4. Stereographic Projection

There are two special projections: one onto the x-axis, the other onto the y-axis. Both are well-known. Using those projections one can define functions sine and cosine. However, there is another projection, less known to students, a projection from a circle to the x-axis. It is called the **stereographic projection**. We will use it to provide geometric interpretations of multiplication, division of real numbers, the tangent function, and basic trigonometric formulae.

***Problem* 4.1.** Find the intersection of the line joining $(0,1)$ and $(1,3)$ with the x-axis.

Hint(s) to 4.1: Find an equation of the line passing through the two points. How does one find its intersection with the x-axis?

Outline(s) of solution(s) to 4.1: A non-vertical line is the graph of a linear function $f(x)$. A function $f(x)$ is called **linear** if the ratio $(f(x_2)-f(x_1))/(x_2-x_1)$ is always constant if $x_2 \neq x_1$. That ratio is the slope m of the geometric line. One can compute m from the data. Now, setting $x_2 = x$ and $x_1 = 0$ gives $(f(x)-1)/(x-0) = m$ as an equation of our line. Its x-intercept is the point such that $f(x) = 0$, so solve $(f(x) - 1)/(x - 0) = m$ in that case.

Answer to 4.1: $-.5$

***Problem* 4.2.** Find the intersection of the line joining $(0, 0, 1)$ and $(1, 1, 3)$ with the xy-plane.

Hint(s) to 4.2: Find an equation of the line passing through the two points. How does one find its intersection with the xy-plane?

Outline(s) of solution(s) to 4.2: A non-vertical line on space is the graph of a linear function $f(t)$. A function $f(t)$ is called **linear** if the ratio $(f(t_2) - f(t_1))/(t_2 - t_1)$ is always constant if $t_2 \neq t_1$. Physically, that constant is the velocity v of a particle traversing our line. One can compute v from the data assuming that at $t = 0$ the particle is at point $(0, 0, 1)$ and at $t = 1$ it is at the other point. Now, setting $t_2 = t$ and $t_1 = 0$ gives $(f(t) - (0, 0, 1))/(t - 0) = v$ as an equation of our line. Its xy-intercept is the point t such that the third coordinate of $f(t)$ equals 0, so solve $(f(t) - (0, 0, 1))/(t - 0) = v$ for such t, then find $f(t)$.

Answer to 4.2: $(-.5, -.5)$

***Problem* 4.3.** Suppose $b \neq 1$. Show that the line joining $(0,1)$ and (a,b) intersects the x-axis at $(\frac{a}{1-b}, 0)$.

Hint(s) to 4.3: Find an equation of the line passing through the two points. How does one find its intersection with the x-axis?

Outline(s) of solution(s) to 4.3: A non-vertical line is the graph of a linear function $f(x)$. A function $f(x)$ is called **linear** if the ratio $(f(x_2)-f(x_1))/(x_2-x_1)$ is always constant if $x_2 \neq x_1$. That ratio is the slope m of the geometric line. One can compute m from the data. Now, setting $x_2 = x$ and $x_1 = 0$ gives $(f(x)-1)/(x-0) = m$ as an equation of our line. Its x-intercept is the point such that $f(x) = 0$, so solve $(f(x) - 1)/(x - 0) = m$ in that case.

***Problem* 4.4.** Suppose $a \neq 0$ and $b \neq 1$. Show that $\frac{a}{1-b} = \frac{b}{a}$ if (a, b) is on the circle centered at $(0, 1/2)$ with radius $r = 1/2$.

Hint(s) to 4.4: What does it mean that a point (x, y) lies on the circle centered at $(0, 1)$ of radius $r = 1/2$? Can you write an equation of that circle? Is it equivalent to $\frac{x}{1-y} = \frac{y}{x}$ if $x \neq 0$?

Outline(s) of solution(s) to 4.4: A point (x, y) lies on the circle centered at $(0, 1)$ of radius $r = 1/2$ if its distance to $(0, 1)$ is $1/2$. Algebraically, it is the same as

$$x^2 + (y - 1/2)^2 = (1/2)^2.$$

Expanding and simplifying gives

$$x^2 + y^2 - y = 0$$

which is equivalent to $\frac{x}{1-y} = \frac{y}{x}$ if $x \neq 0$.

***Problem* 4.5.** Consider the stereographic projection from the North Pole of the circle centered at $(0, 1/2)$ with radius $r = 1/2$ onto the x-axis. Show that the image of the point with argument $\alpha \neq \pi/2$ is $\tan(\alpha)$.

Hint(s) to 4.5:

1. (Algebraic) If a point (x, y) has argument α, how can $\tan(\alpha)$ be expressed using x and y? How does that compare to the stereographic projection of (x, y)?

2. (Geometric) Draw a picture of the stereographic projection from the North Pole N via $A = (x, y)$ to $B = (t, 0)$. Can you calculate all the angles on the picture? What are their tangents?

Outline(s) of solution(s) to 4.5:

1. (Algebraic) If a point (x, y) has argument α, then $\tan(\alpha) = y/x$. The stereographic projection of (x, y) is $\frac{x}{1-y}$ and that equals $\frac{y}{x} = \tan(\alpha)$ by 4.4.

2. (Geometric) Draw a picture of the stereographic projection from the North Pole N via $A = (x, y)$ to $B = (t, 0)$. If α is the argument of A, then the angle $\setminus ONB$ equals α. Since $\tan(\setminus ONB) = t$, we are done.

***Problem* 4.6.** Consider the stereographic projection from the North Pole of the circle centered at $(0, 1/2)$ with radius $r = 1/2$ onto the x-axis. Show that one can construct $\frac{-1}{a}$, $a > 0$, by going straight from $(a, 0)$ to $(0, 1)$ and then turning left via 90 degrees and going straight until the x-axis is met again.

Hint(s) to 4.6: Draw a picture of the stereographic projection from the North Pole N via $B = (x, y)$ to $A = (a, 0)$. Then draw a line through N perpendicular to NA. Can you calculate all the angles on the picture? What are their tangents?

Outline(s) of solution(s) to 4.6: Draw a picture of the stereographic projection from the North Pole N via $B = (x, y)$ to $A = (a, 0)$. Then draw a line through N perpendicular to NA. Let t be the x-intercept of that line. If α is the argument of B, then the angle $\backslash ONA$ equals α and the angle $\backslash OCN = \alpha$, where $C = (t, 0)$. Since $\tan(\backslash OCN) = -1/t$, we are done.

***Problem* 4.7.** Consider the stereographic projection from the North Pole of the circle centered at $(0, 1/2)$ with radius $r = 1/2$ onto the x-axis. Show that one can construct $\frac{a}{b}$, $a, b > 0$, by going straight from $(a, 0)$ to the y-axis on the line forming the angle $\pi/2+\beta$ with the x-axis, where b is the projection of the point with argument β.

Hint(s) to 4.7: Draw a picture of the stereographic projection from the North Pole N via $X = (x, y)$ to $B = (b, 0)$. Then draw a line through $A = (a, 0)$ parallel to NB. That line forms the angle $\pi/2 + \beta$ with the x-axis. Can you calculate its y-intercept?

Outline(s) of solution(s) to 4.7: Draw a picture of the stereographic projection from the North Pole N via $X = (x, y)$ to $B = (b, 0)$. Then draw a line through $A = (a, 0)$ parallel to NB. That line forms the angle $\pi/2 + \beta$ with the x-axis. Its y-intercept equals a/b.

***Problem* 4.8.** Consider the stereographic projection from the North Pole of the circle centered at $(0, 1/2)$ with radius $r = 1/2$ onto the x-axis. Show that one can construct $a \cdot b$, $a, b > 0$, by going straight from $(0, a)$ to the x-axis on the line forming the angle $\pi - \beta$ with the y-axis, where b is the projection of the point with argument β.

Hint(s) to 4.8: Draw a picture of the stereographic projection from the North Pole N via $X = (x, y)$ to $B = (b, 0)$. Then draw a line through $A = (0, a)$ parallel to NB. That line forms the angle $\pi/2 - \beta$ with the y-axis. Can you calculate its x-intercept?

Outline(s) of solution(s) to 4.8: Draw a picture of the stereographic projection from the North Pole N via $X = (x, y)$ to $B = (b, 0)$. Then draw a line through $A = (0, a)$ parallel to NB. That line forms the angle $\pi/2 - \beta$ with the y-axis. Its x-intercept equals $a \cdot b$.

***Problem* 4.9.** Consider the stereographic projection from the North Pole of the circle centered at $(0, 1/2)$ with radius $r = 1/2$ onto the x-axis. If $a > 0$, compute the altitudes of the triangle ABC, $A = (-a, 0)$, $B = (a, 0)$, $C = (0, 1)$, in two different ways and conclude that $\sin(2 \cdot \alpha) = 2 \cdot \sin(\alpha) \cdot \cos(\alpha)$ for positive angles $\alpha < \pi/2$.

Hint(s) to 4.9: Let $h = AD$ be the altitude of triangle ABC from vertex A onto the side CB. Consider both triangles ADC and ADB. Can you relate their angles to α, the argument of the point mapped to B by the stereographic projection?

Outline(s) of solution(s) to 4.9: Let $h = AD$ be the altitude of triangle ABC from vertex A onto the side CB. Consider both triangles ADC and ADB. Let α be the argument of the point mapped to B by the stereographic projection. The angle $\setminus ACB = 2 \cdot \alpha$, the angle $\setminus ABD = \pi/2 - \alpha$. Therefore $BC = 1/\cos(\alpha)$, $h = NA \cdot \sin(2 \cdot \alpha)$, and $h = AB \cdot \cos(\alpha) = 2\tan(\alpha) \cdot \cos(\alpha) = 2 \cdot \sin(\alpha)$. Comparing the two formulae for h gives $\sin(2 \cdot \alpha) = 2 \cdot \sin(\alpha) \cdot \cos(\alpha)$.

***Problem* 4.10.** Consider the stereographic projection from the North Pole of the circle centered at $(0, 1/2)$ with radius $r = 1/2$ onto the x-axis. If $a > 0$, compute the bases of altitudes of the triangle ABC, $A = (-a, 0)$, $B = (a, 0)$, $C = (0, 1)$, in two different ways and conclude that $\cos(2 \cdot \alpha) = 2 \cdot \cos^2(\alpha) - 1$ for positive angles $\alpha < \pi/2$.

Hint(s) to 4.10: Let $h = AD$ be the altitude of triangle ABC from vertex A onto the side CB. Consider both triangles ADC and ADB. Can you relate their angles to α, the argument of the point mapped to B by the stereographic projection? Can you compute BD and CD using both triangles?

Outline(s) of solution(s) to 4.10: Let $h = AD$ be the altitude of triangle ABC from vertex A onto the side CB. Consider both triangles ADC and ADB. Let α be the argument of the point mapped to B by the stereographic projection. The angle $\setminus ACB = 2 \cdot \alpha$, the angle $\setminus ABD = \pi/2 - \alpha$. Therefore $CA = 1/\cos(\alpha)$, so $CD = CA \cdot \cos(2 \cdot \alpha) = \cos(2 \cdot \alpha)/\cos(\alpha)$ and $BD = AB \cdot \sin(\alpha) = 2 \cdot \tan(\alpha) \cdot \sin(\alpha)$. Since $CB = BD + DC$, one gets $\cos(2 \cdot \alpha) = 1 - 2 \cdot \sin^2(\alpha)$.

Problem **4.11.** Consider the stereographic projection from the North Pole of the circle centered at $(0,1/2)$ with radius $r=1/2$ onto the x-axis. If $a,b>0$, compute the altitudes of the triangle ABC, $A=(-a,0)$, $B=(b,0)$, $C=(0,1)$, in two different ways and conclude that $\sin(\alpha+\beta)=\sin(\alpha)\cdot\cos(\beta)+\cos(\alpha)\cdot\sin(\beta)$ for positive angles $\alpha,\beta<\pi/2$.

Hint(s) to 4.11: Let $h = AD$ be the altitude of triangle ABC from vertex A onto the side CB. Consider both triangles ADC and ADB. Can you relate their angles to β, the argument of the point mapped to B by the stereographic projection and α, the argument of the point mapped to $-A$ by the stereographic projection?

Outline(s) of solution(s) to 4.11: Let $h = AD$ be the altitude of triangle ABC from vertex A onto the side CB. Consider both triangles ADC and ADB. Let β be the argument of the point mapped to B by the stereographic projection. Let α be the argument of the point mapped to $-A$ by the stereographic projection. The angle $\angle ACB = \alpha+\beta$, the angle $\angle ABD = \pi/2 - \beta$. Therefore $BC = 1/\cos(\alpha)$, $h = NA\cdot\sin(\alpha+\beta)$, and $h = AB\cdot\cos(\beta) = 2\tan(\alpha)\cdot\cos(\beta)$. Comparing the two formulae for h gives $\sin(\alpha+\beta) = \sin(\alpha)\cdot\cos(\beta)+\cos(\alpha)\cdot\sin(\beta)$.

***Problem* 4.12.** Consider the stereographic projection from the North Pole of the circle centered at $(0, 1/2)$ with radius $r = 1/2$ onto the x-axis. If $a, b > 0$, compute the bases of altitudes of the triangle ABC, $A = (-a, 0)$, $B = (b, 0)$, $C = (0, 1)$, in two different ways and conclude that $\cos(\alpha+\beta) = \cos(\alpha)\cdot\cos(\beta) - \sin(\alpha)\cdot\sin(\beta)$ for positive angles $\alpha, \beta < \pi/2$.

Hint(s) to 4.12: Let $h = AD$ be the altitude of triangle ABC from vertex A onto the side CB. Consider both triangles ADC and ADB. Can you relate their angles to β, the argument of the point mapped to B by the stereographic projection and α, the argument of the point mapped to $-A$ by the stereographic projection?

Outline(s) of solution(s) to 4.12: Let $h = AD$ be the altitude of triangle ABC from vertex A onto the side CB. Consider both triangles ADC and ADB. Let β be the argument of the point mapped to B by the stereographic projection. Let α be the argument of the point mapped to $-A$ by the stereographic projection. The angle $\setminus ACB = \alpha + \beta$, the angle $\setminus ABD = \pi/2 - \beta$. Therefore $AC = 1/\cos(\alpha)$, $CD = AC \cdot \cos(\alpha + \beta)$, and $DB = AB \cdot \sin(\beta) = (\tan(\alpha) + \tan(\beta)) \cdot \sin(\beta)$. Analyzing $CB = CD + DB$ gives $\cos(\alpha + \beta) = \cos(\alpha) \cdot \cos(\beta) - \sin(\alpha) \cdot \sin(\beta)$.

***Problem* 4.13.** Consider the stereographic projection from the North Pole of the circle centered at $(0, 1/2)$ with radius $r = 1/2$ onto the x-axis. If $b > a > 0$, consider the intersection C of the vertical line $x = b$ with the line passing through the North Pole and perpendicular to the line joining North Pole and $(a, 0)$. Show that $C = (b, 1+a\cdot b)$ and conclude that $\tan(\beta-\alpha) = \frac{\tan(\beta)-\tan(\alpha)}{1+\tan(\alpha)\cdot\tan(\beta)}$ for positive angles $\alpha < \beta < \pi/2$.

Hint(s) to 4.13: Draw a picture of the stereographic projection from the North Pole N via $B' = (x, y)$ to $B = (b, 0)$ and of the stereographic projection from the North Pole N via $A' = (x, y)$ to $A = (a, 0)$. Then draw a line through N perpendicular to NB and mark its intersection C with line $x = b$. What is the angle between NA and NB in terms of arguments α of A' and β of B'? Do you see that all points N, A, B, and C lie on one circle?

Outline(s) of solution(s) to 4.13: Draw a picture of the stereographic projection from the North Pole N via $B' = (x, y)$ to $B = (b, 0)$ and of the stereographic projection from the North Pole N via $A' = (x, y)$ to $A = (a, 0)$. Then draw a line through N perpendicular to NB and mark its intersection C with line $x = b$. The angle between NA and NB can be expressed in terms of arguments α of A' and β of B' as $\beta - \alpha$. Since all points N, A, B, and C lie on one circle with diameter NB, the angle between CA and CB is also $\beta - \alpha$. Apply that information to the right triangle ABC.

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