11. **Scalar product and vector product for complex numbers**

**Problem 11.1.** Given two complex numbers \( z \) and \( w \), the scalar product \( S(z, w) \) is defined as \( (z \cdot \bar{w} + \bar{z} \cdot w)/2 \). Show that \( S(z, w) = \text{Re}(\bar{z} \cdot w) \). Conclude that \( S(z, w) = |z| \cdot |w| \cdot \cos(\alpha) \), where \( \alpha \) is the angle from \( z \) to \( w \) measured in counterclockwise direction.
Problem 11.2. Given two complex numbers $z$ and $w$, the scalar product $S(z, w)$ is defined as $(z \cdot \bar{w} + \bar{z} \cdot w)/2$. Show that $|z + w|^2 = |z|^2 + |w|^2 + 2 \cdot S(z, w)$. Derive the Cosine Theorem from that equality.
Problem 11.3. Given two complex numbers \( z \) and \( w \), the scalar product \( S(z, w) \) is defined as \( (z \cdot \bar{w} + \bar{z} \cdot w)/2 \). Show algebraically that \( S(z, w) = S(w, z) \).
Problem 11.4. Given two complex numbers \( z \) and \( w \), the scalar product \( S(z, w) \) is defined as \((z \cdot \bar{w} + \bar{z} \cdot w)/2\). Show algebraically that \( S(z, a \cdot w + b \cdot v) = a \cdot S(z, w) + b \cdot S(z, v) \) provided \( a \) and \( b \) are real.
**Problem 11.5.** Given two complex numbers $z$ and $w$, the vector product $V(z, w)$ is defined as $i \cdot (z \cdot \bar{w} - \bar{z} \cdot w)/2$. Show that $V(z, w) = Im(\bar{z} \cdot w)$. Conclude that $V(z, w) = |z| \cdot |w| \cdot \sin(\alpha)$, where $\alpha$ is the angle from $z$ to $w$ measured in counterclockwise direction. Conclude that $|V(z, w)|$ is the area of parallelogram formed by $z$ and $w$. 
Problem 11.6. Given two complex numbers \( z \) and \( w \), the vector product \( V(z, w) \) is defined as
\[
 i \cdot (z \cdot \bar{w} - \bar{z} \cdot w)/2.
\]
Show algebraically that \( V(z, w) = -V(w, z) \).
Problem 11.7. Given two complex numbers $z$ and $w$, the vector product $V(z, w)$ is defined as $i \cdot (z \cdot \bar{w} - \bar{z} \cdot w)/2$. Show algebraically that $V(z, a \cdot w + b \cdot v) = a \cdot V(z, w) + b \cdot V(z, v)$ provided $a$ and $b$ are real.
**Problem 11.8.** Given two complex numbers $z$ and $w$, the scalar product $S(z, w)$ is defined as $(z \cdot \bar{w} + \bar{z} \cdot w)/2$. If $z = x_1 + y_1 \cdot i$ and $w = x_2 + y_2 \cdot i$, show that $S(z, w) = x_1 \cdot x_2 + y_1 \cdot y_2$. 
**Problem 11.9.** Given two complex numbers $z$ and $w$, the vector product $V(z, w)$ is defined as $i \cdot (z \cdot \bar{w} - \bar{z} \cdot w)/2$. If $z = x_1 + y_1 \cdot i$ and $w = x_2 + y_2 \cdot i$, show that $V(z, w) = x_1 \cdot y_2 - x_2 \cdot y_1$, the determinant of the matrix $[[x_1, y_1], [x_2, y_2]]$. 
Problem 11.10. Find the area of the triangle with vertices $P(-1, 1), Q(1, -1)$ and $R(1, 1)$. 
Hint(s) to 11.10: How are triangles related to parallelograms? How do we compute areas of parallelograms?
Answer to 11.10: 2
Problem 11.11. Find the remaining two vertices $Q$ and $S$ of the square whose diagonal joins points $P = (1, -1)$ and $R = (3, 1)$. 
Hint(s) to 11.11: Can you find the center $C$ of the square? How does one get vector $CQ$ from the vector $CP$?
Answer to 11.11: $Q = (1, 1), S = (3, -1)$
Problem 11.12. If \( z_1/z_2 = a + b \cdot i \), then \( z_1 = a \cdot z_2 + b \cdot (i \cdot z_2) \), \( a \cdot z_2 \) is parallel to \( z_2 \), and \( b \cdot (i \cdot z_2) \) is perpendicular to \( z_2 \). Express vector \( \vec{v} = [-2, 4] \) as \( \vec{v} = \vec{v}_1 + \vec{v}_2 \), where \( \vec{v}_1 \) is parallel to \([1, 1]\) and \( \vec{v}_2 \) is perpendicular to \([1, 1]\). Report \( \vec{v}_2 \).
Answer to 11.12: $[-3, 3]$
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