9. **Triangle Inequality for Complex Numbers**

**Problem 9.1.** Suppose $z_1$ and $z_2$ are complex numbers such that $z_1 + z_2$ is real. Prove that $|z_1 + z_2|$ is not greater than $|z_1| + |z_2|$ by using similar inequality for real numbers.
Problem 9.2. Suppose $z_1$ and $z_2$ are non-zero complex numbers such that $z_1 + z_2$ is real. Prove that if $|z_1 + z_2| = |z_1| + |z_2|$, then $z_1$ and $z_2$ are real numbers whose ratio is positive.
Problem 9.3. Prove algebraically that $|z_1 + z_2|$ is not greater than $|z_1| + |z_2|$. Do not use Triangle Inequality. This is Triangle Inequality.
Problem 9.4. Suppose \( z_1 \) and \( z_2 \) are non-zero complex numbers. Prove that if \(|z_1 + z_2| = |z_1| + |z_2|\), then \( z_1/z_2 \) is a positive real number.
10. **Multiplication of complex numbers**

*Problem 10.1.* Prove geometrically that $i \cdot z$ is $z$ rotated counterclockwise by 90 degrees.
Problem 10.2. \([1, 1]\) is rotated clockwise by 90 degrees. Find the resulting vector.
**Hint(s) to 10.2:** What is the result of rotating $e_1$ and $e_2$? What is the result of rotating $a \cdot e_1 + b \cdot e_2$?
Answer to 10.2: $[1, -1]$
**Problem 10.3.** Prove geometrically that \((\cos(\theta) + i \cdot \sin(\theta)) \cdot z\) is \(z\) rotated counterclockwise by \(\theta\) radians.
\textbf{Problem 10.4.} The geometrical interpretation of \( z \cdot (\cos(\alpha) + i \cdot \sin(\alpha)) \) is \( z \) rotated counterclockwise by angle \( \alpha \). \([3, 4]\) is rotated \textbf{clockwise} by 6 degrees. Find the resulting vector.
**Hint(s) to 10.4**: Switch to complex numbers and stick into a calculator.
Answer to 10.4: $[3.4, 3.66]$
Problem 10.5. Sketch a picture illustrating multiplication of unit complex numbers.
Problem 10.6. Show that \((1 + 2 \cdot i)^{2001} + (1 - 2 \cdot i)^{2001}\) is a real number.
Hint(s) to 10.6: Show that the number equals its conjugate.
Problem 10.7. Prove that $z/(z^2 + 1)$ is a real number if $z$ lies on the unit circle.
Hint(s) to 10.7: Show that the number equals its conjugate.
Problem 10.8. Prove that \( \frac{z}{(z + 1)^2} \) is a real number if \( z \) lies on the unit circle.
Hint(s) to 10.8: Show that the number equals its conjugate.
**Problem 10.9.** Show that $i \cdot (1 + z)/(1 - z)$ is a real number if $z \neq 1$ is a unit complex number.
**Hint(s) to 10.9:** Show that the number equals its conjugate.
**Problem 10.10.** Show that \((1 - z^2) \cdot w / (w^2 - z^2)\) is a real number if \(z \neq w\) are unit complex numbers.
**Hint(s) to 10.10:** Show that the number equals its conjugate.