Math 460 Homework 2 Solutions

1. Let σ be reflection in the line $y = x$, and let τ be reflection in the line $x = 1$. The composite transformation $\sigma \circ \tau \circ \sigma$ is a reflection; what is its mirror line?

We have

$$\sigma : \begin{bmatrix} x \\ y \end{bmatrix} \mapsto \begin{bmatrix} y \\ x \end{bmatrix} \quad , \quad \tau : \begin{bmatrix} x \\ y \end{bmatrix} \mapsto \begin{bmatrix} 2-x \\ y \end{bmatrix} .$$

Composing these transformations gives

$$\begin{bmatrix} x \\ y \end{bmatrix} \underset{\sigma}{\mapsto} \begin{bmatrix} y \\ x \end{bmatrix} \underset{\tau}{\mapsto} \begin{bmatrix} 2-y \\ x \end{bmatrix} \underset{\sigma}{\mapsto} \begin{bmatrix} x \\ 2-y \end{bmatrix} ,$$

whence $\sigma\tau\sigma$ is reflection in the line $y = 1$.

2. Let σ, τ be reflections in lines ℓ_1, ℓ_2 respectively. Determine the mirror line of $\sigma \circ \tau \circ \sigma$. *Hint:* find the fixed points of this composite transformation.

Let x be a point on the line $\sigma(\ell_2)$. Then $\sigma(x)$ lies on ℓ_2, whence $\sigma(x)$ is fixed by τ and $\sigma\tau\sigma(x) = \sigma\sigma(x) = x$. Therefore the set of fixed points of $\sigma\tau\sigma$ contains the line $\sigma(\ell_2)$. However, an orientation-reversing isometry of the Euclidean plane is either a reflection or a glide-reflection, and since the set of fixed points of $\sigma\tau\sigma$ is non-empty, $\sigma\tau\sigma$ must be a reflection, with mirror-line $\sigma(\ell_2)$.

3. Let σ, τ be inversions in circles C_1, C_2 respectively. The composite $\sigma \circ \tau \circ \sigma$ is an inversion; identify the circle in which it inverts points.

Let x be a point on $\sigma(C_2)$ ($\sigma(C_2)$ is either a circle or possibly a stright line.) Then $\sigma(x)$ lies on C_2, whence $\sigma(x)$ is fixed by τ and $\sigma\tau\sigma(x) = \sigma\sigma(x) = x$. Therefore the set of fixed points of $\sigma\tau\sigma$ contains $\sigma(C_2)$. It is given that $\sigma\tau\sigma$ is an inversion; its inverting circle is its set of fixed points, namely $\sigma(C_2)$ (exceptionally $\sigma(C_2)$ could be a straight line, *i.e.* a "circle" of infinite radius.)

4. Let C_1 be the circle of radius 1 centered at the origin, and let C_2 be the circle of radius 1 centered at the point $(3, 0)$. Let σ, τ be inversions in the circles C_1, C_2 respectively. Show that if P is any point not on the x-axis, then P is not fixed by $\tau \circ \sigma$. Find all points fixed by $\tau \circ \sigma$.

(This neat argument was used in some people's homework.) Let Q_1, Q_2 be the centers of C_1, C_2 respectively, and suppose that P is fixed by $\tau\sigma$. First we eliminate some trivial cases. We note that $\tau\sigma(Q_1) = \tau(\infty) = Q_2 \neq Q_1$, and that $\tau\sigma(Q_2)$ cannot equal Q_2, since $\sigma(Q_2) \neq \infty$. Therefore P cannot equal either of Q_1, Q_2. Furthermore, if $P \in C_1$, then $\tau\sigma(P) = \tau(P)$ cannot equal P as the circles C_1, C_2 are disjoint; similarly, if $P \in C_2$, then $\sigma(P) \notin C_2$, whence $\tau\sigma(P) \notin C_2$ and thus $P \neq \tau\sigma(P)$. Therefore we may also assume that P does not lie on either circle.

It follows that the points $Q_1, P, \sigma(P)$ are distinct and collinear, and that the points $Q_2, P, \tau(P)$ are distinct and collinear. But $\tau\sigma(P) = P \Longrightarrow \sigma(P) = \tau(P)$, so the points $Q_1, P, \sigma(P), Q_2$ are collinear. In particular, P must lie on the line joining Q_1, Q_2, namely the x-axis.

To locate the fixed points of $\sigma\tau$, we consider each inversion as acting on the x-axis, and write $\sigma(x) = \frac{1}{x}$, $\tau(x) = 3 - \frac{1}{3-x}$. Solving $\sigma(x) = \tau(x)$ gives us

$$\frac{1}{x} = 3 - \frac{1}{3-x} \iff x = \frac{3-x}{8-3x} \iff x^2 - 3x + 1 = 0 \quad .$$

Solving this quadratic, we find that there are two fixed points on the x-axis, with x-coordinates $\dfrac{3 \pm \sqrt{5}}{2}$.

5. Repeat Q4, but with C_2 the circle of radius 1 centered at the point $(2, 0)$.

The argument showing that all fixed points of $\tau\sigma$ lie on the x-axis is almost identical, the only difference being that this time the circles C_1, C_2 meet at the point $(1, 0)$. However, this point does lie on the x-axis, so the conclusion is unaltered.

We locate the fixed point(s) of $\tau\sigma$ similarly. The inversion σ is as in Question 4, and we have $\tau(x) = 2 - \frac{1}{2-x}$. The equation to solve is

$$\frac{1}{x} = 2 - \frac{1}{2-x} \iff x = \frac{2-x}{3-2x} \iff x^2 - 2x + 1 = 0 \quad ,$$

but this time there is a repeated root $x = 1$. We deduce that the only fixed point of $\tau\sigma$ is $(1, 0)$.