Seminar 5: *Equation-solving V*
A+S 101-003: High school mathematics from a more advanced point of view

1. We'll continue with quadratic equations today.
2. We need a volunteer to take notes and send them to the email list as an attachment.
3. You received Kristen's great notes on last week's discussion. (If you need a hard copy, ask Dr S.) Here are a couple of things Kristen brought to light:
   - Solving a quadratic equation must have involved some ingenuity.
   - There are many ways to represent/analyze solutions to an equation including: Numeric, Algebraic, Graphical...the famous N.A.G. trio. It's nice to be able to use all three in tandem– understanding one supports understanding of the others.
   - Equations can speak..really.
   - We can derive the quadratic formula. In fact, the first part of the notes today involves a "discovery approach" to finding that formula.

Discovering quadratics?

Here's a bit of a re-cap of what we did last time. **Our main goal** Find the (numeric) solution(s) to $x^2 = -2x + 5$. We've observed twice now that the naive approach - "isolate the variable" - doesn't seem to lead to the numeric solutions.

We've back off and attempted easier quadratics. We looked at $x^2 = 4$, which we can solve using multiplication facts. We solved and found the solution set to be $\{-2, 2\}$.

We observed we could extend the scope of our equation-solving capabilities by considering similar equations, those of the form "something squared is equal to a non-negative constant". For example, we can solve $(x-1)^2 = 4$ essentially by *substituting*. We didn't use the word "substituting" but that's what we did (in our heads).

We let $u = x - 1$. We solved $u^2 = 4$: so $u = -2, 2$. Then we back-substituted and found that $x - 1 = u = 2$, so $x = 3$. And with $u = -2$, we have $x - 1 = u = -2$, so $x = -1$. So the solution set to $(x-1)^2 = 4$ is $\{-1, 3\}$.

Problem. Solve $(2x-1)^2 = 9$

Of course the last two equations we have that in common the left-hand side is a linear expression which is squared and the right-hand side is a non-negative real number.

Problem. Alas, $x^2 = -2x + 5$ is not of that form. But it might be equivalent to such an equation. Can we tweak it and get it in a nice equivalent form?

Let's do this one: $2x^2 - 5 = 9x$

Discussion question Are there circumstances under which a high school class that could do the above problem by forming a square (completing the square) derive the quadratic formula, in class, via a (reasonably unhurried) directed discovery?

Substitutions, briefly. More later..

Skill at substituting is rare. A person needs time to reflect on what goes on. For example, to solve $(x-1)^2 = 4$, we let the equation speak to us and observed that we were looking for a number (here's where the substitution comes in) whose square is 4. The equation spoke to us. Let u be that number. So of course $u = -2$ or $u = 2$. And we can get the solution in one more step.

Find solutions to $u^4 + 2 = 3u^2$. It's not a quadratic, of course. Or is it?

Maxima and minima of quadratics, briefly

You're graphing a quadratic function $f(x) = x^2$. Does it have a maximum value; that is, can $f(x)$ be maximized? Explain your response.

Does it have a minimum?

Where does the minimum occur? Can you explain with a graph? **without a graph**?

Hmm..$f(x) = x^2$..so the range, the set of all possible "outputs" consists of squares. Thus, the range consists of non-negative numbers. The least non-negative number is 0 *and it is in the range ($f(0) = 0$). So the minimum value of $f(x)$ is* 0 *and it occurs at* 0.

Problem Consider $g(x) = (x-3)^2 + 5$. Does $g(x)$ have a max? Does it have a min? Can you explain your proposed solution *without using a graph*?

Consider $h(x) = -(x-3)^2 + 5$. Does $h(x)$ have a max? Does it have a min? Can you explain with a graph? Without a graph?

Can you create a real-world optimization problem that illustrates the usefulness and beauty of understanding quadratic functions?

NAG, NAG, NAG..

What's a *root* of a quadratic function? What connection does a root a quadratic have with its graph? Can you explain that connection?

Find the roots of

1. $f(x) = x^2 - 3x + 2$

2. $g(x) = -x^2 + 4x + 1$

What connection does the factorization of a quadratic q have to with the roots of the function $q(x)$? For example, let $q(x) = x^2 - 3x + 2$. Factor $q(x)$. Determine its roots.

Suppose we have factored a quadratic $q(x) = (x - c)(x - d)$. What are the roots of $q(x)$? Explain. How can you be sure?

Let's work in Z_{12}. Let $f(x) = (x - 2)(x - 4)$ What are its roots?

Absolute values and inequalities too

Let's recall the 50 cent definition of absolute value: $|x| = x$, if $x \geq 0$; otherwise, $|x| = -x$ (when $x < 0$).

How would you find solutions to $|x| = 2$?

To $|x - 1| = 2$?

To $|2x - 1| - 2 = 0$?

Now you want to find solutions to $|x| > 1$. How many solutions are there? Can you provide a graphical description of the solution set?

Solve $|2x - 1| \geq 3$.

Connect absolute values and the idea of distance on the number line.

How would you solve $|x^2 - 4| \geq 0$?