4. **Stereographic Projection**

There are two special projections: one onto the $x$-axis, the other onto the $y$-axis. Both are well-known. Using those projections one can define functions sine and cosine. However, there is another projection, less known to students, a projection from a circle to the $x$-axis. It is called the **stereographic projection**. We will use it to provide geometric interpretations of multiplication, division of real numbers, the tangent function, and basic trigonometric formulae.
**Problem 4.1.** Find the intersection of the line joining \((0, 1)\) and \((1, 3)\) with the \(x\)-axis.
**Hint(s) to 4.1:** Find an equation of the line passing through the two points. How does one find its intersection with the $x$-axis?
Outline(s) of solution(s) to 4.1: A non-vertical line is the graph of a linear function $f(x)$. A function $f(x)$ is called \textbf{linear} if the ratio $(f(x_2) - f(x_1))/(x_2 - x_1)$ is always constant if $x_2 \neq x_1$. That ratio is the slope $m$ of the geometric line. One can compute $m$ from the data. Now, setting $x_2 = x$ and $x_1 = 0$ gives $(f(x) - 1)/(x - 0) = m$ as an equation of our line. Its $x$-intercept is the point such that $f(x) = 0$, so solve $(f(x) - 1)/(x - 0) = m$ in that case.
Answer to 4.1: \(-.5\)
Problem 4.2. Find the intersection of the line joining $(0, 0, 1)$ and $(1, 1, 3)$ with the $xy$-plane.
**Hint(s) to 4.2:** Find an equation of the line passing through the two points. How does one find its intersection with the $xy$-plane?
Outline(s) of solution(s) to 4.2: A non-vertical line on space is the graph of a linear function $f(t)$. A function $f(t)$ is called linear if the ratio $(f(t_2) - f(t_1))/(t_2 - t_1)$ is always constant if $t_2 \neq t_1$. Physically, that constant is the velocity $v$ of a particle traversing our line. One can compute $v$ from the data assuming that at $t = 0$ the particle is at point $(0, 0, 1)$ and at $t = 1$ it is at the other point. Now, setting $t_2 = t$ and $t_1 = 0$ gives $(f(t) - (0, 0, 1))/(t - 0) = v$ as an equation of our line. Its $xy$-intercept is the point $t$ such that the third coordinate of $f(t)$ equals 0, so solve $(f(t) - (0, 0, 1))/(t - 0) = v$ for such $t$, then find $f(t)$. 
Answer to 4.2: \((-0.5, -0.5)\)
Problem 4.3. Suppose $b \neq 1$. Show that the line joining $(0, 1)$ and $(a, b)$ intersects the $x$-axis at \( \left( \frac{a}{1-b}, 0 \right) \).
**Hint(s) to 4.3:** Find an equation of the line passing through the two points. How does one find its intersection with the $x$-axis?
Outline(s) of solution(s) to 4.3: A non-vertical line is the graph of a linear function \( f(x) \). A function \( f(x) \) is called **linear** if the ratio \( (f(x_2) - f(x_1))/(x_2 - x_1) \) is always constant if \( x_2 \neq x_1 \). That ratio is the slope \( m \) of the geometric line. One can compute \( m \) from the data. Now, setting \( x_2 = x \) and \( x_1 = 0 \) gives \( (f(x) - 1)/(x - 0) = m \) as an equation of our line. Its \( x \)-intercept is the point such that \( f(x) = 0 \), so solve \( (f(x) - 1)/(x - 0) = m \) in that case.
Problem 4.4. Suppose $a \neq 0$ and $b \neq 1$. Show that $\frac{a}{1-b} = \frac{b}{a}$ if $(a, b)$ is on the circle centered at $(0, 1/2)$ with radius $r = 1/2$. 
Hint(s) to 4.4: What does it mean that a point \((x, y)\) lies on the circle centered at \((0, 1)\) of radius \(r = 1/2\)? Can you write an equation of that circle? Is it equivalent to \(\frac{x}{1-y} = \frac{y}{x}\) if \(x \neq 0\)?
Outline(s) of solution(s) to 4.4: A point \((x, y)\) lies on the circle centered at \((0, 1)\) of radius \(r = 1/2\) if its distance to \((0, 1)\) is 1/2. Algebraically, it is the same as

\[
x^2 + (y - 1/2)^2 = (1/2)^2.
\]

Expanding and simplifying gives

\[
x^2 + y^2 - y = 0
\]

which is equivalent to \(\frac{x}{1-y} = \frac{y}{x}\) if \(x \neq 0\).
**Problem 4.5.** Consider the stereographic projection from the North Pole of the circle centered at \((0, 1/2)\) with radius \(r = 1/2\) onto the \(x\)-axis. Show that the image of the point with argument \(\alpha \neq \pi/2\) is \(\tan(\alpha)\).
**Hint(s) to 4.5:**

1. (Algebraic) If a point \((x, y)\) has argument \(\alpha\), how can \(\tan(\alpha)\) be expressed using \(x\) and \(y\)? How does that compare to the stereographic projection of \((x, y)\)?

2. (Geometric) Draw a picture of the stereographic projection from the North Pole \(N\) via \(A = (x, y)\) to \(B = (t, 0)\). Can you calculate all the angles on the picture? What are their tangents?
Outline(s) of solution(s) to 4.5:

1. (Algebraic) If a point \((x, y)\) has argument \(\alpha\), then \(\tan(\alpha) = \frac{y}{x}\). The stereographic projection of \((x, y)\) is \(\frac{x}{1-y}\) and that equals \(\frac{y}{x} = \tan(\alpha)\) by 4.4.

2. (Geometric) Draw a picture of the stereographic projection from the North Pole \(N\) via \(A = (x, y)\) to \(B = (t, 0)\). If \(\alpha\) is the argument of \(A\), then the angle \(\angle ONB\) equals \(\alpha\). Since \(\tan(\angle ONB) = t\), we are done.
Problem 4.6. Consider the stereographic projection from the North Pole of the circle centered at $(0, 1/2)$ with radius $r = 1/2$ onto the $x$-axis. Show that one can construct $\frac{-1}{a}$, $a > 0$, by going straight from $(a, 0)$ to $(0, 1)$ and then turning left via 90 degrees and going straight until the $x$-axis is met again.
Hint(s) to 4.6: Draw a picture of the stereographic projection from the North Pole $N$ via $B = (x, y)$ to $A = (a, 0)$. Then draw a line through $N$ perpendicular to $NA$. Can you calculate all the angles on the picture? What are their tangents?
Outline(s) of solution(s) to 4.6: Draw a picture of the stereographic projection from the North Pole $N$ via $B = (x, y)$ to $A = (a, 0)$. Then draw a line through $N$ perpendicular to $NA$. Let $t$ be the $x$-intercept of that line. If $\alpha$ is the argument of $B$, then the angle $\angleONA$ equals $\alpha$ and the angle $\angleOCN = \alpha$, where $C = (t, 0)$. Since $\tan(\angleOCN) = -1/t$, we are done.
**Problem 4.7.** Consider the stereographic projection from the North Pole of the circle centered at $(0, 1/2)$ with radius $r = 1/2$ onto the $x$-axis. Show that one can construct $\frac{a}{b}$, $a, b > 0$, by going straight from $(a, 0)$ to the $y$-axis on the line forming the angle $\pi/2 + \beta$ with the $x$-axis, where $b$ is the projection of the point with argument $\beta$. 
**Hint(s) to 4.7:** Draw a picture of the stereographic projection from the North Pole $N$ via $X = (x, y)$ to $B = (b, 0)$. Then draw a line through $A = (a, 0)$ parallel to $NB$. That line forms the angle $\pi/2 + \beta$ with the $x$-axis. Can you calculate its $y$-intercept?
Outline(s) of solution(s) to 4.7: Draw a picture of the stereographic projection from the North Pole \( N \) via \( X = (x, y) \) to \( B = (b, 0) \). Then draw a line through \( A = (a, 0) \) parallel to \( NB \). That line forms the angle \( \pi/2 + \beta \) with the \( x \)-axis. Its \( y \)-intercept equals \( a/b \).
Problem 4.8. Consider the stereographic projection from the North Pole of the circle centered at 
$(0, 1/2)$ with radius $r = 1/2$ onto the $x$-axis. Show that one can construct $a \cdot b$, $a, b > 0$, 
by going straight from $(0, a)$ to the $x$-axis on the line forming the angle $\pi - \beta$ with the $y$-
axis, where $b$ is the projection of the point with argument $\beta$. 
**Hint(s) to 4.8:** Draw a picture of the stereographic projection from the North Pole $N$ via $X = (x, y)$ to $B = (b, 0)$. Then draw a line through $A = (0, a)$ parallel to $NB$. That line forms the angle $\pi/2 - \beta$ with the $y$-axis. Can you calculate its $x$-intercept?
Outline(s) of solution(s) to 4.8: Draw a picture of the stereographic projection from the North Pole $N$ via $X = (x, y)$ to $B = (b, 0)$. Then draw a line through $A = (0, a)$ parallel to $NB$. That line forms the angle $\pi/2 - \beta$ with the $y$-axis. Its $x$-intercept equals $a \cdot b$. 
**Problem 4.9.** Consider the stereographic projection from the North Pole of the circle centered at $(0, 1/2)$ with radius $r = 1/2$ onto the $x$-axis. If $a > 0$, compute the altitudes of the triangle $ABC$, $A = (-a, 0)$, $B = (a, 0)$, $C = (0, 1)$, in two different ways and conclude that $\sin(2\cdot \alpha) = 2 \cdot \sin(\alpha) \cdot \cos(\alpha)$ for positive angles $\alpha < \pi/2$. 
**Hint(s) to 4.9:** Let $h = AD$ be the altitude of triangle $ABC$ from vertex $A$ onto the side $CB$. Consider both triangles $ADC$ and $ADB$. Can you relate their angles to $\alpha$, the argument of the point mapped to $B$ by the stereographic projection?
Outline(s) of solution(s) to 4.9: Let $h = AD$ be the altitude of triangle $ABC$ from vertex $A$ onto the side $CB$. Consider both triangles $ADC$ and $ADB$. Let $\alpha$ be the argument of the point mapped to $B$ by the stereographic projection. The angle $\angle ACB = 2 \cdot \alpha$, the angle $\angle ABD = \pi/2 - \alpha$. Therefore $BC = 1/\cos(\alpha)$, $h = NA \cdot \sin(2 \cdot \alpha)$, and $h = AB \cdot \cos(\alpha) = 2 \tan(\alpha) \cdot \cos(\alpha) = 2 \cdot \sin(\alpha)$. Comparing the two formulae for $h$ gives $\sin(2 \cdot \alpha) = 2 \cdot \sin(\alpha) \cdot \cos(\alpha)$. 
**Problem 4.10.** Consider the stereographic projection from the North Pole of the circle centered at 
$(0, 1/2)$ with radius $r = 1/2$ onto the $x$-axis. If $a > 0$, compute the bases of altitudes of the triangle $ABC$, $A = (-a, 0)$, $B = (a, 0)$, $C = (0, 1)$, in two different ways and conclude that $\cos(2 \cdot \alpha) = 2 \cdot \cos^2(\alpha) - 1$ for positive angles $\alpha < \pi/2$. 
**Hint(s) to 4.10:** Let $h = AD$ be the altitude of triangle $ABC$ from vertex $A$ onto the side $CB$. Consider both triangles $ADC$ and $ADB$. Can you relate their angles to $\alpha$, the argument of the point mapped to $B$ by the stereographic projection? Can you compute $BD$ and $CD$ using both triangles?
Outline(s) of solution(s) to 4.10: Let $h = AD$ be the altitude of triangle $ABC$ from vertex $A$ onto the side $CB$. Consider both triangles $ADC$ and $ADB$. Let $\alpha$ be the argument of the point mapped to $B$ by the stereographic projection. The angle $\angle ACB = 2 \cdot \alpha$, the angle $\angle ABD = \pi/2 - \alpha$. Therefore $CA = 1/\cos(\alpha)$, so $CD = CA \cdot \cos(2 \cdot \alpha) = \cos(2 \cdot \alpha)/\cos(\alpha)$ and $BD = AB \cdot \sin(\alpha) = 2 \cdot \tan(\alpha) \cdot \sin(\alpha)$. Since $CB = BD + DC$, one gets $\cos(2 \cdot \alpha) = 1 - 2 \cdot \sin^2(\alpha)$. 
Problem 4.11. Consider the stereographic projection from the North Pole of the circle centered at $(0, 1/2)$ with radius $r = 1/2$ onto the $x$-axis. If $a, b > 0$, compute the altitudes of the triangle $ABC$, $A = (-a, 0)$, $B = (b, 0)$, $C = (0, 1)$, in two different ways and conclude that $\sin(\alpha + \beta) = \sin(\alpha) \cdot \cos(\beta) + \cos(\alpha) \cdot \sin(\beta)$ for positive angles $\alpha, \beta < \pi/2$. 
Hint(s) to 4.11: Let $h = AD$ be the altitude of triangle $ABC$ from vertex $A$ onto the side $CB$. Consider both triangles $ADC$ and $ADB$. Can you relate their angles to $\beta$, the argument of the point mapped to $B$ by the stereographic projection and $\alpha$, the argument of the point mapped to $-A$ by the stereographic projection?
Outline(s) of solution(s) to 4.11: Let $h = AD$ be the altitude of triangle $ABC$ from vertex $A$ onto the side $CB$. Consider both triangles $ADC$ and $ADB$. Let $\beta$ be the argument of the point mapped to $B$ by the stereographic projection. Let $\alpha$ be the argument of the point mapped to $-A$ by the stereographic projection. The angle $\angle ACB = \alpha + \beta$, the angle $\angle ABD = \pi/2 - \beta$. Therefore $BC = 1/\cos(\alpha)$, $h = NA \cdot \sin(\alpha + \beta)$, and $h = AB \cdot \cos(\beta) = 2 \tan(\alpha) \cdot \cos(\beta)$. Comparing the two formulae for $h$ gives $\sin(\alpha + \beta) = \sin(\alpha) \cdot \cos(\beta) + \cos(\alpha) \cdot \sin(\beta)$. 
**Problem 4.12.** Consider the stereographic projection from the North Pole of the circle centered at $(0, \frac{1}{2})$ with radius $r = 1/2$ onto the $x$-axis. If $a, b > 0$, compute the bases of altitudes of the triangle $ABC$, $A = (-a, 0)$, $B = (b, 0)$, $C = (0, 1)$, in two different ways and conclude that $\cos(\alpha + \beta) = \cos(\alpha) \cdot \cos(\beta) - \sin(\alpha) \cdot \sin(\beta)$ for positive angles $\alpha, \beta < \pi/2$. 
**Hint(s) to 4.12:** Let $h = AD$ be the altitude of triangle $ABC$ from vertex $A$ onto the side $CB$. Consider both triangles $ADC$ and $ADB$. Can you relate their angles to $\beta$, the argument of the point mapped to $B$ by the stereographic projection and $\alpha$, the argument of the point mapped to $-A$ by the stereographic projection?
Outline(s) of solution(s) to 4.12: Let $h = AD$ be the altitude of triangle $ABC$ from vertex $A$ onto the side $CB$. Consider both triangles $ADC$ and $ADB$. Let $\beta$ be the argument of the point mapped to $B$ by the stereographic projection. Let $\alpha$ be the argument of the point mapped to $-A$ by the stereographic projection. The angle $\angle ACB = \alpha + \beta$, the angle $\angle ABD = \pi/2 - \beta$. Therefore $AC = 1/\cos(\alpha)$, $CD = AC \cdot \cos(\alpha + \beta)$, and $DB = AB \cdot \sin(\beta) = (\tan(\alpha) + \tan(\beta)) \cdot \sin(\beta)$. Analyzing $CB = CD + DB$ gives $\cos(\alpha + \beta) = \cos(\alpha) \cdot \cos(\beta) - \sin(\alpha) \cdot \sin(\beta)$. 
Problem 4.13. Consider the stereographic projection from the North Pole of the circle centered at $(0, 1/2)$ with radius $r = 1/2$ onto the $x$-axis. If $b > a > 0$, consider the intersection $C$ of the vertical line $x = b$ with the line passing through the North Pole and perpendicular to the line joining North Pole and $(a, 0)$. Show that $C = (b, 1 + a \cdot b)$ and conclude that $\tan(\beta - \alpha) = \frac{\tan(\beta) - \tan(\alpha)}{1 + \tan(\alpha) \cdot \tan(\beta)}$ for positive angles $\alpha < \beta < \pi/2$. 
**Hint(s) to 4.13:** Draw a picture of the stereographic projection from the North Pole $N$ via $B' = (x, y)$ to $B = (b, 0)$ and of the stereographic projection from the North Pole $N$ via $A' = (x, y)$ to $A = (a, 0)$. Then draw a line through $N$ perpendicular to $NB$ and mark its intersection $C$ with line $x = b$. What is the angle between $NA$ and $NB$ in terms of arguments $\alpha$ of $A'$ and $\beta$ of $B'$? Do you see that all points $N, A, B,$ and $C'$ lie on one circle?
Outline(s) of solution(s) to 4.13: Draw a picture of the stereographic projection from the North Pole $N$ via $B' = (x, y)$ to $B = (b, 0)$ and of the stereographic projection from the North Pole $N$ via $A' = (x, y)$ to $A = (a, 0)$. Then draw a line through $N$ perpendicular to $NB$ and mark its intersection $C$ with line $x = b$. The angle between $NA$ and $NB$ can be expressed in terms of arguments $\alpha$ of $A'$ and $\beta$ of $B'$ as $\beta - \alpha$. Since all points $N$, $A$, $B$, and $C$ lie on one circle with diameter $NB$, the angle between $CA$ and $CB$ is also $\beta - \alpha$. Apply that information to the right triangle $ABC$. 
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