Seminar 13: *Sequences, limits: aiming at calculus*
A+S 101-003: High school mathematics from a more advanced point of view

1. We will need a volunteer to take class notes.
2. Last time we discussed sequences, limits, and limits of functions. We discussed the classic application of limit: determine the instanteous velocity of a particle with a given position function $s(t)$.
3. Dr. Jones indicated that sequences need careful attention in high school mathematics. He suggested that teachers have a rich set of natural examples and that time be spent developing student intuitions about finding the general term of a sequence (eg. given the first few terms of a sequence, let students guess the next term and then guess the general term). Dr. Jones pointed out that the notion of a limit was one that needed to be approached (no pun intended!!) with some care; the expression 'limit' itself does not always suggest its mathematical meaning to a high school student beginning the topic.
4. We continue with limits today.

Sequences, limits continued

One often hears mathematics described as the "science of patterns". Nothing I can think of lends itself better to representing patterns than a *sequence*, $a_1, a_2, \ldots$. We most often talk about sequences of real numbers (so a_1, a_2 and so on are real numbers). A sequence of real numbers can be formally defined as a function $f : \mathbf{N} \to \mathbf{R}$, where $\mathbf{N}$ denotes the natural numbers $(1, 2, \ldots,)$ and $\mathbf{R}$ denotes the real numbers: so there's a first real number $f(1)$ (usually written something like a_1, a second real number $f(2)$ (often a_2) and so on.

Sometimes we're able to find the general rule (which amounts to finding a formula or an algorithm that determines the *general term* a_n). Some examples are provided below. See if you can find a formula or rule for the *n-th term* which we'll denote by a_n.

1. $1, \frac{1}{2}, \frac{1}{4}, \frac{1}{8}, \ldots$

   $a_n =$

2. $1, -1, 1, -1, 1, -1, \ldots$

   $a_n =$

3. $\frac{1}{2}, \frac{2}{3}, \frac{3}{4}, \frac{4}{5}, \ldots$

   $a_n =$

4. $(1 + \frac{1}{2})^2, (1 + \frac{1}{3})^3, (1 + \frac{1}{4})^4, \ldots$

   $a_n =$

5. Last time, we drew a sequence of the *triangular numbers* via geometric figures, points arrayed as triangles. We observed that the number of points in the sequence of figures was given by $1, 3, 6, 10, \ldots$. The sequence can be defined *recursively* by the rule $a_n = a_{n-1} + n$. Show that $a_n = \binom{n}{2}$.

Sometimes a sequence has a *limit* a: as n gets higher and higher, a_n "approaches" the limit a. The idea is that for all really large n, the distance between a (the proposed limit) and a_n is really small. By the way, "the distance between a and a_n" is nicely described by the symbol $|a - a_n|$.

If you want to see the limit idea graphically, draw the horizontal line $y = a$ and graph the sequence $a_n = f(n)$: if a_n *converges* to a, as you move to the right, your graph should begin to "hug" the line $y = a$.

With this informal definition of limit of a sequence, which of the sequences above are you fairly certain has a limit?

It took a long, long time for mathematicians and others to find the "right" formal definition of "limit". Here it is:

Let $(a_n)_{n=1,2,...,}$ be a sequence. Then that sequence has limit a if for all real numbers $\epsilon > 0$, there exists a natural number k such that $|a_k - a| < \epsilon$.

I'll say a little bit about the formal definition–it's kinda neat as formal definitions go.

Problem 1 Find a sequence which converges to 5.

Problem 2 A sequence $(a_n)_{n=1,2,...}$ is *bounded* if there exists a positive real M such that for all n, we have $|a_n| \leq M$.

- Draw a "picture" of the above definition: draw graphs of a bounded sequence and an unbounded sequence.

- If a sequence is bounded, does it necessarily have a limit? If you think the above statement is false, come up with a nice simple counterexample.

Most of what has been discussed in the course falls under the heading of "discrete mathematics". Here is a description of "discrete mathematics" from the Math Forum:

"Discrete mathematics deals with discrete phenomena and finite processes, as opposed to the continuous functions and infinite limits that are the mainstay of calculus and classical analysis. It comprises many diverse topics, some familiar to secondary school teachers, like matrices and finite probability, and others not so familiar, like difference equations and graph theory. Amidst this diversity of topics, the unifying theme of discrete mathematics is "algorithmic problem solving," that is, solving problems by devising and analyzing algorithms that construct the solution."

Sequences are generally regarded as part of discrete math; once we begin to talk about "limits of sequences" we're leaving discrete and entering the realm of "continuous mathematics".

We completely leave behind discrete mathematics with the topic *limits of functions*. For example, we could ask: as t approaches 4, what "happens" to $f(t) = \frac{t^2-16}{t-4}$? Or we could ask a difficult, interesting one such as: as t gets close to 0, what happens to e^t? Another one: as t gets close to 0, what happens to $e^{-\frac{1}{t}}$? Let's examine a "position / velocity" example.

A particle travels along a *continuum* according to the function $s(t) = \sqrt{t}$, where t is in seconds and $s(t)$ is in feet. For example, its position at time 1 second is 1 and its position at time 3 seconds is $\sqrt{3}$. The particle is apparently moving to the right.

Problem 3

1. What is the average velocity of the particle between in the time interval $[1, 3]$?

2. How can we define *instantaneous velocity* of the particle at time, say, 3 seconds?

3. We know the position function $s(t)$ of the particle. Can we find a function $v(t)$ which provides the instanteous velocity of said particle at time t seconds?

4. We'll talk about this one at more length next week: We've got this other particle. We've measured its velocity and know pretty much that its velocity function is given by $v(t) = t$. We know also that at time $t = 0$ its position was -4. Can we determine its position at all time t? (That is, can we determine its position function $s(t)$?)

5. We know how to find areas of a lot of *polygons*. But if you draw a parabola $y = x^2$ and consider the area under it (and above the x-axis) between $x = 0$ and $x = 1$, you're stuck..with approximating the area (which is to say, not stuck at all). Here's an approach to sneaking up on the area under that curve. Of course it involves..a sequence and its limit!