12. Polar form of complex numbers and Euler Formula

Since multiplication of z by a unit complex number $\cos(\theta)+i\cdot\sin(\theta)$ corresponds to rotating z by angle θ in the counterclockwise direction, one gets

$$(\cos(\alpha)+i\cdot\sin(\alpha))\cdot(\cos(\theta)+i\cdot\sin(\theta)) =$$
$$\cos(\alpha+\theta)+i\cdot\sin(\alpha+\theta)$$

That formula resembles $a^m \cdot a^n = a^{m+n}$. It turns out that $\cos(\theta)+i\cdot\sin(\theta)$ can be interpreted as $e^{i\cdot\theta}$ which is called the **Euler Formula**. In this section we try to justify the formula from the point of view of differential equations.

It turns out that Euler Formula implies all the major identities of trigonometry.

***Problem* 12.1.** (A) Let $z = a + b \cdot i$, where $a^2 = 2$, a is negative, $b^2 = 2$, and b is positive. Sketch z. Find $arg(z)$.

Answer to 12.1: $-\pi/4 \bmod 2\pi$

***Problem* 12.2.** (A) Let $z = a + b \cdot i$, where $a^2 = 6$, a is negative, $b^2 = 2$, and b is positive. Sketch z. Find $arg(z)$.

Answer to 12.2: $-\pi/6 \bmod 2\pi$

***Problem* 12.3.** (A) Let $z = a + b \cdot i$, where $a^2 = 2$, a is positive, $b^2 = 6$, and b is positive. Sketch z. Find $arg(z)$.

Answer to 12.3: $\pi/3 \bmod 2\pi$

***Problem* 12.4.** (A) Let $z = -8 \cdot i$. Sketch z. Find square roots of z using polar form.

Answer to 12.4: $2 - 2 \cdot i, -2 + 2 \cdot i$

***Problem* 12.5.** Use polar form to show that every complex number $z \neq 0$ has two square roots.

***Problem* 12.6.** Use polar form to show that every complex number $z \neq 0$ has multiplicative inverse $1/z$.

***Problem* 12.7.** (A) Define $f(x) = \cos(2 \cdot x) + i \cdot \sin(2 \cdot x)$. Find $f'(x)/f(x)$.

Answer to 12.7: $2 \cdot i$

***Problem* 12.8.** (H) Using Euler's Formula show that $\cos(2x) = 2 \cdot \cos^2(x) - 1$.

Hint(s) to 12.8: Expand $(\cos(x)+i\cdot\sin(x))^2 = \cos(2x) + i \cdot \sin(2x)$.

***Problem* 12.9.** (H) Using Euler's Formula show that $\sin(2x) = 2 \cdot \cos(x) \cdot \sin(x)$.

Hint(s) to 12.9: Expand $(\cos(x)+i\cdot\sin(x))^2 = \cos(2x) + i \cdot \sin(2x)$.

***Problem* 12.10.** (H) Using Euler's Formula find $\sin(x + y)$.

Hint(s) to 12.10: Expand $(\cos(x)+i\cdot\sin(x))\cdot(\cos(y)+i\cdot\sin(y))$.

***Problem* 12.11.** (H) Using Euler's Formula find $\cos(x + y)$.

Hint(s) to 12.11: Expand $(\cos(x)+i\cdot\sin(x))\cdot(\cos(y)+i\cdot\sin(y))$.

***Problem* 12.12.** (HA) Find the cosine of the angle between vectors $[1, 1]$ and $[-1, 1]$.

Hint(s) to 12.12: Hint using complex numbers: How is the angle between u and v related to v/u? How can one find $\cos(arg(z))$ without knowing $arg(z)$?

Hint using dot products: Let α be the angle between u and v. $u \cdot v = \mid u \mid \cdot \mid v \mid \cdot \cos(\alpha)$.

Answer to 12.12: 0

***Problem* 12.13.** (HA) $u = [1, 1]$ is rotated counterclockwise until it points in the direction of $v = [-1, 1]$. Find the angle of rotation in degrees.

Hint(s) to 12.13: Complex Numbers Hint: How is the angle related to the argument of v/u? How to find $arg(z)$?

Linear Algebra Hint: Let α be the angle. Then, $u \cdot v =\mid u \mid \cdot \mid v \mid \cdot \cos(\alpha)$ and $det(u,v) =\mid u \mid \cdot \mid v \mid \cdot \sin(\alpha)$. Find $\tan(\alpha)$.

Answer to 12.13: 90

***Problem* 12.14.** (HA) The face of an antique clock shows only numbers 3, 6, 9, and 12. It has only the small hand whose tip is located at $(2, -1)$ in the coordinate system centered in the middle of the face with the x-axis pointing toward 3 and the y-axis pointing toward 12. Find the time being shown by the clock assuming it is before noon.

Hint(s) to 12.14: Let α be the angle from the x-axis to the short hand. Construct a linear function $time(\alpha)$.

Answer to 12.14: 3:53.13 AM

***Problem* 12.15.** (A) Find the symmetric image of $P(-3,-1)$ about the line passing through points $Q(0,0)$ and $R(-2,1)$.

Answer to 12.15: $(-1, 3)$

***Problem* 12.16.** (HA) The length of $\vec{u}$ is 1, the length of $\vec{v}$ is 2, and the angle between the two vectors is 30 degrees. Find the length of $\vec{u}+\vec{v}$.

Hint(s) to 12.16: Complex Numbers Hint: One may assume u is on the x-axis. What is u? What is v?

Linear Algebra Hint: Let α be the angle between u and v. $u \cdot u = \mid u \mid^2$ for any vector u. $\mid u + v \mid^2 = (u+v) \cdot (u+v) = u \cdot u + 2 \cdot u \cdot v + v \cdot v$.

Answer to 12.16: 2.91

***Problem* 12.17.** (HA) Let $R(1,1)$. On the segment joining $P(1,5)$ and $Q(13,1)$ find the point X such that RX bisects the angle between segments RP and RQ.

Hint(s) to 12.17: Hint: Let $\vec{RQ} = u$, $\vec{RP} = v$, and $\vec{RX} = w$. Because w bisects the angle between u and v, it must be a positive multiple of $u/|u| + v/|v|$ (why?). Because X lies on the seqment PQ, $\vec{PX} = s \cdot \vec{PQ} = s(u - v)$. Moreover, $\vec{PX} = w - v$, so $w - v = su - sv$ and $w = su + (1 - s)v$. Can you find $(1 - s)/s$?

Hint using dot product: If RX bisects the angle between segments RP and RQ, then $\cos\alpha = \cos\beta$, where α is the angle between RP and RX and β is the angle between RQ and RX. Thus,

$$\frac{\vec{RP} \cdot \vec{RX}}{RP \cdot RX} = \frac{\vec{RQ} \cdot \vec{RX}}{RQ \cdot RX}.$$

X being on the segment PQ means that $\vec{PX} = k \cdot \vec{PQ}$, where $0 \le k \le 1$. Thus, $X = P + k \cdot \vec{PQ}$ and we can solve for k using $\frac{\vec{RP} \cdot \vec{RX}}{RP} = \frac{\vec{RQ} \cdot \vec{RX}}{RQ}$.

Answer to 12.17: $(4, 4)$

***Problem* 12.18.** (A) Observe that $(3 - i \cdot 4) \cdot (\cos(x) + i \cdot \sin(x)) = (3 \cdot \cos(x) + 4 \cdot \sin(x)) + i \cdot (-4 \cdot \cos(x) + 3 \cdot \sin(x))$. Geometrically, it is vector $[3, -4]$ rotated counterclockwise by angle x. Maximize $3 \cdot \cos x + 4 \cdot \sin x$ by noticing that it amounts to rotating of $[3, -4]$ so that it lies on the x-axis.

Answer to 12.18: 5

***Problem* 12.19.** (A) Express $1 \cdot \cos(x) + 1 \cdot \sin(x)$ as $\mathrm{Re}(r \cdot e^{(x+\theta)i})$ with $r > 0$.

Answer to 12.19: r $= 1.414$, $\theta = -.785 + 2\pi n$

***Problem* 12.20.** By considering $(a + b \cdot i) \cdot (\cos(\theta) + i \cdot \sin(\theta))$, show that $b \cdot \cos(\theta) + a \cdot \sin(\theta) = (a^2 + b^2)^{.5} \cdot \sin(\theta + arctan(b/a))$.

***Problem* 12.21.** (A) Observe that $(-2+i\cdot 1)\cdot(\cos(x)+i\cdot\sin(x)) = (-2\cdot\cos(x)-1\cdot\sin(x))+i\cdot(1\cdot\cos(x)-2\cdot\sin(x))$. Geometrically, it is vector $[-2,1]$ rotated counterclockwise by angle x. Solve $1\cdot\cos x - 2\cdot\sin x = 0$ in the interval $[0,2\pi]$ (x is in radians) by noticing that it amounts to rotating of $[-2,1]$ so that it lies on the x-axis.

Answer to 12.21: $x = 0.464,\ 3.605$

***Problem* 12.22.** (A) Observe that $(-1+i\cdot 2)\cdot(\cos(x)+i\cdot\sin(x)) = (-1\cdot\cos(x)-2\cdot\sin(x))+i\cdot(2\cdot\cos(x)-1\cdot\sin(x))$. Geometrically, it is vector $[-1,2]$ rotated counterclockwise by angle x. Solve $2\cdot\cos x - 1\cdot\sin x = 0$ in degrees by noticing that it amounts to rotating of $[-1,2]$ so that it lies on the x-axis.

Answer to 12.22: $x = 63.4 + k \cdot 180$

***Problem* 12.23.** (A) Solve $f'(x) = 2 \cdot i \cdot f(x)$, $f(0) = 1$.

Answer to 12.23: $1 \cdot \exp(2 \cdot i \cdot x)$

***Problem* 12.24.** (A) Find the function $f(x)$ (x is real) so that the real part of $\exp(f(x))$ equals $\cos(2 \cdot x) \cdot \exp(1 \cdot x)$.

Answer to 12.24: $(1 + 2 \cdot i) \cdot x$

***Problem* 12.25.** (H) State Euler's Formula.

Hint(s) to 12.25: $\exp(i \cdot x) = \cos(x) + i \cdot \sin(x)$.

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