Seminar 8: *Binomial theorem, Pascal's triangle, and finite probabilities*
A+S 101-003: High school mathematics from a more advanced point of view

1. We will need a volunteer to take class notes.
2. Last time Kristin presented on "Connections betweeen the harmonic series and music".
3. We discussed future topics. Brad's talk referenced the binomial theorem and some finite combinatorics. In our last discussion we decided to pursue those topics, with an emphasis connections to the high school class. The discussion today will begin these ideas.

Combinations and permutations

We'll discuss these topics using a "directed discovery" approach. So we begin not with a definition, but with a problem.

Consider the following *theoretical probability* problem: a coin will be flipped 4 times. A typical outcome: $TTHT$.
Problem 1 How many possible outcomes are there?

Problem 1.1 What's the theoretical probability of the heads" sequence?

Problem 1.2 What's the probability that exactly one head is in the sequence? Note: we have to count number of sequences with exactly one head.

Problem 2 What is the theoretical probability that exactly two heads will occur in a sequence?

Problem 3 Now suppose we modify things slightly– each outcome is a sequence of ten flips of a coin. How many possible outcomes are there?

Problem 4 What's the probability that exactly one head is in the sequence?

Problem 5 What's the probability that exactly 5 heads is in the sequence? (Might be tough to do a brute force count..)

Thought problem Suppose we flip a coin n times. What's the probability that exactly r heads are in the sequence? (where $n \geq r \geq 0$).

Obviously we're counting things. Here's a way to look at it, independent of the coin-tossing experiment: we have n positions, each position will be labelled with "H" or "T". To ask " how many sequences have exactly r H's is to ask "how many r-subsets (subsets with r elements) does a set with n elements have? Or, "how many ways are there to combine r objects from a set of n objects"?

The number of ways we can "combine r objects out of a set of n objects" is denoted $C(n,r)$ ("n choose r"). Reminder: some texts/people use $\binom{n}{r}$ rather than $C(n,r)$. The latter is much more common in middle and high school texts.

One of our goals will be to discover a formula for $C(n,r)$ and to discover relationships between "various $C(n,r)$" ; some of those relationships are best understood via *Pascal's Triangle.* But before we begin discovering the formula (and discussing Pascal's Triangle), we'll see that the $C(n,r)$ come up in completely different context (algebra).

Problem 6 Consider $(x+1)^4$. Express it as a polynomial in standard form; determine the coefficients a_0, a_1, a_2, a_3, a_4 where $(x+1)^4 = \sum_{i=0}^{i=4} a_i x^i$.

We have $(x+1)^4 = (x+1)(x+1)(x+1)(x+1)$. We can multiply it out by considering all possible ways to "thread through" the factors: we'll get how many summands (before we even begin to add like terms)?

How many summands will be of the form "x^4"?

How many summands will be of the form "x^3"?

How many summands will be of the form x^2"?

Little Binomial Theorem Let n be a whole number. Then $(x+1)^n =$ (fill in the blank)

Binomial Theorem Let n be a whole number. Then $(x+y)^n =$ (fill in the blank)

Now you really know those $C(n,r)$'s really have it going- how can we "find" them?