Sweeping the **cd**-Index and the Toric h-Vector

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1 Introduction

By sweeping a hyperplane across a simple convex d-polytope P, the h-vector of the dual simplicial d-polytope P^*, $h(P^*) = (h_0, \ldots, h_d)$, can be computed—the edges in P are oriented in the direction of the sweep and h_i equals the number of vertices of indegree i. Moreover, the nonempty faces of P can be partitioned to reflect explicitly the formula converting the h-vector into the f-vector of P^*. For a general convex polytope, in place of the h-vector, one often considers the flag f-vector and flag h-vector as well their encoding into the **cd**-index, and also the toric h-vector (which does not contain the full information of the flag h-vector). In this paper we derive formulas for the **cd**-index of a polytope P and its dual P^*, and for the toric h-vector of P^*, from a sweeping of P (Theorems 1, 2, 3, 4 and 6), by interpreting the corresponding s-shelling [13] of P^*. Moreover, we describe a partition of the faces of the complete truncation of P to reflect explicitly the nonnegativity of its **cd**-index and what its components are counting (Section 3.7). One corollary is a quick way to compute the toric h-vector directly from the **cd**-index (Theorem 5) that turns out to be a reformulation of the formula in [2]. We also propose an "extended toric" h-vector that fully captures the information in the flag h-vector (Section 4.4).

Refer to [4, 6, 7, 8, 10, 11, 14], for example, for background information on polytopes and their face numbers.

2 The h-Vector

We begin by reviewing some facts about the h-vector of a simplicial polytope. For a convex d-dimensional polytope (d-polytope) P let $f_i = f_i(P)$ denote the number of i-faces

(i-dimensional faces) of P, $i = -1, \ldots, d$. (Note that $f_{-1} = 1$, counting the empty set, and $f_d = 1$, counting P itself.) The vector $f(P) = (f_0, \ldots, f_{d-1})$ is the *f-vector* of P. Faces of dimension 0, 1, and $d-1$ are called, respectively, *vertices*, *edges*, and *facets* of P. The set of vertices of P will be denoted $\mathrm{vert}(P)$. A d-polytope is *simplicial* if every face is a simplex. A d-polytope is *simple* if every vertex is contained in exactly d edges. A dual to a simplicial polytope is simple, and vice versa.

Let $P \subset \mathbf{R}^d$ be a simple d-polytope. Choose a vector $p \in \mathbf{R}^d$ such that the inner product $p \cdot v$ is different for each vertex v of P. Sweep the hyperplane $H = \{x \in \mathbf{R}^d : p \cdot x = q\}$ across P by letting the parameter q range from $-\infty$ to ∞. (Recall that if P contains the origin in its interior, then ordering the vertices of P using a sweeping hyperplane corresponds to ordering the facets of the polar dual P^* using a line shelling induced by a line through the origin.) Orient each edge of P in the direction of increasing value of $p \cdot x$. For each vertex v of P define (with slight abuse of notation) $h_v(P^*) = x^{d-i}$ and $f_v(P^*) = (x+1)^{d-i}$, where i is the indegree of v. Then the h-vector $h(P^*) = (h_0(P^*), \ldots, h_d(P^*))$ and the f-vector $f(P^*)$ of the dual P^* of P are given by:

$$\sum_v h_v(P^*) = h(P^*, x) = \sum_{i=0}^{d} h_i(P^*) x^{d-i}$$

and

$$\sum_v f_v(P^*) = f(P^*, x) = \sum_{i=0}^{d} f_{i-1}(P^*) x^{d-i}.$$

Moreover, each face of P will have a unique minimal vertex with respect to this orientation. If we associate with vertex v of indegree i the set of faces having v as the minimal vertex, then there are $\binom{d-i}{j}$ such faces of dimension j, $j = 0, \ldots, d-i$. In this way we can partition the nonempty faces of P (including P itself) to combinatorially represent the formula $f(P^*, x) = h(P^*, x+1)$ and visualize what the components of the h-vector are counting, recalling that $f_{i-1}(P^*) = f_{d-i}(P)$, $i = -1, \ldots, d$.

Because the h-vector is independent of the choice of sweeping hyperplane, reversing a sweep shows that the h-vector is symmetric, $h_i(P^*) = h_{d-i}(P^*)$, $i = 0, \ldots, d$, a representation of the *Dehn-Sommerville equations*. In fact, the affine span of the set $\{h(P) : P$ is a simplicial d-polytope$\}$ has dimension $\lfloor d/2 \rfloor$, and the g-Theorem [5, 12] completely characterizes this set.

3 The cd-Index

Unfortunately, the situation is not yet so tidy for general convex d-polytopes, not even in four dimensions. Two objects of study that each, in its own way, generalizes the simplicial h-vector, are the **cd**-index and the toric h-vector. Stanley [13] introduced the notion of s-shellings to demonstrate the nonnegativity of the **cd**-index. We will consider a sweeping of a polytope P and examine the calculations associated with the s-shelling of its dual P^* to visualize the nonnegativity of the components of the **cd**-index, and what is being counted. We will display a partition of the faces of the complete truncation of P to give a combinatorial interpretation of the dependence of the flag f-vector upon the **cd**-index.

3.1 Definitions

Let P be a convex d-polytope. Using the notation $[d-1] = \{0, \ldots, d-1\}$, for every subset $S = \{s_1, \ldots, s_k\} \subseteq [d-1]$ where $s_1 < \cdots < s_k$, define an *S-chain* to be a chain of faces of P of the form $F_1 \subset \cdots \subset F_k$ where F_i is face of P of dimension s_i, $i = 1, \ldots, k$. Let $f_S(P)$ be the number of S-chains. The vector $\overline{f}(P) = (f_S(P))_{S\subseteq[d-1]}$ is the *flag f-vector* of P.

Now define

$$h_S = h_S(P) = \sum_{T\subseteq S} (-1)^{|S|-|T|} f_T(P),\ S \subseteq [d-1]. \tag{1}$$

The vector $\overline{h}(P) = (h_S(P))_{S\subseteq[d-1]}$ is the *flag h-vector* or *extended h-vector* of P, introduced by Bayer and Billera [1]. Like the ordinary h-vector it is symmetric: $h_S(P) = h_{\overline{S}}(P)$, $S \subseteq [d-1]$, where $\overline{S}$ is the complement of S with respect to $[d-1]$. These are known as the *generalized Dehn-Sommerville equations.*

Bayer and Billera [1] showed that the affine span of the set $\{\overline{h}(P) : h \text{ is a convex } d\text{-polytope}\}$ has dimension $F_d - 1$, where F_d is the dth Fibonacci number. Bayer and Klapper [3] proved that the flag h-vector can be encoded into the **cd**-index, which precisely reflects this dimension. Associate with each subset $S \subseteq [d-1]$ the word $w_S = w_0 \cdots w_{d-1}$ in the noncommuting indeterminates $\mathbf{a}$ and $\mathbf{b}$, where $w_i = \mathbf{a}$ if $i \notin S$ and $w_i = \mathbf{b}$ if $i \in S$. The **ab**-*index* of P is then the polynomial

$$\Psi(P) = \Psi(P, \mathbf{a}, \mathbf{b}) = \sum_{S\subseteq[d-1]} h_S(P) w_S.$$

The existence of the **cd**-index asserts that there is a polynomial in the noncommuting indeterminates $\mathbf{c}$ and $\mathbf{d}$, $\Phi(P) = \Phi(P, \mathbf{c}, \mathbf{d})$, such that setting $\mathbf{c} = \mathbf{a} + \mathbf{b}$ and $\mathbf{d} = \mathbf{ab} + \mathbf{ba}$ we have $\Phi(P, \mathbf{c}, \mathbf{d}) = \Phi(P, \mathbf{a} + \mathbf{b}, \mathbf{ab} + \mathbf{ba}) = \Psi(P, \mathbf{a}, \mathbf{b})$. Note that $\mathbf{c}$ has degree one, $\mathbf{d}$ has degree two, and $\Phi(P)$ has degree d. There are F_d **cd**-words of degree d and one of them, $\mathbf{c}^d$, will always have coefficient 1. Therefore the remaining $F_d - 1$ terms of the **cd**-index capture the dimension of the affine span of the flag f-vectors of d-polytopes.

3.2 Example: The Octahedron

Omitting brackets for subsets of $\{0, 1, 2\}$, for the octahedron we have:

S	f_S	h_S	w_S
$\emptyset$	1	1	**aaa**
0	6	5	**baa**
1	12	11	**aba**
2	8	7	**aab**
01	24	7	**bba**
02	24	11	**bab**
12	24	5	**abb**
012	48	1	**bbb**

Using the above ordering of subsets,

$\overline{f}(P) = (1, 6, 12, 8, 24, 24, 24, 48)$,
$\overline{h}(P) = (1, 5, 11, 7, 7, 11, 5, 1)$,
$\Psi(P) = \mathbf{aaa} + 5\mathbf{baa} + 11\mathbf{aba} + 7\mathbf{aab} + 7\mathbf{bba} + 11\mathbf{bab} + 5\mathbf{abb} + \mathbf{bbb}$, and
$\Phi(P) = \mathbf{c}^3 + 4\mathbf{dc} + 6\mathbf{cd}$.

3.3 Computing the cd-Index with s-Shellings

Stanley [13] proved that the coefficients of the **cd**-index are nonnegative by introducing the notion of *s-shellings*. Consider any convex d-polytope P and a shelling of the boundary ∂P^* of the dual of P. As the facets of P^* are sequentially added we obtain a sequence of $(d-1)$-dimensional complexes $P_1, \ldots, P_n$, where P_1 is just a single facet of P^* and P_n is the full boundary complex ∂P^*. For each of the complexes P_m (except the last) a single $(d-1)$-dimensional cap is added, incident to each of the faces in the $(d-2)$-dimensional boundary complex of P_m, to obtain the spherical complex P'_m. (The complex P'_n is defined simply to be P_n, which is also combinatorially equivalent to P'_{n-1}.) Stanley showed that the sequence of the **cd**-indices of the complexes $P'_1, \ldots, P'_n$ is nondecreasing. Let's see how this works.

Consider adding facet Q to P_m to obtain P_{m+1} for some $m = 0, \ldots, n-1$ (taking P_0 to be the empty complex). What is the contribution of Q to the change in the **cd**-index from P'_m to P'_{m+1}? Define this contribution of Q during the shelling of ∂P^* to be $\Phi_Q(P^*) =$

$\Phi(P'_{m+1}) - \Phi(P'_m)$. (Of course, if $m = n - 1$ then $\Phi_Q(P^*) = 0$ since P'_{n-1} and P'_n are combinatorially equivalent.) We compute these changes by considering three cases:

Case 0: Counting chains in P'_1. Here only the first facet Q in the shelling of ∂P^* is involved. For every S-chain in Q we obtain three chains in P'_1:

1. Include the S-chain itself.
2. Append Q to get an $S \cup \{d-1\}$-chain.
3. Append the cap of P'_1 to get an $S \cup \{d-1\}$-chain.

Thus the S-chain contributes 1 to $\overline{f}_S$ and 2 to $\overline{f}_{S\cup\{d-1\}}$. Equation (1) then implies that the contributions to $\overline{h}_S$ and $\overline{h}_{S\cup\{d-1\}}$ are each 1. Since such a contribution occurs for each chain of Q we see that $\Psi(P'_1)$ equals $\Psi(Q)(\mathbf{a}+\mathbf{b})$. Thus $\Phi_Q(P^*) = \Phi(Q)\mathbf{c}$.

Now assume $m > 0$ and let $F_1, \ldots, F_k, \ldots, F_\ell$ be a shelling of ∂Q, where $F_1, \ldots, F_k$ is the initial shelling of ∂Q that is contained in P_m. Define the following objects:

- Z_i is the $(d-2)$-dimensional complex that is (naturally associated with) the union of $F_1, \ldots, F_i$, $i = k, \ldots, \ell$.
- $\hat{F}_i$ is the $(d-2)$-dimensional cap added to Z_i to obtain the $(d-2)$-dimensional spherical complex Z'_i, $i = k, \ldots, \ell - 1$.
- $\overline{Z}_i$ is the $(d-1)$-dimensional complex obtained by adding a single $(d-1)$-dimensional cell T_i to Z'_i, incident to each of the faces of Z'_i, $i = k, \ldots, \ell$.
- $\hat{Z}_i$ is the $(d-1)$-dimensional cap added to the complex $P_m \cup \overline{Z}_i$ to obtain the $(d-1)$-dimensional spherical complex $(P_m \cup \overline{Z}_i)'$, $i = k, \ldots, \ell$.
- R is the $(d-3)$-dimensional complex that is the boundary of the $(d-2)$-dimensional complex $F_1 \cup \cdots \cup F_k$.

We will add the facet Q to P'_m by moving through the sequence $P'_m, (P_m \cup \overline{Z}_k)', \ldots, (P_m \cup \overline{Z}_\ell)'$.

Case 1: Moving from P'_m to $(P_m \cup \overline{Z}_k)'$. Let $G_1 \subset \cdots \subset G_j$ be an S-chain in P'_m.

1. If G_j is not the cap of P'_m then this chain remains unaffected.
2. If G_j is the cap of P'_m but G_{j-1} is not a face in the complex $(F_1 \cup \cdots \cup F_k) \setminus R$ then this chain is replaced by the S-chain $G_1 \subset \cdots \subset G_{j-1} \subset \hat{Z}_k$.
3. If G_j is the cap of P'_m and G_{j-1} is a face in the complex $(F_1 \cup \cdots \cup F_j) \setminus R$ then this chain is replaced by the S-chain $G_1 \subset \cdots \subset G_{j-1} \subset T_k$.

4. If G_j is in R then four new chains are created:
   (a) Append T_k to get an $(S \cup \{d-1\})$-chain.
   (b) Append $\hat{F}_k$ and T_k to get an $(S \cup \{d-2, d-1\})$-chain.
   (c) Append $\hat{F}_k$ to get an $(S \cup \{d-2\})$-chain.
   (d) Append $\hat{F}_k$ and $\hat{Z}_k$ to get an $(S \cup \{d-2, d-1\})$-chain.

   Thus $\overline{f}_{S\cup\{d-2\}}$ increases by 1, $\overline{f}_{S\cup\{d-1\}}$ increases by 1, and $\overline{f}_{S\cup\{d-2,d-1\}}$ increases by 2. Equation (1) then implies that $\overline{h}_{S\cup\{d-2\}}$ and $\overline{h}_{S\cup\{d-1\}}$ each increases by 1. Since such a change occurs for each chain in R, we see that Ψ changes by adding $\Psi(R)(\mathbf{ab}+\mathbf{ba})$. Thus the **cd**-index increases by $\Phi(R)\mathbf{d}$.

Case 2: Moving from $(P_m \cup \overline{Z}_{i-1})'$ to $(P_m \cup \overline{Z}_i)'$, $i = k+1, \ldots, \ell$. Let $G_1 \subset \cdots \subset G_j$ be an S-chain in $(P_m \cup \overline{Z}_{i-1})'$.

1. If the S-chain includes the cell $\hat{F}_{i-1}$, then replace that cell with either $\hat{F}_i$ or with F_i, analogous to Case 1(2) and (3) above. Also, if the S-chain includes the cells T_{i-1} or $\hat{Z}_{i-1}$, replace these cells, respectively, with T_i or $\hat{Z}_i$. Otherwise the S-chain remains unaffected.

2. For every new S-chain created in moving from Z'_{i-1} to Z'_i three new chains are created in $(P_m \cup \overline{Z}_i)'$:
   (a) Add the new S-chain itself.
   (b) Append T_i to get an $(S \cup \{d-1\})$-chain.
   (c) Append $\hat{Z}_i$ to get an $(S \cup \{d-1\})$-chain.

   Hence $\overline{f}_S$ increases by 1 and $\overline{f}_{S\cup\{d-1\}}$ increases by 2. Equation (1) then implies that $\overline{h}_S$ and $\overline{h}_{S\cup\{d-1\}}$ each increases by 1. Since such a change occurs for each chain contributing to the increase in $\overline{f}(Q)$, we see that Ψ changes by adding $\Psi_{F_i}(Q)(\mathbf{a}+\mathbf{b})$. Thus the **cd**-index increases by $\Phi_{F_i}(Q)\mathbf{c}$.

Putting this all together, and including the low-dimensional base cases, we have the following result:

Proposition 1 *Let P^* be a convex d-polytope.*

1. If $d = 0$ then P^* has one facet, $Q = \emptyset$, and $\Phi_Q(P^*) = \Phi(P^*) = 1$.

2. *If $d = 1$ then P^* has two facets Q_1 and Q_2. With this shelling order, $\Phi_{Q_1}(P^*) = \mathbf{c}$, $\Phi_{Q_2}(P^*) = 0$, and $\Phi(P^*) = \mathbf{c}$.*

3. *If $d > 1$ then in the notation above,*

$$\Phi_Q(P^*) = \Phi(R)\mathbf{d} + \sum_{i=k+1}^{\ell} \Phi_{F_i}(Q)\mathbf{c} \tag{2}$$

and

$$\Phi(P^*) = \sum_{Q} \Phi_Q(P^*). \tag{3}$$

Corollary 1 (Stanley) *For a convex d-polytope P^* the coefficients of $\Phi(P^*)$ are nonnegative.*

3.4 Sweeping the cd-Index

It is straightforward to dualize the above formulas. Assume that P contains the origin in its interior, P^* is its polar dual, we have a sweeping of P by the hyperplane family $H = \{x \in \mathbf{R}^d : p \cdot x = q\}$, ordering the vertices $v_1, \ldots, v_n$ in order of increasing q, and the shelling of the polar dual P^* is the line shelling dual to the hyperplane sweep. For any particular value of q define $H^+ = \{x \in \mathbf{R}^d : p \cdot x > q\}$ and $H^- = \{x \in \mathbf{R}^d : p \cdot x < q\}$. Refer to the notation of the previous section.

Each vertex v of P will contribute an amount $\Phi_v(P^*)$ to $\Phi(P^*)$. Truncate v to obtain a $(d-1)$-polytope Q_v, making the truncation in general enough direction so that the inner product $p \cdot w$ is different for each vertex w of Q_v. This polytope will be dual to Q.

Let H_v be the hyperplane in the family that contains v. Let R_v be the $(d-2)$-polytope $Q_v \cap H_v$. This polytope will be dual to R. (Note that R_v is empty for the first and last vertices.)

Theorem 1 *For any convex d-polytope P,*

1. *If d=0 then P has one vertex v and $\Phi_v(P^*) = \Phi(P^*) = 1$.*

2. *If $d > 0$ then*

$$\Phi_v(P^*) = \Phi((R_v)^*)\mathbf{d} + \sum_{w \in \mathrm{vert}(Q_v) \cap H_v^+} \Phi_w((Q_v)^*)\mathbf{c}, \; v \in \mathrm{vert}(P), \tag{4}$$

and

$$\Phi(P^*) = \sum_{v \in \mathrm{vert}(P)} \Phi_v(P^*). \tag{5}$$

(Note that $Q_{v_1} \cap H_v^+ = Q_{v_1}$ and $Q_{v_n} \cap H_v^+ = \emptyset$.)

Since the face lattices of P and P^* are anti-isomorphic, it is immediate that the **ab**-index and the **cd**-index of P are obtained by reversing the letters in each word of the respective indices of P^*. Thus we can obtain the **cd**-index of a polytope P by sweeping P itself and pre-multiplying by **c** and **d**:

Theorem 2 *For any convex d-polytope P,*

1. *If d=0 then P has one vertex v and* $\Phi_v(P) = \Phi(P) = 1$.
2. *If* $d > 0$ *then*

$$\Phi_v(P) = \mathbf{d}\Phi(R_v) + \sum_{w \in \mathrm{vert}(Q_v) \cap H_v^+} \mathbf{c}\Phi_w(Q_v), \;\; v \in \mathrm{vert}(P), \tag{6}$$

and

$$\Phi(P) = \sum_{v \in \mathrm{vert}(P)} \Phi_v(P). \tag{7}$$

The nonnegativity of the coefficients of $\Phi(P)$ is immediate.

3.5 A Symmetric Formula

Since the **cd**-index is independent of the sweeping used, we can symmetrize the formula in Theorem 2 by taking the average of the results from a sweep and its opposite.

Theorem 3 *For any convex d-polytope P,*

1. *If d=0 then P has one vertex v and* $\Phi_v(P) = \Phi(P) = 1$.
2. *If* $d > 0$ *then*

$$\Phi_v(P) = \frac{1}{2}[c\Phi(Q_v) + (2\mathbf{d} - \mathbf{c}^2)\Phi(R_v)], \;\; v \in \mathrm{vert}(P), \tag{8}$$

and

$$\Phi(P) = \sum_{v \in \mathrm{vert}(P)} \Phi_v(P). \tag{9}$$

(Note that there is also an obvious analogous symmetric version of Theorem 1.)

We will prove this using the notation and shelling context of Section 3.3. First observe that if we create the $(d-2)$-dimensional spherical complex $\overline{R}$ by adding two $(d-2)$-cells to the complex R, each incident to each of the faces of R, analogous arguments to those in Section 3.3, Case 3, imply that $\Phi(\overline{R}) = \Phi(R)\mathbf{c}$.

For each facet F of Q, define $\overrightarrow{\Phi}_F(Q)$ to be the contribution by F to the **cd**-index of Q in the shelling order $F_1, \ldots, F_\ell$, and $\overleftarrow{\Phi}_F(Q)$ to be the contribution by F to the **cd**-index of Q in the shelling order $F_\ell, \ldots, F_1$. Then

$$\Phi(Q) = \sum_{i=1}^{\ell} \overrightarrow{\Phi}_{F_i}(Q) = \sum_{i=1}^{\ell} \overleftarrow{\Phi}_{F_i}(Q),$$

$$\Phi((F_1 \cup \cdots \cup F_k)') = \sum_{i=1}^{k} \overrightarrow{\Phi}_{F_i}(Q),$$

and

$$\Phi((F_{k+1} \cup \cdots \cup F_\ell)') = \sum_{i=k+1}^{\ell} \overleftarrow{\Phi}_{F_i}(Q).$$

Now as complexes, $(F_1 \cup \cdots \cup F_k)'$ and $(F_{k+1} \cup \cdots \cup F_\ell)'$ together equal the boundary complex of Q with an extra copy of $\overline{R}$, so

$$\Phi((F_1 \cup \cdots \cup F_k)') + \Phi((F_{k+1} \cup \cdots \cup F_\ell)') = \Phi(Q) + \Phi(\overline{R}) = \Phi(Q) + \Phi(R)\mathbf{c}.$$

Thus

$$\begin{aligned}
\sum_{i=k+1}^{\ell} \overrightarrow{\Phi}_{F_i}(Q) + \sum_{i=1}^{k} \overleftarrow{\Phi}_{F_i}(Q) &= 2\Phi(Q) - \sum_{i=1}^{k} \overrightarrow{\Phi}_{F_i}(Q) - \sum_{i=k+1}^{\ell} \overleftarrow{\Phi}_{F_i}(Q) \\
&= 2\Phi(Q) - (\Phi(Q) + \Phi(R)\mathbf{c}) \\
&= \Phi(Q) - \Phi(R)\mathbf{c}.
\end{aligned}$$

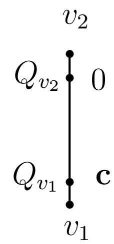

Figure 1: Sweeping the **cd**-Index of a Line Segment

Equation (4) then becomes

$$\begin{aligned}
\Phi_v(P^*) &= \tfrac{1}{2}[\Phi(P^*) + \Phi(P^*)] \\
&= \tfrac{1}{2}[\Phi(R_v)\mathbf{d} + \sum_{w \in \mathrm{vert}(Q_v) \cap H_v^+} \overrightarrow{\Phi}_w\ (Q_v)\mathbf{c} + \Phi(R_v)\mathbf{d} + \sum_{w \in \mathrm{vert}(Q_v) \cap H_v^-} \overleftarrow{\Phi}_w\ (Q_v)\mathbf{c}] \\
&= \tfrac{1}{2}[2\Phi(R_v)\mathbf{d} + \Phi(Q_v)\mathbf{c} - \Phi(R_v)\mathbf{c}^2] \\
&= \tfrac{1}{2}[\Phi(Q_v)\mathbf{c} + \Phi(R_v)(2\mathbf{d} - \mathbf{c}^2)].
\end{aligned}$$

Theorem 3 now follows as Theorem 2 does from Proposition 1. Though it might not be obvious from the statement of the theorem, note that $\Phi_v(P)$ in the theorem is necessarily nonnegative since it is the sum of two nonnegative quantities. □

3.6 Examples

1. The line segment ($d = 1$). See Figure 1.

   If P is a line segment with two vertices swept in the order v_1, v_2, then Q_{v_i} is a point and R_{v_i} is empty, $i = 1, 2$. By Theorem 2: Q_{v_1} is in $H_{v_1}^+$, $\Phi_{v_1}(P) = \mathbf{c}\Phi(Q_{v_1}) + \mathbf{d}\Phi(R_{v_1}) = \mathbf{c}(1) + \mathbf{d}(0) = \mathbf{c}$; and Q_{v_2} is in $H_{v_2}^-$, $\Phi_{v_2}(P) = \mathbf{c}(0) + \mathbf{d}(0) = 0$. By Theorem 3: $\Phi_{v_i}(P) = \frac{1}{2}[\mathbf{c}\Phi(Q_{v_i}) + (2\mathbf{d} - \mathbf{c}^2)\Phi(R_{v_i})] = \frac{1}{2}[\mathbf{c}(1) + (2\mathbf{d} - \mathbf{c}^2)(0)] = \frac{1}{2}\mathbf{c}$, $i = 1, 2$. Either way, $\Phi(P) = \mathbf{c}$.

2. The n-gon ($d = 2$). See Figure 2.

   If P is an n-gon with vertices swept in the order $v_1, \ldots, v_n$, then Q_{v_i} is a line segment, $i = 1, \ldots, n$; R_{v_1} and R_{v_n} are empty; and R_{v_i} is a point, $i = 2, \ldots, n-1$. By Theorem 2:

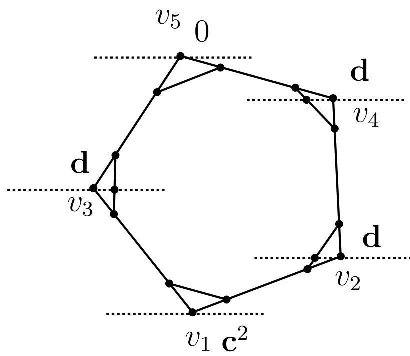

Figure 2: Sweeping the **cd**-Index of a Polygon

$Q_{v_1} \subset H^+_{v_1}$, $Q_{v_n} \subset H^-_{v_n}$, and only the top vertex of Q_{v_i} is in $H^+_{v_i}$, $i = 2, \ldots, n-1$. So $\Phi_{v_1}(P) = \mathbf{c}\Phi(Q_{v_1}) + \mathbf{d}\Phi(R_{v_1}) = \mathbf{c}(\mathbf{c}) + \mathbf{d}(0) = \mathbf{c}^2$, $\Phi_{v_n}(P) = \mathbf{c}(0) + \mathbf{d}\Phi(R_{v_n}) = \mathbf{c}(0) + \mathbf{d}(0) = 0$, and $\Phi_{v_i}(P) = \mathbf{c}(0) + \mathbf{d}\Phi(R_{v_i}) = \mathbf{c}(0) + \mathbf{d}(1) = \mathbf{d}$, $i = 2, \ldots, n-1$. By Theorem 3: For $i = 1$ or $i = n$, $\Phi_{v_i}(P) = \frac{1}{2}[\mathbf{c}\Phi(Q_{v_i}) + (2\mathbf{d} - \mathbf{c}^2)\Phi(R_{v_i})] = \frac{1}{2}[\mathbf{c}(\mathbf{c}) + (2\mathbf{d} - \mathbf{c}^2)(0)] = \frac{1}{2}\mathbf{c}^2$; and for $i = 2, \ldots, n-1$, $\Phi_{v_i}(P) = \frac{1}{2}[\mathbf{c}\Phi(Q_{v_i}) + (2\mathbf{d} - \mathbf{c}^2)\Phi(R_{v_i})] = \frac{1}{2}[\mathbf{c}^2 + (2\mathbf{d} - \mathbf{c}^2)] = \mathbf{d}$, $i = 2, \ldots, n-1$. Either way, $\Phi(P) = \mathbf{c}^2 + (n-2)\mathbf{d}$.

3. The octahedron.

   If P is the octahedron with vertices swept in the order $v_1, \ldots, v_6$ as indicated in Figure 3, then Q_{v_i} is a square, $i = 1, \ldots, 6$; R_{v_1} and R_{v_6} are empty; and R_{v_i} is a line segment, $i = 2, \ldots, 5$. By Theorem 2: All of the vertices of Q_{v_1} are in $H^+_{v_1}$; only the top three vertices of Q_{v_2} are in $H^+_{v_2}$; only the top two vertices of Q_{v_i} are in $H^+_{v_i}$, $i = 3, 4$; only the top vertex of Q_{v_5} is in $H^+_{v_5}$; and none of the vertices of Q_{v_6} are in $H^+_{v_6}$. So $\Phi_{v_1}(P) = \mathbf{c}(\mathbf{c}^2 + 2\mathbf{d}) + \mathbf{d}(0) = \mathbf{c}^3 + 2\mathbf{cd}$, $\Phi_{v_2}(P) = \mathbf{c}(2\mathbf{d}) + \mathbf{d}(\mathbf{c}) = 2\mathbf{cd} + \mathbf{dc}$, $\Phi_{v_3}(P) = \Phi_{v_4}(P) = \mathbf{c}(\mathbf{d}) + \mathbf{d}(\mathbf{c}) = \mathbf{cd} + \mathbf{dc}$, $\Phi_{v_5}(P) = \mathbf{c}(0) + \mathbf{d}(\mathbf{c}) = \mathbf{dc}$, and $\Phi_{v_6}(P) = \mathbf{c}(0) + \mathbf{d}(0) = 0$. By Theorem 3: $\Phi_{v_i}(P) = \frac{1}{2}[\mathbf{c}(\mathbf{c}^2 + 2\mathbf{d}) + (2\mathbf{d} - \mathbf{c}^2)(0)] = \frac{1}{2}\mathbf{c}^3 + \mathbf{cd}$, $i = 1$ and $i = 6$; and $\Phi_{v_i}(P) = \frac{1}{2}[\mathbf{c}(\mathbf{c}^2 + 2\mathbf{d}) + (2\mathbf{d} - \mathbf{c}^2)(\mathbf{c})] = \mathbf{cd} + \mathbf{dc}$, $i = 2, \ldots, 5$. Either way, $\Phi(P) = \mathbf{c}^3 + 6\mathbf{cd} + 4\mathbf{dc}$ (and we can reverse the letters in each word of $\Phi(P)$ to get the **cd**-index of the cube, $\mathbf{c}^3 + 6\mathbf{dc} + 4\mathbf{cd}$).

4. The square-based pyramid. See Figure 4.

   If P is the square-based pyramid with vertices swept in the order $v_1, \ldots, v_5$ as indicated in Figure 4, then Q_{v_i} is a triangle, $i = 1, 2, 4, 5$; Q_{v_3} is a square; R_{v_1} and R_{v_5} are empty;

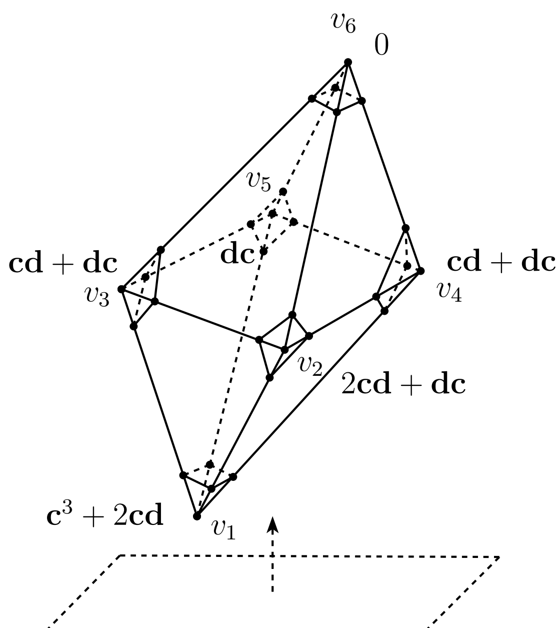

Figure 3: Sweeping the **cd**-Index of an Octahedron

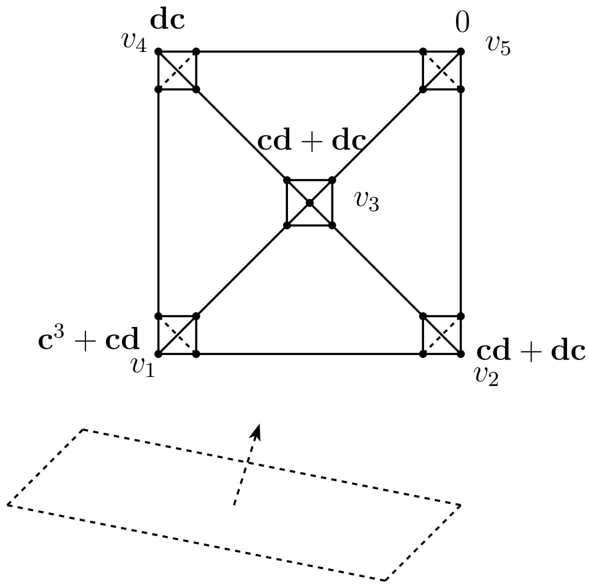

Figure 4: Sweeping the **cd**-Index of a Pyramid (View from Above)

and R_{v_i} is a line segment, $i = 2, 3, 4$. By Theorem 2: All of the vertices of Q_{v_1} are in $H^+_{v_1}$; only the top two vertices of Q_{v_2} are in $H^+_{v_2}$; only the top two vertices of Q_{v_3} are in $H^+_{v_3}$; only the top vertex of Q_{v_4} is in $H^+_{v_4}$; and none of the vertices of Q_{v_5} are in $H^+_{v_5}$. So $\Phi_{v_1}(P) = \mathbf{c}(\mathbf{c}^2 + \mathbf{d}) + \mathbf{d}(0) = \mathbf{c}^3 + \mathbf{cd}$, $\Phi_{v_2}(P) = \mathbf{c}(\mathbf{d}) + \mathbf{d}(\mathbf{c}) = \mathbf{cd} + \mathbf{dc}$, $\Phi_{v_3}(P) = \mathbf{c}(\mathbf{d}) + \mathbf{d}(\mathbf{c}) = \mathbf{cd} + \mathbf{dc}$, $\Phi_{v_4}(P) = \mathbf{c}(0) + \mathbf{d}(\mathbf{c}) = \mathbf{dc}$, and $\Phi_{v_5}(P) = \mathbf{c}(0) + \mathbf{d}(0) = 0$. By Theorem 3: $\Phi_{v_i}(P) = \frac{1}{2}[\mathbf{c}(\mathbf{c}^2 + \mathbf{d}) + (2\mathbf{d} - \mathbf{c}^2)(0)] = \frac{1}{2}\mathbf{c}^3 + \frac{1}{2}\mathbf{cd}$, $i = 1$ and $i = 5$; $\Phi_{v_i}(P) = \frac{1}{2}[\mathbf{c}(\mathbf{c}^2 + \mathbf{d}) + (2\mathbf{d} - \mathbf{c}^2)(\mathbf{c})] = \frac{1}{2}\mathbf{cd} + \mathbf{dc}$, $i = 2, 4$; and $\Phi_{v_3}(P) = \frac{1}{2}[\mathbf{c}(\mathbf{c}^2 + 2\mathbf{d}) + (2\mathbf{d} - \mathbf{c}^2)(\mathbf{c})] = \mathbf{cd} + \mathbf{dc}$, Either way, $\Phi(P) = \mathbf{c}^3 + 3\mathbf{cd} + 3\mathbf{dc}$.

3.7 Partitioning the Complete Truncation

To visualize what the **cd**-index is counting we return to the notation of Sections 3.3 and 3.4, where the vertices $v_1, \ldots, v_n$ of P are ordered by a sweep of the hyperplane family $H = \{x \in \mathbf{R}^d : p \cdot x = q\}$ and the facets of its polar dual P^* are given the shelling order induced by the associated line shelling. Consider a choice of the parameter q such that for the hyperplane H we have $v_1, \ldots, v_m \in H^-$ and $v_{m+1}, \ldots, v_n \in H^+$. Observe that we can create a complex U'_m dual to P'_m by taking $P \cap H^-$, creating an unbounded polyhedron U_m

by applying a projective transformation that sends the facet $P \cap H$ onto the hyperplane at infinity, and finally introducing a new vertex at infinity incident to each of the unbounded faces of U_m. Each $\{s_1, \ldots, s_j\}$-chain in P'_m naturally corresponds to an $\{s'_j, \ldots, s'_1\}$-chain in U'_m, where $s'_i = d - s_i - 1$, $i = 0, \ldots, j$. As the sweep (and the s-shelling) progresses, chains in U'_m that involve the "top" vertex at infinity are eventually replaced by chains in P involving one of the vertices $v_{m+1}, \ldots, v_n$ as the "top" vertex (corresponding to (3) in Case 1).

Truncate all of the faces of P by first truncating the vertices of P and giving each resulting $(d-1)$-face the label 0, then truncating the original edges of P and giving each resulting $(d-1)$-face the label 1, then truncating the original 2-faces of P, etc., until finally truncating the original facets of P. The resulting polytope, $T(P)$, called the *complete truncation* of P, is dual to the complete barycentric subdivision of P^*, and its faces are in one-to-one correspondence with the chains of P. In fact, each nonempty face G of $T(P)$ corresponds to an S-chain of P, where $\sigma(G) = S$ is the set of labels of all of the facets of $T(P)$ containing G. The polytope $T(P)$ itself is labeled by the empty set. The goal of this section is to partition the faces of $T(P)$ into parts so that the **cd**-index of each part is a single **cd**-word summing to the **cd**-index of P.

When constructing $T(P)$, make the truncations in sufficiently general direction so that each of the vertices w of $T(P)$ has a different value of $p \cdot w$. For each nonempty face G of $T(P)$ of positive dimension $\dim(G)$ let $j = \min\{i : i \notin \sigma(G)\}$ and w be the vertex of G with greatest value of $p \cdot w$. Define the *top co-face* of G to be the unique face $\tau(G)$ of G of dimension $\dim(G) - 1$ that contains w and has label $\sigma(G) \cup \{j\}$. In this way "top" vertices of faces in P are replaced by top co-faces.

We can now read off the faces of $T(P)$ corresponding to (duals of) the chains created in Case 0, Case 1(4), and Case 2(2). Let v be a vertex of P and H_v, Q_v, and R_v be as before. Take $T(Q_v)$ to be the complete truncation of Q_v, regarded as a facet of $T(P)$, and $T(R_v)$ to be the complete truncation of R_v, realized as $T(Q_v) \cap H_v$.

1. Consider any face of $T(R_v)$. It is of the form $G \cap H_v$, where G is a face of $T(Q_v)$. Let $S = \sigma(G)$. Note that $0 \in S$ because $\sigma(T(Q_v)) = \{0\}$ and $1 \notin S$ because H_v does not intersect the relative interiors of any edges containing v. Associate with G the following collection of faces $D(G) = \{G_1, G_2, G_3, G_4\}$:

   (a) $G_1 = G$, which has label S,

   (b) The top co-face $G_2 = \tau(G)$ of G, which has label $S \cup \{1\}$,

   (c) The unique face G_3 of $T(P)$ containing G_2 and having label $S \setminus \{0\}$, and

   (d) The top co-face $G_4 = \tau(G_3)$ of G_3, which has label $S \cup \{1\}$.

   These four cases correspond, respectively, to cases (4a)–(4d) of Case 1.

2. Consider any face G of $T(Q_v)$. Let $S = \sigma(G)$. Note that $0 \in S$ because $\sigma(T(Q_v)) = \{0\}$. Associate with G the following collection of faces $C(G) = \{G_1, G_2, G_3\}$:

   (a) $G_1 = G$, which has label S,

   (b) The unique face G_2 of $T(P)$ containing G_1 and having label $S \setminus \{0\}$, and

   (c) The top co-face $G_3 = \tau(G_2)$ of G_2, which has label S. These three cases correspond, respectively, to cases (2b), (2a), and (2c) of Case 2, as well as cases (2), (1), and (3) when $m = 0$.

The operations D and C can be extended to sets of faces of $T(P)$ as well as collections of sets of faces of $T(P)$ in the obvious way.

We can now describe the partition $\Pi(T(P))$ of the nonempty faces of $T(P)$ associated with $\Phi(P)$ by associating a collection of parts $\Pi_v(T(P))$ of this partition with each vertex v of P.

1. If $d = 0$ then P has one vertex v, $\Phi(P) = 1$, $T(P)$ is combinatorially equivalent to P, and $\Pi_v(T(P)) = \Pi(T(P)) = \{\{T(P)\}\}$.

2. If $d = 1$ then P has two vertices v_1 and v_2, $\Phi(P) = c$, $T(P)$ is combinatorially equivalent to P, and with this sweeping order, $\Pi_{v_1}(T(P)) = \{\{v_1, v_2, T(P)\}\}$, $\Pi_{v_2}(T(P)) = \emptyset$, and $\Pi(T(P)) = \Pi_{v_1}(T(P))$.

3. If $d > 1$ then

$$\Pi_v(T(P)) = D[\Pi(T(R_v))] \cup C[\bigcup_{w \in \mathrm{vert}(Q_v) \cap H_v^+} \Pi_w(T(Q_v))],\ v \in \mathrm{vert}(P),$$

   and

$$\Pi(T(P)) = \bigcup_{v \in \mathrm{vert}(P)} \Pi_v(T(P)).$$

3.8 Examples

The complete truncation of a line segment is a (shortened) line segment. There is only one part in the partition of its faces, consisting of the two endpoints and the line segment itself, together representing **c**.

Figure 5 shows the complete truncation of the pentagon of Figure 2, and the partition of its faces into four parts.

Figure 6 is the complete truncation of the squared-based pyramid of Figure 4, together with the facet labels (the base octagon has label 0). Figure 7 shows the partition of the faces

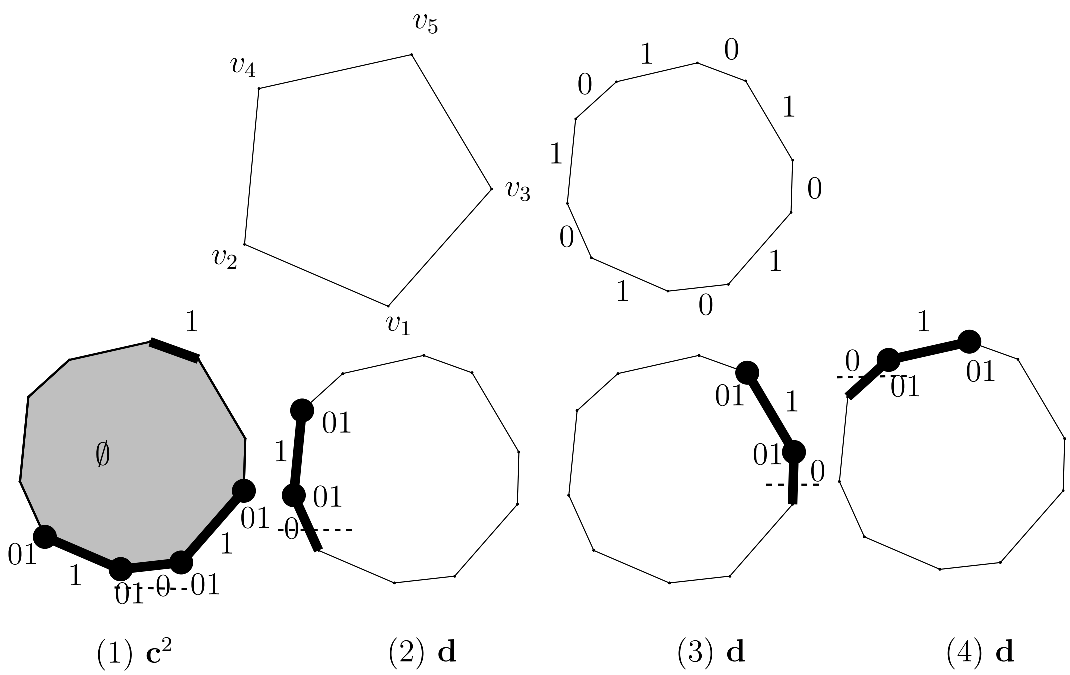

Figure 5: Partitioning a Truncated Polygon

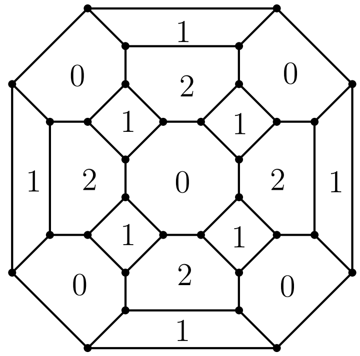

Figure 6: Truncated Pyramid

of this complete truncation. Parts (1) and (2) are associated with vertex v_1 of the original pyramid—note that the part (1) also includes the truncated base of the pyramid (the outer octagon) as well as the truncated pyramid itself. Parts (3) and (4) are associated with vertex v_2, parts (5) and (6) with vertex v_3, and part (7) with vertex v_4.

4 The Toric h-Vector

4.1 Definitions

The *toric h-vector* of (the boundary complex of) a convex d-polytope P, $h(\partial P) = (h_0, \ldots, h_d)$, is a linear combination of the components of the flag h-vector that is a non-negative, symmetric, generalization of the h-vector of a simplicial polytope. The component $h_i = h_i(\partial P)$ is the rank of the $(2d-2i)$th middle perversity intersection homology group of the associated toric variety in the case that P is rational (has a realization with rational vertices). The g-Theorem [12] implies that the h-vector of a simplicial polytope is unimodal. Karu [9] proved that this is also the case for the toric h-vector of a general polytope P, even when P is not rational. For a summary of some other results on the toric h-vector see [4].

To define the toric h-vector recursively, let $h(\partial P, x) = \sum_{i=0}^{d} h_i x^{d-i}$ and $g(\partial P, x) = \sum_{i=0}^{\lfloor d/2 \rfloor} g_i x^i$ where $g_0 = g_0(\partial P) = h_0$ and $g_i = g_i(\partial P) = h_i - h_{i-1}$, $i = 1, \ldots, \lfloor d/2 \rfloor$. Then

$$g(\emptyset, x) = h(\emptyset, x) = 1,$$

and

$$h(\partial P, x) = \sum_{G \text{ face of } \partial P} g(\partial G, x)(x-1)^{d-1-\dim G}. \qquad (10)$$

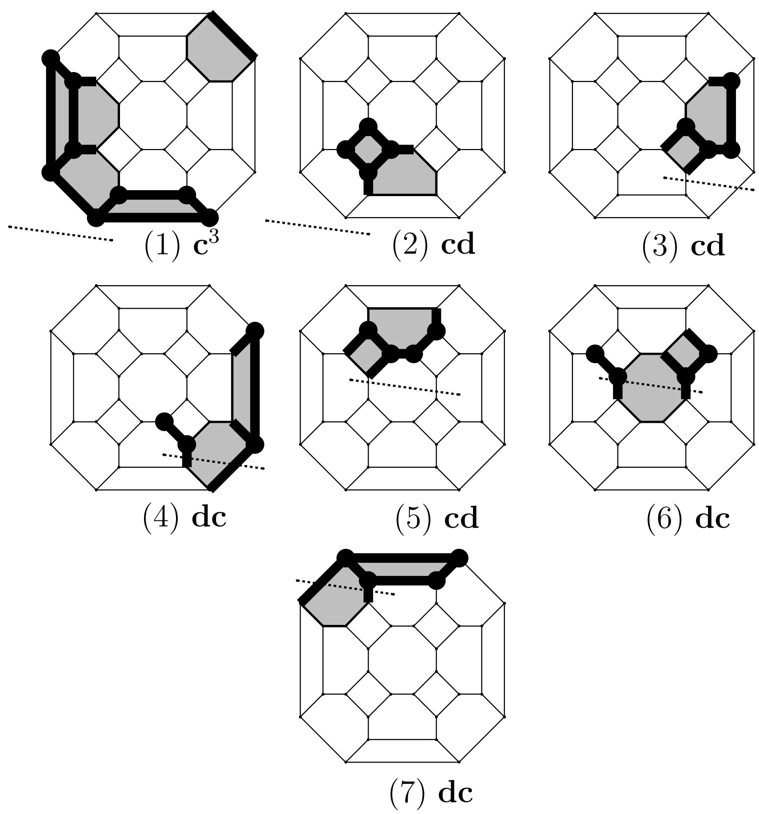

Figure 7: Partitioning a Truncated Pyramid (View from Above)

In the case that P is simplicial the toric h-vector of ∂P agrees with the simplicial h-vector of P.

For example, the toric h-vectors of the boundary complexes of a point, line segment, n-gon, octahedron, and cube are, respectively, (1), $(1,1)$, $(1,n-2,1)$, $(1,3,3,1)$, and $(1,5,5,1)$.

4.2 Sweeping the Toric h-Vector

Using the notation of Section 3.3 we will first compute the changes in the toric h vector during the s-shelling of P^*. Define functions $\mathbf{c} : \mathbf{R}^{d+1} \to \mathbf{R}^{d+2}$ and $\mathbf{d} : \mathbf{R}^{d+1} \to \mathbf{R}^{d+3}$ by

$$(h_0, \ldots, h_d)\mathbf{c} = \begin{cases} (g_0, g_1, \ldots, g_{\lfloor d/2 \rfloor}, g_{\lfloor d/2 \rfloor}, \ldots, g_1, g_0) & \text{if } d \text{ is even} \\ (g_0, g_1, \ldots, g_{\lfloor d/2 \rfloor}, 0, g_{\lfloor d/2 \rfloor}, \ldots, g_1, g_0) & \text{if } d \text{ is odd} \end{cases}$$

and

$$(h_0, \ldots, h_d)\mathbf{d} = \begin{cases} (0, \ldots, 0, g_{\lfloor d/2 \rfloor}, 0, \ldots, 0) & \text{if } d \text{ is even} \\ (0, \ldots, 0) & \text{if } d \text{ is odd} \end{cases}$$

where as before $g_0 = h_0$ and $g_i = h_i - h_{i-1}$, $i = 1, \ldots, \lfloor d/2 \rfloor$.

Define $h_Q(\partial P^*)$ to be the contribution by Q to the toric h-vector of P^* during the s-shelling of P^*. We now have an analogue to Proposition 1:

Proposition 2 *Let P^* be a convex d-polytope.*

1. *If $d = 0$ then P^* has one facet, $Q = \emptyset$, and $h_Q(\partial P) = h(\partial P) = (1)$.*

2. *If $d = 1$ then P^* has two facets Q_1 and Q_2. With this shelling order, $h_{Q_1}(\partial P^*) = (1,1)$, $h_{Q_2}(\partial P^*) = (0,0)$, and $h(\partial P^*) = (1,1)$.*

3. *If $d > 1$ then in the notation of Section 3.3 and regarding $\mathbf{c}$ and $\mathbf{d}$ as functions,*

$$h_Q(\partial P^*) = h(\partial R)\mathbf{d} + \sum_{i=k+1}^{\ell} h_{F_i}(\partial Q)\mathbf{c}$$

and

$$h(\partial P^*) = \sum_Q h_Q(\partial P^*).$$

We prove this by considering the three cases of Section 3.3.

Case 0: Counting chains in P'_1. For any $(d-1)$-polytope Q define $L(\partial Q)$ (the *lens* on ∂Q) to be the $(d-1)$-dimensional spherical complex obtained by appending two $(d-1)$-cells to ∂Q, each incident to every face of ∂Q. (In this case the two added cells are Q and the cap of P_1'.) Then

$$\begin{aligned} h(L(\partial Q),x) &= \sum_{G \text{ face of } L(\partial Q)} g(\partial G,x)(x-1)^{d-1-\dim G} \\ &= 2g(\partial Q,x) + \sum_{G \text{ face of } \partial Q} g(\partial G,x)(x-1)^{d-2-\dim G}(x-1) \\ &= 2g(\partial Q,x) + h(\partial Q,x)(x-1). \end{aligned}$$

If d is odd (so $d-1$ is even) and $h(\partial Q) = (h_0,\ldots,h_{r-1},h_r,h_{r-1},\ldots,h_0)$, then

$$\begin{aligned} h(L(\partial Q)) &= 2(h_0,h_1-h_0,\ldots,h_r-h_{r-1},0,\ldots,0) \\ &\quad +(-h_0,h_0-h_1,\ldots,h_{r-1}-h_r,h_r-h_{r-1},\ldots,h_1-h_0,h_0) \\ &= (h_0,h_1-h_0,\ldots,h_r-h_{r-1},h_r-h_{r-1},\ldots,h_1-h_0,h_0) \\ &= h(\partial Q)\mathbf{c}. \end{aligned}$$

If d is even (so $d-1$ is odd) and $h(\partial Q) = (h_0,\ldots,h_{r-1},h_r,h_r,h_{r-1},\ldots,h_0)$, then

$$\begin{aligned} h(L(\partial Q)) &= 2(h_0,h_1-h_0,\ldots,h_r-h_{r-1},0,\ldots,0) \\ &\quad +(-h_0,h_0-h_1,\ldots,h_{r-1}-h_r,0,h_r-h_{r-1},\ldots,h_1-h_0,h_0) \\ &= (h_0,h_1-h_0,\ldots,h_r-h_{r-1},0,h_r-h_{r-1},\ldots,h_1-h_0,h_0) \\ &= h(\partial Q)\mathbf{c}. \end{aligned}$$

Case 1: Moving from P_m' to $(P_m \cup \overline{Z}_k)'$. Let X be the cap of P_m', and T_k, $\hat{Z}_k$, $\hat{F}_k$, and $R = \partial\hat{F}_k$ be as before. Considering faces with multiplicity, taking the union of the complexes ∂T_k and $\partial\hat{Z}_k$ and removing ∂X leaves us with the faces of a complex combinatorially equivalent to $L(R)$, the lens on R (the cell $\hat{F}_k$ appears twice). The change in the h-polynomial can be computed:

$$\begin{aligned} h((P_m \cup \overline{Z}_k)',x) - h(P_m',x) &= g(\partial T_k,x) + g(\partial\hat{Z}_k) + g(R,x)(x-1) - g(\partial X,x) \\ &= g(L(R),x) + g(R,x)(x-1). \end{aligned}$$

If d is odd (so $d-2$ is odd) and $h(R) = (h_0,\ldots,h_{r-1},h_r,h_r,h_{r-1},\ldots,h_0)$, then:

$$\begin{aligned} h(L(R)) &= (h_0,h_1-h_0,h_2-h_1,\ldots,h_r-h_{r-1},0,h_r-h_{r-1},\ldots,h_2-h_1,h_1-h_0,h_0), \\ g(L(R)) &= (h_0,h_1-2h_0,h_2-2h_1+h_0,\ldots,h_r-2h_{r-1}+h_{r-2},-h_r+h_{r-1},0,\ldots,0), \\ g(R) &= (h_0,h_1-h_0,h_2-h_1,\ldots,h_r-h_{r-1},0,\ldots,0), \end{aligned}$$

and $g(R,x)(x-1)$ is the polynomial for

$$(-h_0, -h_1+2h_0, -h_2+2h_1-h_0, \ldots, -h_r+2h_{r-1}-h_{r-2}, h_r-h_{r-1}, 0, \ldots, 0).$$

Hence $g(L(R),x)+g(R,x)(x-1)$ is the polynomial for $(0,\ldots,0)=h(R)\mathbf{d}$.

If d is even (so $d-2$ is even) and $h(R)=(h_0,\ldots,h_{r-1},h_r,h_{r-1},\ldots,h_0)$, then:

$$\begin{aligned} h(L(R)) &= (h_0, h_1-h_0, h_2-h_1, \ldots, h_r-h_{r-1}, h_r-h_{r-1}, \ldots, h_2-h_1, h_1-h_0, h_0), \\ g(L(R)) &= (h_0, h_1-2h_0, h_2-2h_1+h_0, \ldots, h_r-2h_{r-1}+h_{r-2}, 0, \ldots, 0), \\ g(R) &= (h_0, h_1-h_0, h_2-h_1, \ldots, h_r-h_{r-1}, 0, \ldots, 0), \end{aligned}$$

and $g(R,x)(x-1)$ is the polynomial for

$$(-h_0, -h_1+2h_0, -h_2+2h_1-h_0, \ldots, -h_r+2h_{r-1}-h_{r-2}, h_r-h_{r-1}, 0, \ldots, 0).$$

Hence $g(L(R),x)+g(R,x)(x-1)$ is the polynomial for $(0,\ldots,0,h_r-h_{r-1},0,\ldots,0)=h(R)\mathbf{d}$.

Case 2: Moving from $(P_m\cup\overline{Z}_{i-1})'$ to $(P_m\cup\overline{Z}_i)'$, $i=k+1,\ldots,\ell$. Here, the change in the flag f-vector is the same as the change in the flag f-vector when moving from $L(\partial T_{i-1})$ to $L(\partial T_i)$ (just consider the numbers of types of chains gained). Hence by Case 0 the change in the h-vector is $h(\partial T_i)\mathbf{c}-h(\partial T_{i-1})\mathbf{c}$, which is $h_{F_i}(\partial Q)\mathbf{c}$. This completes the proof of the proposition. □

Dualizing, we have the analogue to Theorem 1:

Theorem 4 *For any convex d-polytope P,*

1. *If $d=0$ then P has one vertex v and $h_v(\partial P^*)=h(\partial P^*)=(1)$.*
2. *If $d>0$ then, regarding $\mathbf{c}$ and $\mathbf{d}$ as functions,*

$$h_v(\partial P^*)=h(\partial(R_v)^*)\mathbf{d}+\sum_{w\in\mathrm{vert}(Q_v)\cap H_v^+} h_w(\partial(Q_v)^*)\mathbf{c},\ \ v\in\mathrm{vert}(P),$$

and

$$h(\partial P^*)=\sum_{v\in\mathrm{vert}(P)} h_v(\partial P^*).$$

Induction and duality lead immediately to a formula to obtain the toric h-vector directly from the **cd**-index (which can be seen to be a reformulation of the formula in [2]) and an analogue of Theorem 3 (allowing the functions **c** and **d** to act on the left as well as on the right).

Theorem 5 *Let P be a convex d-polytope. Then, regarding* $\mathbf{c}$ *and* $\mathbf{d}$ *as functions,* $h(\partial P) = (1)\Phi(P)$ *and* $h(\partial P^*) = \Phi(P)(1)$.

Theorem 6 *For any convex d-polytope P,*

1. *If $d = 0$ then P has one vertex v and $h_v(\partial P^*) = h(\partial P^*) = (1)$.*
2. *If $d > 0$ then, regarding* $\mathbf{c}$ *and* $\mathbf{d}$ *as functions,*

$$h_v(\partial P^*) = \frac{1}{2}[h(\partial(Q_v)^*)\mathbf{c} + h(\partial(R_v)^*)(2\mathbf{d} - \mathbf{c}^2)],\ v \in \mathrm{vert}(P),$$

and

$$h(\partial P^*) = \sum_{v \in \mathrm{vert}(P)} h(\partial P^*).$$

4.3 Examples

1. If $d = 0$ and P is a point then $h(\partial P) = (1)\Phi(P) = (1)1 = (1)$.
2. If $d = 1$ and P is a line segment then $h(\partial P) = (1)\mathbf{c} = (1, 1)$.
3. If $d = 2$ and P is an n-gon then

$$\begin{aligned} h(P) &= (1)\Phi(P) \\ &= (1)(\mathbf{c}^2 + (n-2)\mathbf{d}) \\ &= (1,1)\mathbf{c} + (n-2)(0,1,0) \\ &= (1,0,1) + (n-2)(0,1,0) \\ &= (1, n-2, 1). \end{aligned}$$

4. If $d = 3$ and P is the octahedron then

$$\begin{aligned} h(\partial P) &= (1)\Phi(P) \\ &= (1)(\mathbf{c}^3 + 6\mathbf{cd} + 4\mathbf{dc}) \\ &= (1,1)\mathbf{c}^2 + 6(1,1)\mathbf{d} + 4(0,1,0)\mathbf{c} \\ &= (1,0,1)\mathbf{c} + 6(0,0,0,0) + 4(0,1,1,0) \\ &= (1,-1,-1,1) + (0,0,0,0) + (0,4,4,0) \\ &= (1,3,3,1). \end{aligned}$$

and

$$
\begin{aligned}
h(\partial P^*) &= \Phi(P)(1) \\
&= (\mathbf{c}^3 + 6\mathbf{cd} + 4\mathbf{dc})(1) \\
&= \mathbf{c}^2(1,1) + 6\mathbf{c}(0,1,0) + 4\mathbf{d}(1,1) \\
&= \mathbf{c}(1,0,1) + 6(0,1,1,0) + 4(0,0,0,0) \\
&= (1,-1,-1,1) + (0,6,6,0) + (0,0,0,0) \\
&= (1,5,5,1).
\end{aligned}
$$

We can also apply Theorem 6 to the octahedron to compute the h-vector of the cube; refer to Example 3 of Section 3.6, in which $(Q_{v_i})^*$ is a square with h-vector $(1,2,1)$, $i = 1,\ldots,6$, $(R_{v_1})^*$ and $(R_{v_6})^*$ are empty, and $(R_{v_i})^*$ is a line segment with h-vector $(1,1)$, $i = 2,\ldots,5$. Then $h_{v_i}(\partial P^*) = \frac{1}{2}[(1,2,1)\mathbf{c} + (0,0)(2\mathbf{d} - \mathbf{c}^2)] = \frac{1}{2}(1,1,1,1)$, $i = 1$ and $i = 6$; and $h_{v_i}(P^*) = \frac{1}{2}[(1,2,1) + \mathbf{c}(1,1)(2\mathbf{d} - \mathbf{c}^2)] = \frac{1}{2}[(1,1,1,1) + 2(0,0,0,0) - (1,-1,-1,1))] = \frac{1}{2}(0,2,2,0) = (0,1,1,0)$, $i = 2,\ldots,5$. Thus $h(\partial P^*) = (1,5,5,1)$.

4.4 An "Extended Toric" h-Vector

As mentioned before, even though for a d-polytope P the **cd**-index $\Phi(P)$ contains $F_d - 1$ independent pieces of information, the toric h-vector $h(P)$ contains only $\lfloor (d+1)/2 \rfloor$ independent pieces of information. The source of the loss from $\Phi(P)$ to $h(P)$ is evident—the **d** function "erases" information. We can get around this by keeping track of some of the intermediate calculations (those vectors that are about to be acted upon by **d**).

Let W be the set of all **cd**-words w of degree at most d (including the word 1). Denote by $W^{\mathbf{d}}$ the set of all words in W having **d** as the first letter, and include 1 in this set also. For $w \in W$ let $\Phi^w(P)w$ be that portion of $\Phi(P)$ with terms ending in w. Define $h^w(P) = (1)\Phi^w(P)$. Define the *"extended toric" h-vector* of P to be $\hat{h}(P) = (h^w(P) : w \in W^{\mathbf{d}})$.

For example, if P is the octahedron, then $\Phi(P) = \mathbf{c}^3 + 4\mathbf{dc} + 6\mathbf{cd}$. We have:

w	$\Phi^w(P)$	$h^w(P)$
1	$\mathbf{c}^3 + 4\mathbf{dc} + 6\mathbf{cd}$	$(1,3,3,1)$
$\mathbf{c}$	$\mathbf{c}^2 + 4\mathbf{d}$	$(1,4,1)$
$\mathbf{d}$	$6\mathbf{c}$	$(6,6)$
$\mathbf{c}^2$	$\mathbf{c}$	$(1,1)$
$\mathbf{dc}$	4	(4)
$\mathbf{cd}$	6	(6)
$\mathbf{c}^3$	1	(1)

Then $W^{\mathbf{d}} = \{1, \mathbf{d}, \mathbf{dc}\}$ and the extended toric h-vector is $\hat{h}(P) = (h^1(P), h^{\mathbf{d}}(P), h^{\mathbf{dc}}(P)) = ((1,3,3,1),(6,6),(4)))$.

Theorem 7 *For a d-polytope P each $h^w(P)$, $w \in W^{\mathbf{d}}$, is nonnegative, symmetric, and unimodal, and $\hat{h}(P)$ determines $\Phi(P)$.*

To prove this, recall that the toric h-vector of any polytope is nonnegative, symmetric, and unimodal, and by the recursive application of Proposition 1 or Theorem 1 the function $\mathbf{d}$ is always multiplied onto the **cd**-index of some polytope. Hence each $h^w(P)$, $w \in W^{\mathbf{d}}$, being a sum of h-vectors of such polytopes, is nonnegative, symmetric, and unimodal. To show that $\hat{h}(P)$ determines $\Phi(P)$, observe that

1. Any symmetric vector h can be recovered from $h\mathbf{c}$.
2. For any **cd**-word w, $h^{\mathbf{c}w}(P)$ can be recovered from $h^w(P)$ and $h^{\mathbf{d}w}(P)$, since $h^w(P) = (h^{\mathbf{c}w}(P))\mathbf{c} + (h^{\mathbf{d}w}(P))\mathbf{d}$. Therefore, by reverse induction on the degree of w, we can recover all of the vectors $h^w(P)$ from $\hat{h}(P)$.
3. For any **cd**-word w of degree d, the coefficient of w in $\Phi(P)$ is precisely the single entry of $h^w(P)$.

This concludes the proof. □

At this point it remains to be seen whether or not one can get a better understanding of the collection of flag f-vectors of convex d-polytopes from their extended toric h-vectors, or indeed whether one is even justified in giving $\hat{h}(P)$ this name.

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