ABSTRACT. Geometry and Complex Numbers

GEOMETRY AND COMPLEX NUMBERS

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Contents

Key words and phrases. Complex numbers.

1. Introduction

The purpose of these notes is to relate algebraic properties of complex numbers to geometry of the plane. Suggested (but not required) texts are:

(1) ‘Modern geometries: Non-Euclidean, Projective, and Discrete’ by Michael Henle (2nd edition, Prentice Hall).
(2) ‘Complex numbers and geometry’ by Liang-shin Hahn (Mathematical Association of America).

A good way to get cheaply lots of material on complex numbers and geometry is to search internet. For example, one can go to

www.google.com

and search for **complex+numbers+pdf** to find notes in pdf. Please post on Blackboard discussion forum interesting **urls** you find on internet.

GEOMETRY AND COMPLEX NUMBERS URL (1 POINT)

You must submit to the discussion boards one unique (i.e., not given by anyone else before you) URL of a complex numbers website together with a paragraph or two on the content of the site. Try to list a site that you could imagine using (or, better, one you have already used) in your teaching, and include an explanation of how you might use it. If a whole site (or part of a site) is devoted to complex numbers, give the URL of the top-level page, not a particular page within the site.

If the site is a single page (e.g., a single good page on complex numbers in a site that is otherwise not about complex numbers), then you may give the URL of that page alone. The purpose of this assignment is to introduce you to the geometry and complex numbers resources available on the Internet. This is due by Monday, January 26.

PERSONAL PAGE (2 POINTS)

On our course site you have space to set up a personal web page. Please make a page that includes brief descriptions of your current teaching assignment (e.g., where you teach, what grades you teach, what courses you teach), your acquaintance with complex numbers (e.g., any relevant courses you have had, any geometry topics you have taught), and what you hope to get out of this course. Also, please include a photograph of yourself.

You are welcome to include more information if you like (e.g., your favorite mathematician, your favorite pet, a teaching tip, a great website, your understanding of the meaning of life). The purpose of this assignment is to help me get to know you and to help you get to know one another, reducing the isolation inherent in distance education. This assignment is due by Monday, January 26.

QUESTIONNAIRE ON MY WEB SITE (1 POINT)

You must fill out a questionnaire on my web site. This is due by Monday, January 19.

Example: my password is xvy232

When lecturing using whiteboard on Centra or explaining things via Blackboard, I will apply TeX notation:

Math symbol	Text equivalent
x^2	\$x^ 2\$
x^{-1}	\$x^ {-1}\$
x_0	\$x_{0}\$ or \$x_0\$
$\sin(x)$	\$\sin(x)\$

Please use the same notation when submitting homework via email or posting a comment on discussion forum at Blackboard.

2. Solving equations

The purpose of this section is to introduce students to rigorous definition of concepts in mathematics. We ask for definitions of well-known entities (problems 2.1-2.3 and 2.5), and the remaining problems are designed to check if students understand them.

All problems in this section form Homework 1 due via email (in text format) on January 23, 2004.

***Problem* 2.1.** Assume knowledge of addition and multiplication of real numbers. What do we mean by $u^{-1} = v$? Can this equality be verified?

***Problem* 2.2.** Assume knowledge of addition and multiplication of real numbers. What do we mean by $\frac{u}{v} = w$? Can this equality be verified?

***Problem* 2.3.** Suppose $f(x)$ is a given function. What do we mean by the statement:

$$\text{Solve } f(x) = 0?$$

***Problem* 2.4.** Solve $(2\cdot x-6)^{-1}=(3\cdot x-9)^{-1}$. Show your work.

***Problem* 2.5.** Assume knowledge of addition and multiplication of real numbers. What do we mean by $\sqrt{u} = v$? Can this equality be verified?

***Problem* 2.6.** Solve $x^2 = 5$.

***Problem* 2.7.** Solve $(3\cdot x-14)^{1/2} = (x-6)^{1/2}$. Show your work.

***Problem* 2.8.** Solve $(x^2 - 8 \cdot x + 14)^{1/2} = (6 - 2 \cdot x)^{1/2}$. Show your work.

Answer to 2.8: $x = 2$

3. Geometric proofs

The purpose of this section is to introduce students to proofs in mathematics.

The traditional way to do proofs is via the so-called 'axiomatic method': one declares a few basic (obvious) truths called 'axioms', and all other truths are derived from axioms using logic. A good example of this method is the theory of reals developed using the field axioms as in M300.

Since our course is devoted to interaction of geometry and algebra, in this section we derive algebraic properties of reals from geometric properties of the plane. This way one improves the understanding of both algebra and geometry.

***Problem* 3.1.** What is a geometric interpretation of the product of two positive numbers? Using it show that $a \cdot (b+c) = a \cdot b + a \cdot c$ for all positive numbers a, b, and c. What facts did you use in your proof?

Hint(s) to 3.1: The left side of the equality

$$a \cdot (b + c) = a \cdot b + a \cdot c$$

is the area of a certain geometric figure.

Outline(s) of solution(s) to 3.1: Basic geometric interpretation of the product $a \cdot b$ of two positive numbers is the area of a rectangle of base a and height b.

The left side of the equality

$$a \cdot (b + c) = a \cdot b + a \cdot c$$

is the area of a rectangle of base a and height $b + c$. That rectangle is the union of two rectangles: one of base a and height b, the other of base a and height c. Using the formula that

$$area(A \cup B) = area(A) + area(B) - area(A \cap B)$$

one arrives at $a \cdot (b + c) = a \cdot b + a \cdot c$.

***Problem* 3.2.** What is a geometric interpretation of the product of two positive numbers? Using it show that $(a+b)\cdot(c+d) = a\cdot c+a\cdot d+b\cdot c+b\cdot d$ for all positive numbers a, b, c, and d. What facts did you use in your proof?

Hint(s) to 3.2: The left side of the equality

$$(a+b)\cdot(c+d) = a\cdot c + a\cdot d + b\cdot c + b\cdot d$$

is the area of a certain geometric figure.

Outline(s) of solution(s) to 3.2: Basic geometric interpretation of the product $a \cdot b$ of two positive numbers is the area of a rectangle of base a and height b.

The left side of the equality

$$(a + b) \cdot (c + d) = a \cdot c + a \cdot d + b \cdot c + b \cdot d$$

is the area of a rectangle of base $a + b$ and height $c + d$. That rectangle is the union of four rectangles. Using the formula that

$$area(A \cup B) = area(A) + area(B) - area(A \cap B)$$

one arrives at $(a+b) \cdot (c+d) = a \cdot c + a \cdot d + b \cdot c + b \cdot d$.

***Problem* 3.3.** What is a geometric interpretation of the product of two positive numbers? Using it show that $a \cdot b = b \cdot a$ for all positive numbers a and b. What facts did you use in your proof?

Hint(s) to 3.3: The left side of the equality $a \cdot b = b \cdot a$ is the area of a certain geometric figure. Same about the right side. How are those two figures related?

Outline(s) of solution(s) to 3.3: Basic geometric interpretation of the product $a \cdot b$ of two positive numbers is the area of a rectangle of base a and height b.

The left side of the equality $a \cdot b = b \cdot a$ is the area of a rectangle of base a and height b. The right side of the equality $a \cdot b = b \cdot a$ is the area of a rectangle of base b and height a. The geometric reason the two rectangles have the same area is that rotations preserve area. The physical reason the two rectangles have the same area is that there is no preferred orientation in the space. What one observer calls 'base' the other observer calls 'height'.

***Problem* 3.4.** What is a geometric interpretation of the product of two positive numbers? Using it show that $(a+b)^2 = a^2 + 2a \cdot b + b^2$ for all positive numbers a and b. What facts did you use in your proof?

Hint(s) to 3.4: The left side of the equality

$$(a+b)^2 = a^2 + 2a \cdot b + b^2$$

is the area of a certain geometric figure. Decompose it into four parts.

Outline(s) of solution(s) to 3.4: Basic geometric interpretation of the product $a \cdot b$ of two positive numbers is the area of a rectangle of base a and height b.

The left side of the equality

$$(a+b)^2 = a^2 + 2a \cdot b + b^2$$

is the area of a square of base $a+b$. That square naturally decomposes into two squares and two rectangles whose areas represent the right side of the equality.

***Problem* 3.5.** Using a c by c square and four right triangles with hypothenuse c and sides a and b construct a square of height $a+b$. Using the picture show a proof of the Pythagoras Theorem. What facts did you use in your construction and proof?

Hint(s) to 3.5: Start with a square of base $a + b$ and cut off four right triangles with legs a and b.

Outline(s) of solution(s) to 3.5: Start with a square of base $a + b$ and cut off four right triangles with legs a and b. In the middle one obtains a square of base c. Calculating the areas, one gets the correct equality. Notice that two right triangles with legs a and b can be put together to form a rectangle with base a and height b. Therefore the area of such triangle is $a \cdot b/2$. We are using the formula

$$area(A \cup B) = area(A) + area(B) - area(A \cap B)$$

throughout the problem.

***Problem* 3.6.** What is a geometric interpretation of the product of two positive numbers? Using it show that $a^2 - b^2 = (a - b) \cdot (a + b)$ for all positive numbers $a > b$. What facts did you use in your proof?

Hint(s) to 3.6: $a^2 - b^2$ represents the area of a square minus another square. Can you rearrange that figure to form a rectangle?

Outline(s) of solution(s) to 3.6: Basic geometric interpretation of the product $a \cdot b$ of two positive numbers is the area of a rectangle of base a and height b.

$a^2 - b^2$ represents the area of a square A with base a minus another square B with base b located in the upper left corner of A. The top part of that figure can be cut off, turned around 90 degrees, and glued to the remaining part. The resulting figure is a rectangle of base $a + b$ and height $a - b$.

***Problem* 3.7.** Using the Pythagoras Theorem show that for any positive number t that can be expressed as $t = s^2 - r^2$ there is a solution to $x^2 = t$. What facts did you use in your construction and proof?

Hint(s) to 3.7: Construct a right triangle whose leg is a solution to $x^2 = t$.

Outline(s) of solution(s) to 3.7: We need a right triangle with one leg r and the hypothenuse s. The other leg will be a solution to $x^2 = t$. Draw a circle of radius r from one vertex of the hypothenuse. Drop a tangent line to that circle from the other vertex.

***Problem* 3.8.** Show that for any positive numbers t and s their product $s \cdot t$ can be expressed as $s \cdot t = a^2 - b^2$.

Hint(s) to 3.8: a^2-b^2 has a natural decomposition into the product of two numbers. Can one of them be equal to s and the other equal to t?

Outline(s) of solution(s) to 3.8: Notice that $a^2 - b^2 = (a+b) \cdot (a-b)$. We would like to match numbers s, t and $a+b$, $a-b$. Since $a+b > a-b$, we need $a+b = \max(s,t)$ and $a-b = \min(s,t)$ Therefore $s+t = (a+b)+(a-b) = 2a$ which gives $a = (s+t)/2$. Similarly, $(a+b)-(a-b) = 2b$, so $b = (\max(s,t) - \min(s,t))/2$.

***Problem* 3.9.** Show that for any positive number t there is a solution to $x^2 = t$. What facts did you use in your proof?

Hint(s) to 3.9: We would like to express t as $c^2 - a^2$ for some positive numbers $c > a$ and construct square root of t geometrically.

Outline(s) of solution(s) to 3.9: Since $t = t \cdot 1$ one ought to find $c > a$ such that $t = c^2 - a^2$. Notice that $(t+1)^2 - (t-1)^2 = 4 \cdot t$. Putting $c = (t+1)/2$ and $a = |t-1|/2$ (why not $a = (t-1)/2$?) gives $t = c^2 - a^2$.

***Problem* 3.10.** Show geometrically that for any positive numbers b and c there is a positive solution to $x^2 + b \cdot x = c$. What facts did you use in your proof?

Hint(s) to 3.10: $x^2+b\cdot x$ represents the area of a rectangle of base $x+b$ and height x. Can you rearrange that rectangle to form a square minus another square?

Outline(s) of solution(s) to 3.10: $x^2 + b \cdot x$ represents the area of a rectangle of base $x+b$ and height x. That rectangle is the union of a square of base x and a rectangle of base b and height x. Divide the rectangle vertically into two equal parts, turn the right one by 90 degrees and glue at the bottom of the square. If one fills in the square of base $b/2$, the result is a square of base $x + b/2$. Algebraically that means $(x+b/2)^2 = x^2 + b \cdot x + (b/2)^2 = c + (b/2)^2$. Therefore, $x + b/2 = \sqrt{c + (b/2)^2}$ and $x = \sqrt{c + (b/2)^2} - b/2$.

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