Proofs of Divisibility Tests

Here is a basic fact: Suppose you have a positive integer x which, when you write its digits, looks like:

$$a_m \cdots a_4 a_3 a_2 a_1 a_0.$$

So a_0 is the digit in the one's place, a_1 is the digit in the 10's place, a_2 is the digit in the 100's place, etc. Then the number x equals

$$\begin{aligned} x &= a_0 + a_1 \cdot 10 + a_2 \cdot 100 + a_3 \cdot 1000 + a_4 \cdot 10000 + \cdots + a_m \cdot 10^m \\ &= a_0 + a_1 \cdot 10 + a_2 \cdot 10^2 + a_3 \cdot 10^3 + a_4 \cdot 10^4 + \cdots + a_m \cdot 10^m. \end{aligned}$$

Most of the divisibility tests and their proofs come from examining this expression mod n for some n, and replacing the various powers of 10 with their equivalents mod n:

1. Proof of Test for Divisibility by 2. Observe that 10 divided by 2 has a remainder of 0. So $10 \equiv 0 \pmod 2$. Then $10^k \equiv 0^k \equiv 0 \pmod 2$ for $k = 1, 2, 3, \ldots$. Hence

$$\begin{aligned} x &\equiv a_0 + a_1 \cdot 0 + a_2 \cdot 0 + a_3 \cdot 0 + a_4 \cdot 0 + \cdots + a_m \cdot 0 \\ &\equiv a_0 \pmod 2. \end{aligned}$$

   Therefore x is divisible by 2 if and only if its last digit a_0 is divisible by 2, which happens if and only if the last digit is one of $0, 2, 4, 6, 8$.

2. Proof of Test for Divisibility by 4. Observe that 100 divided by 4 has a remainder of 0. So $100 \equiv 0 \pmod 4$. Hence $10^k \equiv 0 \pmod 4$ for $k = 2, 3, 4, \ldots$. Then

$$\begin{aligned} x &\equiv a_0 + a_1 \cdot 10 + a_2 \cdot 0 + a_3 \cdot 0 + a_4 \cdot 0 + \cdots + a_m \cdot 0 \\ &\equiv a_0 + a_1 \cdot 10 \pmod 4. \end{aligned}$$

   Therefore x is divisible by 4 if and only if the number $a_0 + a_1 \cdot 10$ is divisible 4. But $a_0 + a_1 \cdot 10$ is the number formed by keeping only the last two digits of x. So x is divisible by 4 if and only if the number formed by dropping all but the last two digits of x is divisible by 4.

3. Proof of Test for Divisibility by 9. Observe that 10 divided by 9 has a remainder of 1. So $10 \equiv 1 \pmod 9$. Hence $10^k \equiv 1^k \equiv 1 \pmod 9$ for $k = 1, 2, 3, 4, \ldots$. Then

$$\begin{aligned} x &\equiv a_0 + a_1 \cdot 1 + a_2 \cdot 1 + a_3 \cdot 1 + a_4 \cdot 1 + \cdots + a_m \cdot 1 \\ &\equiv a_0 + a_1 + a_2 + a_3 + \cdots + a_m \pmod 9. \end{aligned}$$

   Therefore x is divisible by 9 if and only if the sum of its digits is divisible by 9.

4. Proof of Test for Divisibility by 10. Observe that 10 divided by 10 has a remainder of 0. So $10 \equiv 0 \pmod{10}$. Hence $10^k \equiv 0^k \equiv 0 \pmod{10}$ for $k = 1, 2, 3, \ldots$. Then

$$\begin{aligned} x &\equiv a_0 + a_1 \cdot 0 + a_2 \cdot 0 + a_3 \cdot 0 + a_4 \cdot 0 + \cdots + a_m \cdot 0 \\ &\equiv a_0 \pmod{10}. \end{aligned}$$

Therefore x is divisible by 10 if and only if its last digit a_0 is divisible by 10, which happens if and only if the last digit is 0.

5. Proof of Test for Divisibility by 11. Observe that $10 \equiv -1 \pmod{11}$. Hence $10^k \equiv (-1)^k \pmod{11}$ for $k = 1, 2, 3, 4, \ldots$. Then

$$\begin{aligned} x &\equiv a_0 + a_1 \cdot (-1) + a_2 \cdot (-1)^2 + a_3 \cdot (-1)^3 + a_4 \cdot (-1)^4 + \cdots + a_m \cdot (-1)^m \\ &\equiv a_0 - a_1 + a_2 - a_3 + a_4 + \cdots + a_m(-1)^m \pmod{11}. \end{aligned}$$

Therefore x is divisible by 11 if and only if alternating sum of its digits $a_0 - a_1 + a_2 - a_3 + a_4 + \cdots + a_m(-1)^m$ is divisible by 11.

6. Proof of Test for Divisibility by 12. Since $12 = 2^2 3$ involves more than one prime, let's start a different way. If a number is divisible by 12, then in its prime factorization it must contain $2^2 3$, possibly along with other prime factors (perhaps even some more 2's and 3's). Therefore the number must be divisible by both 3 and 4. Conversely, if a number is divisible by both 3 and 4, then in its prime factorization it must contain $2^2 3$, possibly along with other prime factors (perhaps even some more 2's and 3's). Therefore, a number is divisible by 12 if and only if it is divisible by both 3 and 4, and this is our divisibility test.