations today.

2. We need a volunteer to take notes and send them to the email list as an attachment.

3. You received Kristen’s great notes on last week’s discussion. (If you need a hard copy, ask Dr S.) Here are a couple of things Kristen brought to light:

   - Solving a quadratic equation must have involved some ingenuity.
   - There are many ways to represent/analyze solutions to an equation including: numeric, algebraic, graphical...the famous N.A.G. trio. It’s nice to be able to use all three in tandem—understanding one supports understanding of the others.
   - Equations can speak...really.
   - We can derive the quadratic formula. In fact, the first part of the notes today involves a “discovery approach” to finding that formula.

**Discovering quadratics?**

Here’s a bit of a re-cap of what we did last time. **Our main goal** Find the (numeric) solution(s) to \( x^2 = -2x + 5 \). We’ve observed twice now that the naive approach - “isolate the variable” - doesn’t seem to lead to the numeric solutions.

We’ve back off and attempted easier quadratics. We looked at \( x^2 = 4 \), which we can solve using multiplication facts. We solved and found the solution set to be \( \{-2, 2\} \).

We observed we could extend the scope of our equation-solving capabilities by considering similar equations, those of the form “something squared is equal to a non-negative constant”. For example, we can solve \((x - 1)^2 = 4\) essentially by *substituting*. We didn’t use the word “substituting” but that’s what we did (in our heads).

We let \( u = x - 1 \). We solved \( u^2 = 4 \): so \( u = -2, 2 \). Then we back-substituted and found that \( x - 1 = u = 2 \), so \( x = 3 \). And with \( u = -2 \), we have \( x - 1 = u = -2 \), so \( x = -1 \). So the solution set to \((x - 1)^2 = 4\) is \( \{-1, 3\} \).

**Problem.** Solve \((2x - 1)^2 = 9\)

Of course the last two equations we have that in common the left-hand side is a linear expression which is squared and the right-hand side is a non-negative real number.

**Problem.** Alas, \( x^2 = -2x + 5 \) is not of that form. But it might be equivalent to such an equation. Can we tweak it and get it in a nice equivalent form?
Let’s do this one: $2x^2 - 5 = 9x$

**Discussion question** Are there circumstances under which a high school class that could do the above problem by forming a square (completing the square) derive the quadratic formula, in class, via a (reasonably unhurried) directed discovery?

*Substitutions, briefly. More later.*

Skill at substituting is rare. A person needs time to reflect on what goes on. For example, to solve $(x - 1)^2 = 4$, we let the equation speak to us and observed that we were looking for a number (here’s where the substitution comes in) whose square is 4. The equation spoke to us. Let $u$ be that number. So of course $u = -2$ or $u = 2$. And we can get the solution in one more step.

Find solutions to $u^4 + 2 = 3u^2$. It’s not a quadratic, of course. Or is it?

*Maxima and minima of quadratics, briefly*

You’re graphing a quadratic function $f(x) = x^2$. Does it have a maximum value; that is, can $f(x)$ be maximized? Explain your response.

Does it have a minimum?

Where does the minimum occur? Can you explain with a graph? **without a graph?**

Hmm.. $f(x) = x^2$.. so the range, the set of all possible “outputs” consists of squares. Thus, the range consists of non-negative numbers. The least non-negative number is 0 and it is in the range ($f(0) = 0$). So the minimum value of $f(x)$ is 0 and it occurs at 0.

**Problem** Consider $g(x) = (x - 3)^2 + 5$. Does $g(x)$ have a max? Does it have a min? Can you explain your proposed solution **without using a graph?**

Consider $h(x) = -(x - 3)^2 + 5$. Does $h(x)$ have a max? Does it have a min? Can you explain with a graph? Without a graph?
Can you create a real-world optimization problem that illustrates the usefulness and beauty of understanding quadratic functions?

\[ \text{NAG, NAG, NAG..} \]

What’s a root of a quadratic function? What connection does a root a quadratic have with its graph? Can you explain that connection?

Find the roots of
1. \( f(x) = x^2 - 3x + 2 \)
2. \( g(x) = -x^2 + 4x + 1 \)

What connection does the factorization of a quadratic \( q \) have to with the roots of the function \( q(x) \)? For example, let \( q(x) = x^2 - 3x + 2 \). Factor \( q(x) \). Determine its roots.

Suppose we have factored a quadratic \( q(x) = (x - c)(x - d) \). What are the roots of \( q(x) \)? Explain. How can you be sure?

Let’s work in \( \mathbb{Z}_{12} \). Let \( f(x) = (x - 2)(x - 4) \). What are its roots?

Absolute values and inequalities too

Let’s recall the 50 cent definition of absolute value: \( |x| = x \), if \( x \geq 0 \); otherwise, \( |x| = -x \) (when \( x < 0 \)).

How would you find solutions to \( |x| = 2 \)?

To \( |x - 1| = 2 \)?

To \( |2x - 1| - 2 = 0 \)?

Now you want to find solutions to \( |x| > 1 \). How many solutions are there? Can you provide a graphical description of the solution set?
Solve $|2x - 1| \geq 3$.

Connect absolute values and the idea of distance on the number line.

How would you solve $|x^2 - 4| \geq 0$?