

12. POLAR FORM OF COMPLEX NUMBERS AND EULER FORMULA

Since multiplication of z by a unit complex number $\cos(\theta) + i \cdot \sin(\theta)$ corresponds to rotating z by angle θ in the counterclockwise direction, one gets

$$\begin{aligned} (\cos(\alpha) + i \cdot \sin(\alpha)) \cdot (\cos(\theta) + i \cdot \sin(\theta)) = \\ \cos(\alpha + \theta) + i \cdot \sin(\alpha + \theta) \end{aligned}$$

That formula resembles $a^m \cdot a^n = a^{m+n}$. It turns out that $\cos(\theta) + i \cdot \sin(\theta)$ can be interpreted as $e^{i \cdot \theta}$ which is called the **Euler Formula**. In this section we try to justify the formula from the point of view of differential equations.

It turns out that Euler Formula implies all the major identities of trigonometry.

Problem 12.1. (A) Let $z = a + b \cdot i$, where $a^2 = 2$, a is negative, $b^2 = 2$, and b is positive. Sketch z . Find $\arg(z)$.

Answer to 12.1: $-\pi/4 \bmod 2\pi$

Problem 12.2. (A) Let $z = a + b \cdot i$, where $a^2 = 6$, a is negative, $b^2 = 2$, and b is positive. Sketch z . Find $\arg(z)$.

Answer to 12.2: $-\pi/6 \bmod 2\pi$

Problem 12.3. (A) Let $z = a + b \cdot i$, where $a^2 = 2$, a is positive, $b^2 = 6$, and b is positive. Sketch z . Find $\arg(z)$.

Answer to 12.3: $\pi/3 \bmod 2\pi$

Problem 12.4. (A) Let $z = -8 \cdot i$. Sketch z .
Find square roots of z using polar form.

Answer to 12.4: $2 - 2 \cdot i, -2 + 2 \cdot i$

Problem 12.5. Use polar form to show that every complex number $z \neq 0$ has two square roots.

Problem 12.6. Use polar form to show that every complex number $z \neq 0$ has multiplicative inverse $1/z$.

Problem 12.7. (A) Define $f(x) = \cos(2 \cdot x) + i \cdot \sin(2 \cdot x)$. Find $f'(x)/f(x)$.

Answer to 12.7: $2 \cdot i$

Problem 12.8. (H) Using Euler's Formula show that $\cos(2x) = 2 \cdot \cos^2(x) - 1$.

Hint(s) to 12.8: Expand $(\cos(x) + i \cdot \sin(x))^2 = \cos(2x) + i \cdot \sin(2x)$.

Problem 12.9. (H) Using Euler's Formula show that $\sin(2x) = 2 \cdot \cos(x) \cdot \sin(x)$.

Hint(s) to 12.9: Expand $(\cos(x) + i \cdot \sin(x))^2 = \cos(2x) + i \cdot \sin(2x)$.

Problem 12.10. (H) Using Euler's Formula find $\sin(x + y)$.

Hint(s) to 12.10: Expand $(\cos(x) + i \cdot \sin(x)) \cdot (\cos(y) + i \cdot \sin(y))$.

Problem 12.11. (H) Using Euler's Formula find $\cos(x + y)$.

Hint(s) to 12.11: Expand $(\cos(x) + i \cdot \sin(x)) \cdot (\cos(y) + i \cdot \sin(y))$.

Problem 12.12. (HA) Find the cosine of the angle between vectors $[1, 1]$ and $[-1, 1]$.

Hint(s) to 12.12: Hint using complex numbers:

How is the angle between u and v related to v/u ?

How can one find $\cos(\arg(z))$ without knowing $\arg(z)$?

Hint using dot products: Let α be the angle between u and v . $u \cdot v = |u| \cdot |v| \cdot \cos(\alpha)$.

Answer to 12.12: 0

Problem 12.13. (HA) $u = [1, 1]$ is rotated counterclockwise until it points in the direction of $v = [-1, 1]$. Find the angle of rotation in degrees.

Hint(s) to 12.13: Complex Numbers Hint:
How is the angle related to the argument of v/u ?
How to find $\arg(z)$?

Linear Algebra Hint: Let α be the angle. Then,
 $u \cdot v = |u| \cdot |v| \cdot \cos(\alpha)$ and $\det(u, v) = |u|$
 $\cdot |v| \cdot \sin(\alpha)$. Find $\tan(\alpha)$.

Answer to 12.13: 90

Problem 12.14. (HA) The face of an antique clock shows only numbers 3, 6, 9, and 12. It has only the small hand whose tip is located at $(2, -1)$ in the coordinate system centered in the middle of the face with the x -axis pointing toward 3 and the y -axis pointing toward 12. Find the time being shown by the clock assuming it is before noon.

Hint(s) to 12.14: Let α be the angle from the x -axis to the short hand. Construct a linear function $time(\alpha)$.

Answer to 12.14: 3:53.13 AM

Problem 12.15. (A) Find the symmetric image of $P(-3, -1)$ about the line passing through points $Q(0, 0)$ and $R(-2, 1)$.

Answer to 12.15: $(-1, 3)$

Problem 12.16. (HA) The length of \vec{u} is 1, the length of \vec{v} is 2, and the angle between the two vectors is 30 degrees. Find the length of $\vec{u} + \vec{v}$.

Hint(s) to 12.16: Complex Numbers Hint:
One may assume u is on the x -axis. What is u ?
What is v ?

Linear Algebra Hint: Let α be the angle between u and v . $u \cdot u = |u|^2$ for any vector u . $|u+v|^2 = (u+v) \cdot (u+v) = u \cdot u + 2 \cdot u \cdot v + v \cdot v$.

Answer to 12.16: 2.91

Problem 12.17. (HA) Let $R(1, 1)$. On the segment joining $P(1, 5)$ and $Q(13, 1)$ find the point X such that RX bisects the angle between segments RP and RQ .

Hint(s) to 12.17: Hint: Let $\vec{RQ} = u$, $\vec{RP} = v$, and $\vec{RX} = w$. Because w bisects the angle between u and v , it must be a positive multiple of $u/|u| + v/|v|$ (why?). Because X lies on the segment PQ , $\vec{PX} = s \cdot \vec{PQ} = s(u - v)$. Moreover, $\vec{PX} = w - v$, so $w - v = su - sv$ and $w = su + (1 - s)v$. Can you find $(1 - s)/s$?

Hint using dot product: If RX bisects the angle between segments RP and RQ , then $\cos \alpha = \cos \beta$, where α is the angle between RP and RX and β is the angle between RQ and RX . Thus,

$$\frac{\vec{RP} \cdot \vec{RX}}{RP \cdot RX} = \frac{\vec{RQ} \cdot \vec{RX}}{RQ \cdot RX}.$$

X being on the segment PQ means that $\vec{PX} = k \cdot \vec{PQ}$, where $0 \leq k \leq 1$. Thus, $X = P + k \cdot \vec{PQ}$ and we can solve for k using $\frac{\vec{RP} \cdot \vec{RX}}{RP} = \frac{\vec{RQ} \cdot \vec{RX}}{RQ}$.

Answer to 12.17: $(4, 4)$

Problem 12.18. (A) Observe that $(3 - i \cdot 4) \cdot (\cos(x) + i \cdot \sin(x)) = (3 \cdot \cos(x) + 4 \cdot \sin(x)) + i \cdot (-4 \cdot \cos(x) + 3 \cdot \sin(x))$. Geometrically, it is vector $[3, -4]$ rotated counterclockwise by angle x . Maximize $3 \cdot \cos x + 4 \cdot \sin x$ by noticing that it amounts to rotating of $[3, -4]$ so that it lies on the x -axis.

Answer to 12.18: 5

Problem 12.19. (A) Express $1 \cdot \cos(x) + 1 \cdot \sin(x)$ as $\operatorname{Re}(r \cdot e^{(x+\theta)i})$ with $r > 0$.

Answer to 12.19: $r = 1.414$, $\theta = -.785 + 2\pi n$

Problem 12.20. By considering $(a + b \cdot i) \cdot (\cos(\theta) + i \cdot \sin(\theta))$, show that $b \cdot \cos(\theta) + a \cdot \sin(\theta) = (a^2 + b^2)^{.5} \cdot \sin(\theta + \arctan(b/a))$.

Problem 12.21. (A) Observe that $(-2 + i \cdot 1) \cdot (\cos(x) + i \cdot \sin(x)) = (-2 \cdot \cos(x) - 1 \cdot \sin(x)) + i \cdot (1 \cdot \cos(x) - 2 \cdot \sin(x))$. Geometrically, it is vector $[-2, 1]$ rotated counterclockwise by angle x . Solve $1 \cdot \cos x - 2 \cdot \sin x = 0$ in the interval $[0, 2\pi]$ (x is in radians) by noticing that it amounts to rotating of $[-2, 1]$ so that it lies on the x -axis.

Answer to 12.21: $x = 0.464, 3.605$

Problem 12.22. (A) Observe that $(-1+i\cdot 2)\cdot (\cos(x) + i\cdot \sin(x)) = (-1\cdot \cos(x) - 2\cdot \sin(x)) + i\cdot (2\cdot \cos(x) - 1\cdot \sin(x))$. Geometrically, it is vector $[-1, 2]$ rotated counterclockwise by angle x . Solve $2\cdot \cos x - 1\cdot \sin x = 0$ in degrees by noticing that it amounts to rotating of $[-1, 2]$ so that it lies on the x -axis.

Answer to 12.22: $x = 63.4 + k \cdot 180$

Problem 12.23. (A) Solve $f'(x) = 2 \cdot i \cdot f(x)$,
 $f(0) = 1$.

Answer to 12.23: $1 \cdot \exp(2 \cdot i \cdot x)$

Problem 12.24. (A) Find the function $f(x)$ (x is real) so that the real part of $\exp(f(x))$ equals $\cos(2 \cdot x) \cdot \exp(1 \cdot x)$.

Answer to 12.24: $(1 + 2 \cdot i) \cdot x$

Problem 12.25. (H) State Euler's Formula.

Hint(s) to 12.25: $\exp(i \cdot x) = \cos(x) + i \cdot \sin(x)$.

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