6. Basic proofs for complex numbers

***Problem* 6.1.** Prove that $Re(z) = (z + \bar{z})/2$.

***Problem* 6.2.** Prove that $Im(z) = (z - \bar{z})/(2i)$.

***Problem* 6.3.** Prove that $\tan(arg(z)) = Im(z)/Re(z)$.

***Problem* 6.4.** Prove that $z \cdot \bar{z} = |z|^2$.

***Problem* 6.5.** Prove that $1/z = \bar{z}/|z|^2$.

***Problem* 6.6.** Prove that the conjugate of $z_1 + z_2$ is $\bar{z}_1 + \bar{z}_2$.

***Problem* 6.7.** Prove that the conjugate of $z_1 \cdot z_2$ is $\bar{z}_1 \cdot \bar{z}_2$.

***Problem* 6.8.** Prove that the conjugate of z_1/z_2 is $\bar{z}_1/\bar{z}_2$.

***Problem* 6.9.** Let C denote the set of all complex numbers $a + b \cdot i$. Define the absolute value $|z|$ of $z = a + b \cdot i$ as the square root of $a^2 + b^2$. Show that $|z_1 \cdot z_2| = |z_1| \cdot |z_2|$.

Hint(s) to 6.9: Just grind it out.

7. Quotients and square roots of complex numbers

***Problem* 7.1.** Find $z = \frac{0+2\cdot i}{1+1\cdot i}$.

Answer to 7.1: $z = 1 + 1 \cdot i$

***Problem* 7.2.** Solve $z^2 = 3+4\cdot i$ without using a calculator.

Answer to 7.2: $z = 2 + 1 \cdot i,\ -2 - 1 \cdot i$

***Problem* 7.3.** Let $z = x + i \cdot y$, where $y > 0$. Choose a positive real number t so that $t^2 = x^2 + y^2$ and pick positive numbers u and v satisfying $u^2 = (t + x)/2$, $v^2 = (t - x)/2$. Show that $x + i \cdot y = (u + i \cdot v)^2$. Can you do similar procedure of finding a square root if y is negative?

***Problem* 7.4.** Prove that for any complex number $w \neq 0$ the equation $z^2 = w$ has exactly two solutions.

***Problem* 7.5.** Solve $z^2 - 4 \cdot z + 5 = 0$ without using a calculator if one of solutions is $z = 2{+}1{\cdot}i$.

Answer to 7.5: $z = 2 + 1 \cdot i,\ 2 - 1 \cdot i$

***Problem* 7.6.** Let $a \neq 0$, b, and c be complex constants. Show that the quadratic equation $a \cdot z^2 + b \cdot z + c = 0$ has one or two roots.

Problem 7.7. Let b and c be complex constants such that $z^2 + b \cdot z + c = 0$ has only real roots. Show that b and c are real.

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