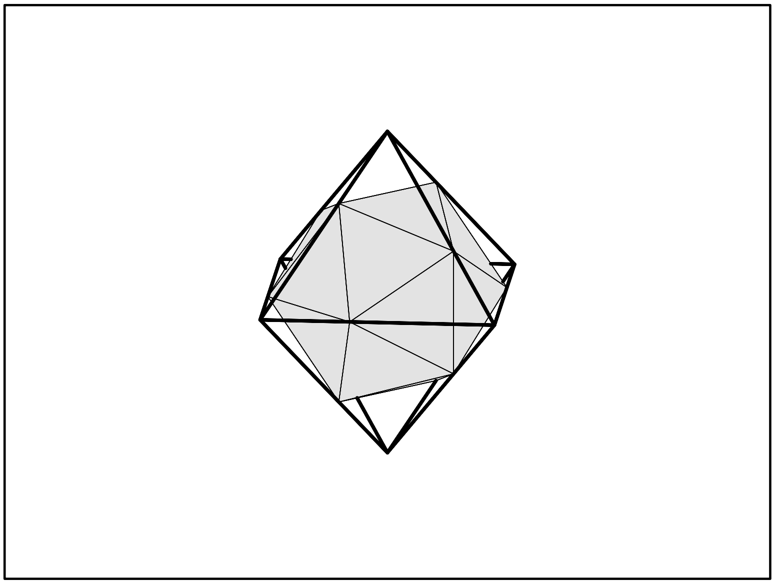
A&S 153
#4 The Icosahedron and Dodecahedron

Here is a way to find coordinates for the vertices of an icosahedron. Look at the diagram below.



Begin with an octahedron. Let's assume the coordinates of its vertices are $(\pm 1, 0, 0)$, $(0, \pm 1, 0)$, $(0, 0, \pm 1)$. On each of its twelve edges we place a new point by dividing the edge into two unequal parts. These new points become the twelve vertices of an icosahedron if they are placed appropriately. The problem is to determine the location of the new points so that all of the resulting triangles pictured are equilateral.

We can, for example, notice that the point P_{61} is closer to P_6 than P_1, but the point P_{36} is closer to P_3 than P_6. Accordingly, we set $P_{61} = P_6 + t(P_1 - P_6) = (0, 0, -1) + t(1, 0, 1) = (t, 0, t-1)$ and $P_{36} = P_3 + t(P_6 - P_3) = (0, 1, 0) + t(0, -1, -1) = (0, 1-t, -t)$, and create similar formulas for the ten other new points. Then by setting certain edge lengths of the icosahedron equal to each other, we can solve for t.

Fortunately, once we have found the coordinates for the icosahedron, those for the dodecahedron are easier: we can choose the centers of each of the triangles of the icosahedron to be the vertices of the dodecahedron.

Let's break down the details a bit.

1. First, record the coordinates of the six vertices of the octahedron.

$$\begin{array}{ll} P_1 & (1,0,0) \\ P_2 & (-1,0,0) \\ P_3 & (0,1,0) \\ P_4 & (0,-1,0) \\ P_5 & (0,0,1) \\ P_6 & (0,0,-1) \end{array}$$

2. Now list the vertices of the icosahedron. There will be one for each edge of the octahedron.

$$P_{13} \quad P_1 + t[P_3 - P_1] = (1,0,0) + t[(0,1,0) - (1,0,0)] \quad = \quad (1-t, t, 0)$$

$$P_{14} \quad P_1 + t[P_4 - P_1] = (1,0,0) + t[(0,-1,0) - (1,0,0)] \quad = \quad (1-t, -t, 0)$$

P_{23}

P_{24}

P_{35}

P_{36}

P_{45}

P_{46}

P_{51}

P_{52}

P_{61}

P_{62}

3. Calculate the lengths of the following segments:

   $P_{35}P_{36}$:

   $P_{35}P_{13}$:

4. Set these two lengths equal to each other and solve for t. Simplify the expression for t so that there are no square roots in the denominator.

5. You now know the coordinates for the vertices of the icosahedron. In order to display it, for example, with Maple, you need to know the twenty polygons (which are all triangles). List them below.

$$P_{13}P_{35}P_{51}$$

$$P_{13}P_{35}P_{36}$$

6. To get the vertices of the dodecahedron, for each of the twenty triangles above you need to know its center. You can get this by adding the coordinates of its three corners, and dividing by 3. You might as well continue using the symbol t even though by now we know its value.

Q_1 $P_{13}P_{35}P_{51}$ $\frac{1}{3}[(1-t,t,0)+(0,1-t,t)+(t,0,1-t)] = (\frac{1}{3},\frac{1}{3},\frac{1}{3})$

Q_2 $P_{13}P_{35}P_{36}$ $\frac{1}{3}[(1-t,t,0)+(0,1-t,t)+(0,1-t,-t)] = (\frac{1-t}{3},\frac{2-t}{3},0)$

Q_3

Q_4

Q_5

Q_6

Q_7

Q_8

Q_9

Q_{10}

Q_{11}

Q_{12}

Q_{13}

Q_{14}

Q_{15}

Q_{16}

Q_{17}

Q_{18}

Q_{19}

Q_{20}

7. Using the above list, find eight vertices of the dodecahedron that form the vertices of an inscribed cube—a cube that sits inside the dodecahedron.

8. Finally, list the twelve pentagons for the dodecahedron. Now you have enough information to display this Platonic solid as well.