1. Let \( \sigma \) be reflection in the line \( y = x \), and let \( \tau \) be reflection in the line \( x = 1 \). The composite transformation \( \sigma \circ \tau \circ \sigma \) is a reflection; what is its mirror line?

We have:

\[
\sigma : \begin{bmatrix} x \\ y \end{bmatrix} \mapsto \begin{bmatrix} y \\ x \end{bmatrix}, \quad \tau : \begin{bmatrix} x \\ y \end{bmatrix} \mapsto \begin{bmatrix} 2 - x \\ y \end{bmatrix}.
\]

Composing these transformations gives

\[
\begin{bmatrix} x \\ y \end{bmatrix} \overset{\sigma}{\mapsto} \begin{bmatrix} y \\ x \end{bmatrix} \overset{\tau}{\mapsto} \begin{bmatrix} 2 - y \\ x \end{bmatrix} \overset{\sigma}{\mapsto} \begin{bmatrix} x \\ 2 - y \end{bmatrix},
\]

whence \( \sigma \tau \sigma \) is reflection in the line \( y = 1 \).

2. Let \( \sigma \), \( \tau \) be reflections in lines \( \ell_1 \), \( \ell_2 \) respectively. Determine the mirror line of \( \sigma \circ \tau \circ \sigma \).

*Hint:* find the fixed points of this composite transformation.

Let \( x \) be a point on the line \( \sigma(\ell_2) \). Then \( \sigma(x) \) lies on \( \ell_2 \), whence \( \sigma(x) \) is fixed by \( \tau \) and \( \sigma \tau \sigma(x) = \sigma \sigma(x) = x \). Therefore the set of fixed points of \( \sigma \tau \sigma \) contains the line \( \sigma(\ell_2) \). However, an orientation-reversing isometry of the Euclidean plane is either a reflection or a glide-reflection, and since the set of fixed points of \( \sigma \tau \sigma \) is non-empty, \( \sigma \tau \sigma \) must be a reflection, with mirror-line \( \sigma(\ell_2) \).

3. Let \( \sigma \), \( \tau \) be inversions in circles \( C_1 \), \( C_2 \) respectively. The composite \( \sigma \circ \tau \circ \sigma \) is an inversion; identify the circle in which it inverts points.

Let \( x \) be a point on \( \sigma(C_2) \) (\( \sigma(C_2) \) is either a circle or possibly a straight line.) Then \( \sigma(x) \) lies on \( C_2 \), whence \( \sigma(x) \) is fixed by \( \tau \) and \( \sigma \tau \sigma(x) = \sigma \sigma(x) = x \). Therefore the set of fixed points of \( \sigma \tau \sigma \) contains \( \sigma(C_2) \). It is given that \( \sigma \tau \sigma \) is an inversion; its inverting circle is its set of fixed points, namely \( \sigma(C_2) \) (exceptionally \( \sigma(C_2) \) could be a straight line, i.e. a "circle" of infinite radius.)
4. Let \( C_1 \) be the circle of radius 1 centered at the origin, and let \( C_2 \) be the circle of radius 1 centered at the point \((3, 0)\). Let \( \sigma, \tau \) be inversions in the circles \( C_1, C_2 \) respectively. Show that if \( P \) is any point not on the x-axis, then \( P \) is not fixed by \( \tau \circ \sigma \). Find all points fixed by \( \tau \circ \sigma \).

(This neat argument was used in some people’s homework.) Let \( Q_1, Q_2 \) be the centers of \( C_1, C_2 \) respectively, and suppose that \( P \) is fixed by \( \tau \sigma \). First we eliminate some trivial cases. We note that \( \tau \sigma(Q_1) = \tau(\infty) = Q_2 \neq Q_1 \), and that \( \tau \sigma(Q_2) \) cannot equal \( Q_2 \), since \( \sigma(Q_2) \neq \infty \). Therefore \( P \) cannot equal either of \( Q_1, Q_2 \). Furthermore, if \( P \in C_1 \), then \( \tau \sigma(P) = \tau(P) \) cannot equal \( P \) as the circles \( C_1, C_2 \) are disjoint; similarly, if \( P \in C_2 \), then \( \sigma(P) \notin C_2 \), whence \( \tau \sigma(P) \notin C_2 \) and thus \( P \neq \tau \sigma(P) \). Therefore we may also assume that \( P \) does not lie on either circle.

It follows that the points \( Q_1, P, \sigma(P) \) are distinct and collinear, and that the points \( Q_2, P, \tau(P) \) are distinct and collinear. But \( \tau \sigma(P) = P \Rightarrow \sigma(P) = \tau(P) \), so the points \( Q_1, P, \sigma(P), Q_2 \) are collinear. In particular, \( P \) must lie on the line joining \( Q_1, Q_2 \), namely the x-axis.

To locate the fixed points of \( \sigma \tau \), we consider each inversion as acting on the x-axis, and write \( \sigma(x) = \frac{1}{x} \), \( \tau(x) = 3 - \frac{1}{3-x} \). Solving \( \sigma(x) = \tau(x) \) gives us

\[
\frac{1}{x} = 3 - \frac{1}{3-x} \iff x = \frac{3-x}{8-3x} \iff x^2 - 3x + 1 = 0
\]

Solving this quadratic, we find that there are two fixed points on the x-axis, with x-coordinates \( \frac{3 \pm \sqrt{5}}{2} \).

5. Repeat Q4, but with \( C_2 \) the circle of radius 1 centered at the point \((2, 0)\).

The argument showing that all fixed points of \( \tau \sigma \) lie on the x-axis is almost identical, the only difference being that this time the circles \( C_1, C_2 \) meet at the point \((1, 0)\). However, this point does lie on the x-axis, so the conclusion is unaltered.

We locate the fixed point(s) of \( \tau \sigma \) similarly. The inversion \( \sigma \) is as in Question 4, and we have \( \tau(x) = 2 - \frac{1}{2-x} \). The equation to solve is

\[
\frac{1}{x} = 2 - \frac{1}{2-x} \iff x = \frac{2-x}{3-2x} \iff x^2 - 2x + 1 = 0
\]

but this time there is a repeated root \( x = 1 \). We deduce that the only fixed point of \( \tau \sigma \) is \((1, 0)\).