Families of Functions

1 Essential Understandings

Excerpts from Developing an Essential Understanding of Functions by Cooney, Beckman, Lloyd, Wilson, and Zbiek

Big Idea 1. The Function Concept

- **Essential Understanding 1a.** Functions are single-valued mappings from one set the domain of the function—to another—its range.
- **Essential Understanding 1b.** Functions apply to a wide range of situations. They do not have to be described by any specific expressions or follow a regular pattern. They apply to cases other than those of "continuous variation." For example, sequences are functions.
- **Essential Understanding 1c.** The domain and range of functions do not have to be numbers. For example, 2-by-2 matrices can be viewed as representing functions whose domain and range are a two-dimensional vector space.

Big Idea 2. Covariation and Rate of Change

- **Essential Understanding 2a.** For functions that map real numbers to real numbers, certain patterns of covariation, or patterns in how two variables change together, indicate membership in a particular family of functions and determine the type of formula that the function has.
- **Essential Understanding 2b.** A rate of change describes how one variable quantity changes with respect to another—in other words, a rate of change describes the covariation between two variables.
- Essential Understanding 2c. A function's rate of change is one of the main characteristics that determine what kinds of real-world phenomena the function can model.

Big Idea 3. Families of Functions

- **Essential Understanding 3a.** Members of a family of functions share the same type of rate of change. This characteristic rate of change determines the kinds of real-world phenomena that the functions in the family can model.
- **Essential Understanding 3b.** Linear functions are characterized by a constant rate of change. Reasoning about the similarity of "slope triangles" allows deducing that linear functions have a constant rate of change and a formula of the type f(x) = mx + b for constants m and b.
- **Essential Understanding 3c.** Quadratic functions are characterized by a linear rate of change, so the rate of change of the rate of change (the second derivative) of a quadratic function is constant. Reasoning about the vertex form of a quadratic allows deducing that the quadratic has a maximum or minimum value and that if the zeros of the quadratic are real, they are symmetric about the *x*-coordinate of the maximum or minimum point.
- Essential Understanding 3d. Exponential functions are characterized by a rate of change that is proportional to the value of the function. It is a property of exponential functions that whenever the input is increased by 1 unit, the output is multiplied by a constant factor. Exponential functions connect multiplication to addition through the equation $a^{b+c} = (a^b)(a^c)$.
- Essential Understanding 3e. Trigonometric functions are natural and fundamental examples of periodic functions. For angles between 0 and 90 degrees, the trigonometric functions can be defined as the ratios of side lengths in right triangles; these functions are well defined because the ratios of side lengths are equivalent in similar triangles. For general angles, the sine and cosine functions can be viewed as the y- and x-coordinates of points on circles or as the projection of circular motion onto the y- and x-axes.
- **Essential Understanding 3f.** Arithmetic sequences can be thought of as linear functions whose domains are the positive integers.
- **Essential Understanding 3g.** Geometric sequences can be thought of as exponential functions whose domains are the positive integers.

Big Idea 4. Combining and Transforming Functions

- Essential Understanding 4a. Functions that have the same domain and that map to the real numbers can be added, subtracted, multiplied, or divided (which may change the domain).
- **Essential Understanding 4b.** Under appropriate conditions, functions can be composed.
- **Essential Understanding 4c.** For functions that map the real numbers to the real numbers, composing a function with "shifting" or "scaling" functions changes the formula and graph of the function in readily predictable ways.
- **Essential Understanding 4d.** Under appropriate conditions, functions have inverses. The logarithmic functions are the inverses of the exponential functions. The square root function is the inverse of the squaring function.

Big Idea 5. Multiple Representations of Functions

- Essential Understanding 5a. Functions can be represented in various ways, including through algebraic means (e.g., equations), graphs, word descriptions, and tables.
- **Essential Understanding 5b.** Changing the way that a function is represented (e.g., algebraically, with a graph, in words, or with a table) does not change the function, although different representations highlight different characteristics, and some may show only part of the function.
- **Essential Understanding 5c.** Some representations of a function may be more useful than others, depending on the context.
- **Essential Understanding 5d.** Links between algebraic and graphical representations of functions are especially important in studying relationships and change.

2 Examples of Lessons for Families of Functions

Linear Functions. CMP, Moving Straight Ahead, Investigation 2. First solve Problem 2.1.

- **Exponential Functions.** *CMP*, *Growing*, *Growing*, *Growing*, Investigation 4. First solve Problem 4.2.
- **Quadratic Functions.** CMP, Frogs, Fleas, and Painted Cubes, Investigation 4. First solve Problem 4.2.

Some Questions.

- 1. For each lesson, solve the particular selected problem above. Then examine the entire lesson, which includes a set of Investigations, a set of ACE problems (applications, connections, extensions), and a final Mathematical Reflection, and discuss the following questions.
- 2. Describe which Essential Understandings are being addressed, and how the lesson does this.
- 3. Which of the CCSSM Standards for Mathematical Practice are being addressed by the lesson?
- 4. How are students' understandings of CCSSM Standards for Mathematical Practice being assessed?
- 5. How does the lesson make connections to real-world problems?
- 6. For each of the above questions, what changes in the lesson might you recommend, and why?

3 GeoGebra

- 1. Download latest version from the website http://www.geogebra.org.
- 2. From this website, select "Help" for the online GeoGebra Manual and Tutorials. Or, instead, select "Materials" from the home page for lots of submitted material and examples.
- 3. Different representations of functions: numerical (spreadsheet window), algebraic (algebra window), graphical (graphing window).

When launching GeoGebra, choose the "Algebra and Graphics" perspective. Then go to View and make sure the following are selected: Algebra, Graphics, Spreadsheet, Input Bar.

Type a function into the Input Bar (at the bottom of the screen), such as f(x)=(x-1)*(x-2). This equation will appear in the Algebra window, and the graph will appear in the Graphics window. In cell A1 of the Spreadsheet enter 1, and in cell A2 enter 2. Highlight both cells, grab the lower right-hand corner of the highlighted cells, and drag down to get a sequence of numbers from, say, 1 to 10.

Enter f(A1) in cell B1. Highlight this, grab the lower right-hand corner of the highlighted cell, and drag down to get a table of values for the function. Now try changing the entries in cells A1 and A2 to see what happens. Change the definition of f(x) in the Algebra window.

Try using the spread sheet to calculate finite differences (for, say, linear, quadratic, and polynomial functions), and finite quotients for exponential functions.

Try using GeoGebra to answer some of the questions the sample lessons for linear, exponential, and quadratic functions.

4. Families of functions.

When launching GeoGebra, choose the "Algebra and Graphics" perspective. Then go to View and make sure the following are selected: Algebra, Graphics, Input Bar.

Click on the Graphics window. Select the Slider Bar icon (second icon from the right at the top—you may have to click the lower right-hand corner to see the various options). Click somewhere out of the way in the Graphics window. Enter the name m and Apply. Then click nearby again in the Graphics window. Enter the name b and Apply. You have now created two sliders, for two different parameters. Type f(x)=m*x+b in the Input Bar at the bottom of the screen. You should see the graph of a straight line.

Select the Move icon (first one on the left at the top). Now change the values in the sliders to see how the function and its graph change. In this way you can generate a family of linear functions.

You can make more sliders for more or richer functions. Try making families of exponential functions with two parameters, or families of quadratic functions with three parameters. For the quadratic functions, try both the form $f(x)=a*x^2+b*x+c$ and the form $f(x)=a*(x-h)^2+k$. What are the advantages of each of these forms?

5. Fitting data with functions.

You can use your parameterized families of functions from above to try to match data. Go to View and select "Spreadsheet". In cells C1 through C4 type (1,1), (2,2), (3,4), and (4,7), respectively. These points should then appear in the Graphics window. Use your sliders to find a function that best matches these points.

GeoGebra has some built-in commands to fit data. With the data you entered above, enter the following commands, one by one, into the Input Bar: FitLine[C1:C4], FitExp[C1:C4], and FitPoly[C1:C4,2]. (The final "2" indicates the degree of the fitting polynomial.)

- 6. Transforming functions—the Key Idea for transforming sets of points in the plane described by an equation:
 - Express the starting figure by an equation in the variables x and y.
 - Express the transformation with equations, describing how to get the new coordinates \overline{x} and \overline{y} for each point in terms of the old coordinates x and y.
 - Solve for x and y in terms of \overline{x} and \overline{y} and substitute into the equation for the figure.
 - The result is the equation for the transformed figure.

Example: Translating a parabola.

If the parabola is defined by the equation $y = x^2$, and the translation is defined by the translation vector (-2, +1); i.e., by the equations:

$$\overline{x} = x - 2,$$

$$\overline{y} = y + 1,$$

give the equation for the transformed set (in terms of \overline{x} and \overline{y}). Solution: Solve for x and y in terms of \overline{x} and \overline{y} :

$$\begin{aligned} x &= \overline{x} + 2\\ y &= \overline{y} - 1 \end{aligned}$$

Substitute into the equation $y = x^2$:

$$(\overline{y}-1) = (\overline{x}+2)^2.$$

Now try these:

- (a) What is the resulting equation when a unit circle about the origin is translated by the vector (3, -4)?
- (b) In general, if a set described by an equation in x and y is translated by the vector (h, k), how do you get the equation of the translated set?
- (c) If a parabola defined by the equation $y = x^2$, and a dilation is defined by

$$\overline{x} = 2x, \\ \overline{y} = 2y,$$

give the equation for the dilated parabola.

(d) If a parabola is defined by the equation $y = x^2$, and a scaling in two different directions is defined by

$$\overline{x} = 3x, \\ \overline{y} = 2y,$$

give the equation for the transformed parabola.

- (e) In general, if a set described by an equation in x and y is scaled with respect to the origin by a factor a parallel to the x-axis and by a factor of b parallel to the y-axis, how do you get the equation of the transformed set?
- (f) What is the resulting equation when a unit circle about the origin is scaled by a factor of $\frac{1}{2}$ parallel to the *x*-axis and by a factor of 3 parallel to the *y*-axis?
- (g) If a function is defined by the equation y = f(x), and the graph of the function is first scaled in the x-direction by a scale factor of a and in the y-direction by a scale factor of b, and after that is translated by the vector (h, k), what is the equation of the resulting function? Solution:

Use the transformations

 $\overline{x} = ax + h$ $\overline{y} = by + k$ $x = \frac{1}{a}(\overline{x} - h)$ $y = \frac{1}{b}(\overline{y} - k)$

Solve for \overline{x} and \overline{y} :

Substitute into the equation for the function:

$$\begin{aligned} \frac{1}{b}(\overline{y}-k) &= f(\frac{1}{a}(x-h)) \\ \text{or} \\ \overline{y} &= bf(\frac{1}{a}(x-h)) + k. \end{aligned}$$

- (h) Try this out with four sliders and some general function. For example, make sliders for a, b, h, k and enter the function f(x)=sin(x) into the Input Bar. The enter the function g(x)=(1/b)*f(a(x-h))+k into the Input Bar. Change the slider values. What happens when they become zero or negative?
- (i) You can use the above Key Idea with other transformations as well, such as reflections and rotations. Start by trying reflections across the x-axis or y-axis, or rotations by 180 degrees about the origin. Then try reflections across the line y = x (which relates to finding the inverse of a function, or rotations by 90 degrees. Much more can be done. In my high school class we used rotations and translations applied to arbitrary quadratic expressions in x and y in order to classify and sketch the conic section that it described.
- 7. Modeling real-life situations matching a picture.

You can use the above to model physical images. For example, find an image of a fountain and save it on your desk top. One possibility is http://farm4.static.flickr.com/3258/2847369024_be4a45303a.jpg. In GeoGebra, click on the lower right-hand corner of the third icon from the right, and choose "Insert Image". Click near the origin and select the image file. Now try to scale and translate a parabola to match one of the fountain arcs.

You can look for other images to model—what would be good choices for linear functions or exponential functions?

Or try some of these:

Bouncing ball: http://people.rit.edu/andpph/photofile-sci/bouncing-ball-7810.jpg. St. Louis Arch:

http://3547.voxcdn.com/photos/0/34/46457_l.jpg. (Model with $y = \frac{a}{2}(e^{x/a} + e^{-x/a})$, which is a catenary.)

Vibrating string (but rotate the image 90 degrees before inserting into GeoGebra: http://people.rit.edu/andpph/text-figures/strings/vibration-setup-8181a.jpg.

Nautilus:

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http://24.media.tumblr.com/tumblr_lv150ex4xN1r58141o3_1280.jpg.
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Try to match an image of the nautilus with a logarithmic spiral. For the logarithmic spiral, given by $r = ae^{b\theta}$, make sliders for a and b and use the command Curve[a*e^(b*t)*cos(t),a*e^(b*t)*sin(t),t,0,25]. Why does this create a logarithmic spiral? Now try matching a spiral on a sunflower: http://farm2.static.flickr.com/1278/694780262_8874b4f225.jpg.

8. Modeling real-life situations — motion. Sliders can be animated too. Example: Make a slider for a parameter a. Enter (a,abs(a)) in the Input Bar. Control-click or Rightclick on the slider bar and select "Animation On". You can now turn the animation on and off by clicking on the small icon in the lower left-hand corner of the Graphics window.

Of course, much more complicated animations are possible, such as planets in a solar system!