Seminar 10: *Discussion III on binomial theorem, Pascal's triangle, and finite probabilities*

A+S 101-003: High school mathematics from a more advanced point of view

1. We will need a volunteer to take class notes.
2. Last time we did a directed discovery and discovered the formuli for $P(n,r)$ and $C(n,r)$. During the discussion, Dr. Jones' comments suggested an alternative (and more conceptual) route to the formula for $C(n,r)$. Let's discuss that today.

   - We had discovered earlier that the number of permutations of n element set is $n!$. With essentially the same set of insights, we found a formula for $P(n,r)$, the number of r letter words in an n-letter alphabet, each such word having no repeated letters. (There are other equivalent ways to define $P(n,r)$. Here's an equivalent way to think about $P(n,r)$: you'll choose r objects from an n-element set, but the order in which you choose them matters.)

     We found that $P(n,r) = \frac{n!}{n-r)!}$. However it's often easier to compute $P(n,r)$ by writing it as a product of r numbers, beginning with n and decreasing by 1.
   - We related the counting of $C(n,r)$ and $P(n,r)$: each choose an r element set (counted once when we count $C(n,r)$) leads to $r!$ objects in the count of $P(n,r)$. We'll draw a partial "counting tree" to illustrate the principle with $C(5,3)$ and $P(5,3)$.

     We can now solve for $C(n,r)$.
   - We'll look briefly at several standard, but interesting, applications to finite probability.
   - We'll (fairly) quickly determine a formula involving binomial coefficients for $(x+y)^n$, where n is a whole number.
   - We'll discuss relationships among "binomial coefficients", including those which allow one to compute the coefficients recursively (essentially, by using Pascal's Triangle).
   - We'll apply Pascal's Triangle to another seemingly unrelated class of problems.

Applications of counting to finite probabilities

Problem 1. You have an urn containing 7 balls. Three are green, four are red. You choose three balls at random. What's probability that all are the same color?

Problem 2. You flip a coin 6 times. What's the probability that you flip exactly 3 heads?

Problem 3. The probability that it rains is 50 percent for the next 6 days. What's the probability that it rains exactly 3 days?

Problem 4. You roll a single die 6 times. What's the probability that each face is rolled exactly once?

Relationships among binomial coefficients

Problem 5. Let $[2] = \{1, 2\}$ and let $[3] = \{1, 2, 3\}$. How many subsets does [2] have? Same question for [3].

Problem 6. Here's some directed discovery. How we can find a formula for the number of elements in $[n]$? Think of each subset as a n-tuple of 0 and 1's, in a way that will be explained now.

For example $\{1, 3\} \subset \{1, 2, 3\}$ will be represented by $(1, 0, 1)$.

Each such n-tuple determines a subset; conversely, each subset determines an n-tuple. We'll make this precise.

So $[n]$ has how many subsets?

We're in position to demonstrate the validity of $\sum_{i=0}^{n} \binom{n}{i} = 2^n$ in multiple ways, which we'll do together below.

Problem 7. You've got 10 marbles: 9 are black, one is magenta. You'll choose 4 marbles.

Explain-conceptually, using your 10 colored marbles- the validity of the equation $\binom{10}{4} = \binom{9}{4} + \binom{9}{3}$ (in a way that a high school student could understand).

Stare at the equation above. Can we generalize "$\binom{10}{4} = \binom{9}{4} + \binom{9}{3}$ "- we'd a general formula that involves binomial coefficients. We'll see and really understand Pascal's Triangle!

Taxicab geometry What's the length of the the shortest paths (along city streets) from First St and First Ave to Fourth St and Fourth Avenue? (All streets, avenues are numbered, in order.)

How many shortest paths can a driver take between those two intersections?