Game, Set, and Match
A Personal Journey

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Introduction

As I reflect, I acknowledge that I was fortunate in several ways:

- I have had a deep and abiding love of math since second grade.
- My father owned some wonderful math books.
- My family was loaded with intellectually curious and encouraging individuals.
- My math teachers were fabulous.

Introduction

Some of my favorite early (pre-college) reading:

- Martin Gardner's many books and articles.
- Ball and Coxeter, *Mathematical Recreations and Essays.*
- Steinhaus, *Mathematical Snapshots.*
- Cundy and Rollett, *Mathematical Models.*
- Holden, *Shapes, Space, and Symmetry.*

Introduction

The cumulative effect was an exposure to a broad range of mathematics far beyond the conventional boundaries of standard high school courses before I set foot in college.

I will focus on a selection of games that I encountered early on, and some that I have come to know more recently, that offer lovely contexts within which to meet some mathematical friends.

GAME!

Fifteen

A set of nine cards labeled 1 through 9 are placed face up on the table. Two players alternately select cards to place face up in front of them. The first player to have, among their cards, a set of three numbers that sum to 15 wins.

Fifteen

Gwen   Daniel

Fifteen

Gwen	Daniel
8	

Fifteen

Gwen	Daniel
8	1

Fifteen

Gwen	Daniel
8	1
4	

Fifteen

Gwen	Daniel
8	1
4	3

Fifteen

Gwen	Daniel
8	1
4	3
2	

Fifteen

Gwen	Daniel
8	1
4	3
2	5

Fifteen

Gwen	Daniel
8	1
4	3
2	5
9	

Fifteen

Gwen	Daniel
8	1
4	3
2	5
9	

Gwen wins!

Fifteen

Gwen	Daniel
8	1
4	3
2	5
9	

Gwen wins!

Now you try it!

Fifteen

Did this game “feel familiar” to you?

Fifteen

Did this game "feel familiar" to you?
Perhaps it should be called Lo Shu...

Fifteen

Did this game "feel familiar" to you?
Perhaps it should be called Lo Shu...
Or Tic-Tac-Toe!

8	1	6
3	5	7
4	9	2

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8	1	6
3	5	7
4	9	2

This was one of my early encounters with the notion of isomorphism.
Martin Gardner, *Mathematical Carnival*, Chapter 16.

(I had previously learned about magic squares in Ball (and Coxeter).)

Three Dimensional Tic-Tac-Toe

Most children are familiar with the fact that when two experienced players play Tic-Tac-Toe, the game ends in a tie. The second player must remember to make the correct move in response to the first move. See *Winning Ways* for a complete analysis via a huge chart.

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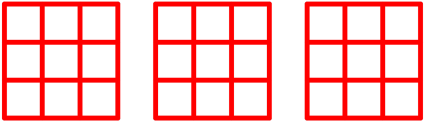
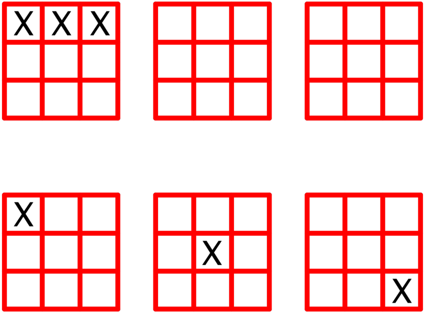
So try playing in three dimensions. Here is a representation:



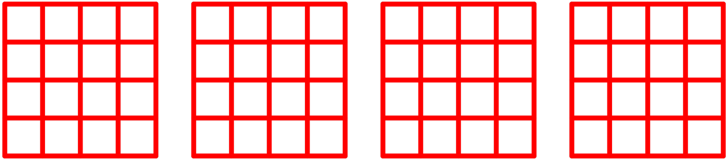
Three Dimensional Tic-Tac-Toe

Some winning combinations:



Three Dimensional Tic-Tac-Toe

It is too easy for the first player to always win in $3 \times 3 \times 3$ tic-tac-toe, so it is more fun to play $4 \times 4 \times 4$ tic-tac-toe.



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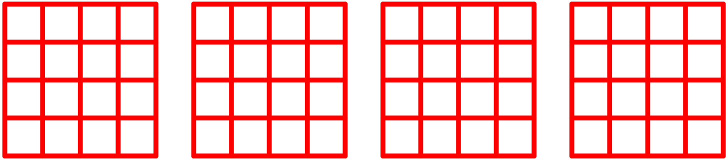

(But Oren Patashnik, with a computer-assisted proof, showed that the first player still has a winning strategy.)

Four Dimensional Tic-Tac-Toe

So let's go to four dimensions.

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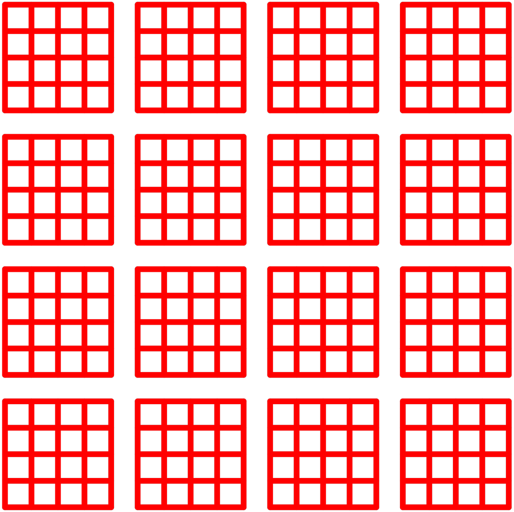

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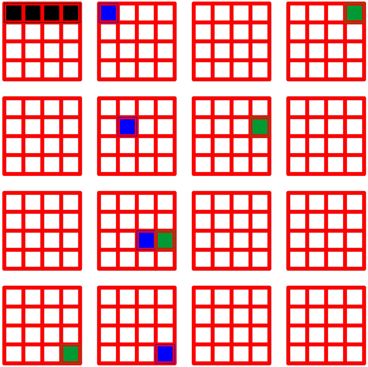


This was one of my early encounters with visualizing four-dimensional objects.

Martin Gardner, *Hexaflexagons and Other Mathematical Diversions*, Chapter 4.

Five Dimensional Tic-Tac-Toe

Now, how about a nice game of five-dimensional, $6 \times 6 \times 6 \times 6 \times 6$, tic-tac-toe?

Wrap-Around Tic-Tac-Toe

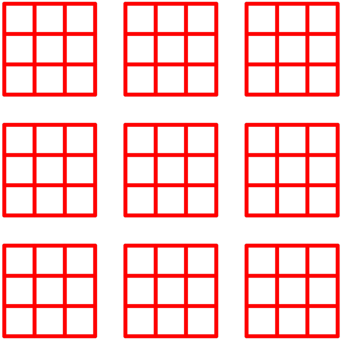
Let's give all squares the same status and power as the center square, so that the center square is no longer so privileged.

	X	
		X
X		

In addition to the ordinary winning moves, include also broken diagonals.

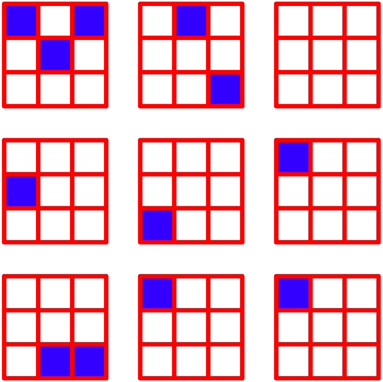
Four Dimensional Wrap-Around Tic-Tac-Toe

Winning combinations of three cells: They are either all in one "major row" of 27 squares or in three different "major rows." The same is true for the "major columns" of 27 squares, the "minor rows" of 27 squares, and the "minor columns" of 27 squares. Now it's probably nearly impossible for the first player to lose!?



Four Dimensional Wrap-Around Tic-Tac-Toe

How many squares can you fill in without creating a winning configuration?



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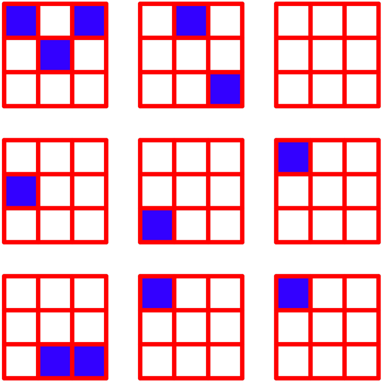
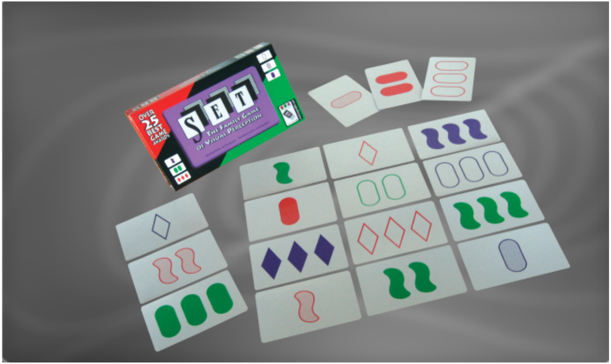


This is a new game: Select, say, 12 squares, and try to find a winning combination. Does this sound like fun?

SET!

SET

It is isomorphic to the game of SET.



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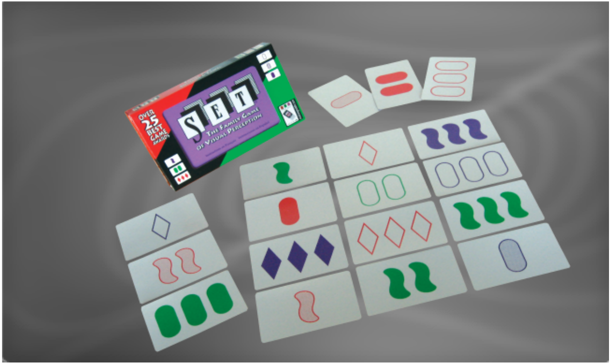
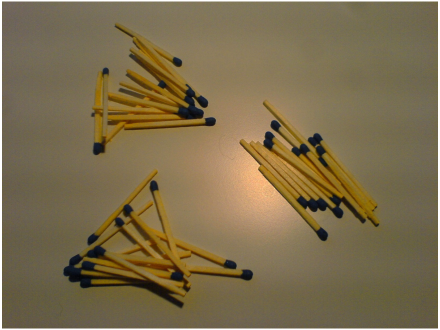


SET opens up nice issues in combinatorics. See, e.g., `http://homepages.warwick.ac.uk/staff/D.Maclagan/papers/set.pdf`. iPad app: `https://itunes.apple.com/us/app/set-pro-hd/id381004916?mt=8`.

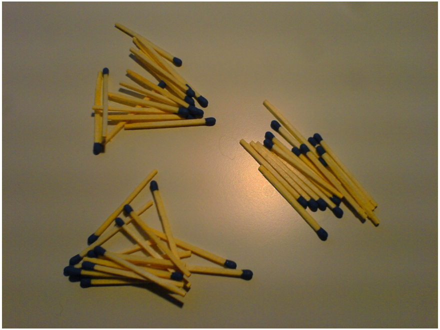
MATCH!

Nim



Two players alternately take a positive number of matches from exactly one of the piles. The player who takes the very last match of all (clears the table) wins.

Nim



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I was introduced to this combinatorial game in high school, and we started with piles of size 3, 5, and 7.
Try it!

Nim

This is an example of a two-person, finite, deterministic game with complete information that cannot end in a tie.

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This is an example of a two-person, finite, deterministic game with complete information that cannot end in a tie.

For such games, there is a winning strategy for either the first player or the second player. A strategy consists in identifying "goal" positions and always ending your turn on a goal position.

(Though in practice, determining goal positions may be very difficult.)

Impartial Games

Nim is also an impartial game, meaning that from any position, both players have the same move options.

How do we characterize these goal positions in impartial games?

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Sprague-Grundy theory. Each position is assigned a nim value, or nimber. Final winning positions are given the nimber 0, and the nimber of every other position is the smallest nonnegative integer that is not a nimber of an immediately succeeding position—this is the mex rule (minimum excluded value).

The goal positions are those with nim value 0.

Nimbers for Nim

Some nimbers for Nim positions:

Pile 1	Pile 2	Pile 3	Nimber
0	0	0	0
0	0	1	1
0	0	2	2
0	0	3	3
0	1	1	0
0	1	2	3
0	1	3	2
0	2	2	0
0	2	3	1
1	1	1	1
1	1	2	2
1	1	3	3
1	2	2	1
1	2	3	0

Nimbers for Nim—Nim Addition

Express the number of matches in each pile in binary, and then sum these binary representations mod 2 for each digit. The result is the binary representation of the position nimber.

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Example: Position 3 5 7

$$\begin{array}{cc} 3 & 011 \\ 5 & 101 \\ 7 & 111 \\ \hline 1 & 001 \end{array}$$

Since the nimber of this position is not zero, this is not a goal position.

Impartial Games

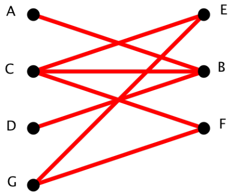
This was one of my early encounters with binary representations, combinatorial games, and recursive definitions, as well as seeing combinatorial problems that are simple to explain but as yet have no solution. It also provides a nice context for proofs by mathematical induction.

Ball and Coxeter.
Martin Gardner, *Hexaflexagons and Other Mathematical Diversions,* Chapter 15; *Mathematical Carnival,* Chapter 16.

The Graph Destruction Game

For many impartial games nimbers may be difficult to determine.

Start with a graph (network). Players alternately erase either a vertex or an edge of the graph. (If a vertex is erased, then all of the edges incident to that vertex must also be erased.) The winner is the player that erases the last remnant of the graph.



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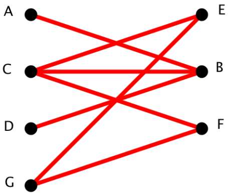
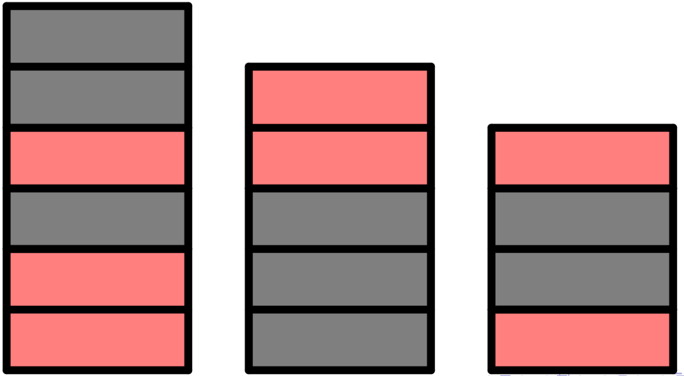


Robert Riehemann (Thomas More College) found a method to compute the nimbers for all bipartite graphs. To the best of my knowledge, the problem is still open for general graphs.

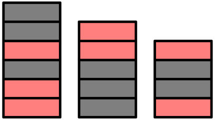
A Partisan Game — Checker Stacks

This partisan (not impartial) game begins with some stacks of red and black checkers. Two players, Red and Black, alternately select a checker of their color from exactly one stack, removing it and all checkers above it. The winner is the player who removes the last checker of all (clears the table).



Checker Stacks

For this game we have numbers instead of nimbers! Assign values to piles in a special way, and then add these values in the ordinary way:



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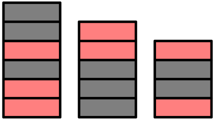


Pile 1: $-2 + \frac{1}{2} - \frac{1}{4} + \frac{1}{8} + \frac{1}{16} = -\frac{25}{16}$

Pile 2: $3 - \frac{1}{2} - \frac{1}{4} = \frac{9}{4}$

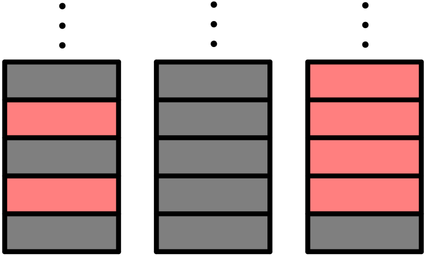
Pile 3: $-1 + \frac{1}{2} + \frac{1}{4} - \frac{1}{8} = -\frac{3}{8}$

Total: $\frac{5}{16}$

If Black (Red) can move to a position with nonnegative (nonpositive) value, Black (Red) can win.

Checker Stacks

What about infinitely tall piles (or infinitely many piles)?



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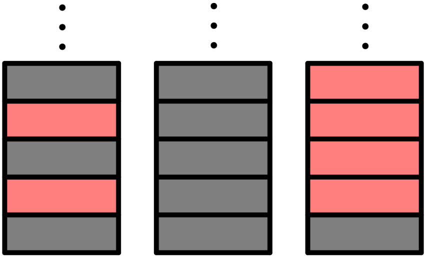

Pile 1: $\frac{2}{3}$

Pile 2:

Checker Stacks

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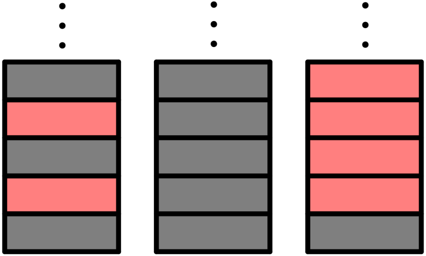

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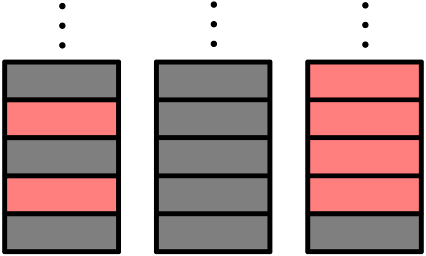

Pile 1: $\frac{2}{3}$

Pile 2: ∞

Pile 3: ε

Surreal Numbers

This serves as an entry point into John Conway's surreal numbers, an extension of the reals including infinite numbers and infinitesimals, which can be further extended to the theory of combinatorial games.

See Donald Knuth, *Surreal Numbers: How Two Ex-Students Turned on to Pure Mathematics and Found Total Happiness.*

Surreal Numbers

Martin Gardner, in *Mathematical Magic Show:* "I believe it is the only time a major mathematical discovery has been published first in a work of fiction." "...It is an astonishing feat of legerdemain. An empty hat rests on a table made of a few axioms of standard set theory. Conway waves two simple rules in the air, then reaches into almost nothing and pulls out an infinitely rich tapestry of numbers that form a real and closed field. Every real number is surrounded by a host of new numbers that lie closer to it than any other "real" value does. The system is truly "surreal.""

John Conway, *On Numbers and Games.*
Berlekamp, Conway, and Guy, *Winning Ways for your Mathematical Plays.*
Gardner, *Penrose Tiles to Trapdoor Ciphers,* Chapter 4.

Adding Games

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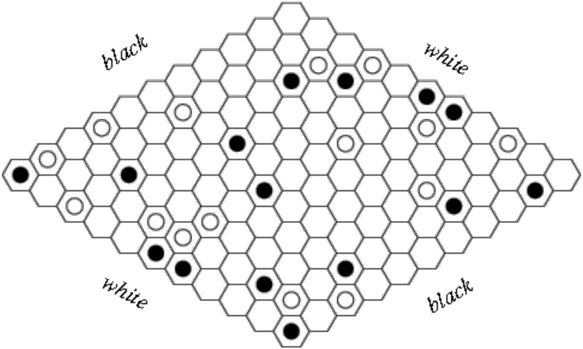
What is "chess minus chess"?

How do you multiply games?

Hex (or Nash)

Players alternately place black and white stones on a diamond-shaped board made of hexagonal tiles. The first player to create a connecting path between the two sides of their color wins.



Online: `http://www.lutanho.net/play/hex.html`
iPad app: `https://itunes.apple.com/gb/app/hex-nash-free/id417433369?mt=8`.

Hex

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Proof: Assume to the contrary that there is a winning strategy for the second player. The first player can then win by the following method. Place a first stone anywhere, ignore its presence, and then play pretending to be the second player, using the second player winning strategy. The extra stone cannot hurt. If later play demands playing in that spot, then mentally acknowledge the presence of the stone, and place a stone somewhere else, ignoring its presence. In this way the first player will win the game, contradicting that the second player can force a win. Therefore the second player does not have a winning strategy, so the first player does.

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But nobody knows what the winning strategy is!

Hex

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Story of playing against Claude Berge. . .

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Story of playing against Claude Berge...

See Gale's paper on "The Game of Hex and Brouwer's Fixed Point Theorem, "
http://www.math.pitt.edu/~gartside/hex_Browuer.pdf.

Misère Hex

This time, the first player to create a path loses!

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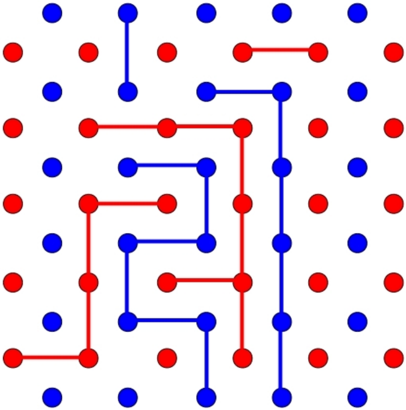
Theorem: The first player has a winning strategy on an $n \times n$ board when n is even, and the second player has a winning strategy when n is odd. Furthermore, the losing player has a strategy that guarantees that every cell of the board must be played before the game ends.

Lagarias and Sleator in Berlekamp and Rodgers, *The Mathemagician and Pied Puzzler.*

Gale

The game of Gale (or Bridg-It) is similar to Hex in that it cannot tie and an identical proof by contradiction justifies the existence of a winning strategy for the first player.



Gale

Unlike Hex, however, an easily described winning strategy is known (Oliver Gross).

Gardner, *New Mathematical Diversions from Scientific American*, Chapter 18.

iPad app for Gale: `https://itunes.apple.com/app/id520227043`

Generalization: Shannon's Switching Game.

Research on Games and Theorems into Games

- Choose or invent a game and try to find the goal positions.
- Turn theorems into games. Example: The six person acquaintanceship theorem.

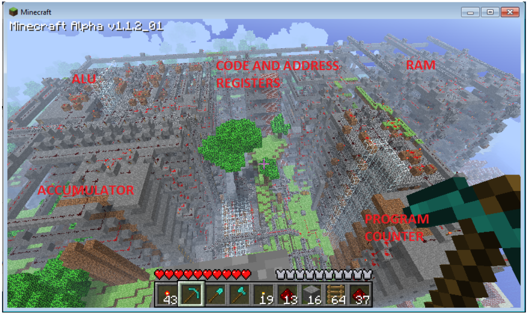
Some More Recent Games

Minecraft
https://minecraft.net .

From the website of this currently popular game: "Minecraft is a game about breaking and placing blocks. At first, people built structures to protect against nocturnal monsters, but as the game grew players worked together to create wonderful, imaginative things."

Minecraft Redstone Computer

You can build digital computers within Minecraft! And many players are learning how to do this. (This is what I would be doing if I were back in high school.)



Minecraft Redstone Computer

Some videos:

http://www.youtube.com/watch?v=_kSnrT75uyk

Tutorial series:
http://www.youtube.com/playlist?list=PLAB22555094B9D077

Someone has even built a computer within Minecraft that itself plays Minecraft!

Games for Research

Game playing is being used as a tool to draw large numbers of people into working on and solving problems connected to scientific research. NPR article:

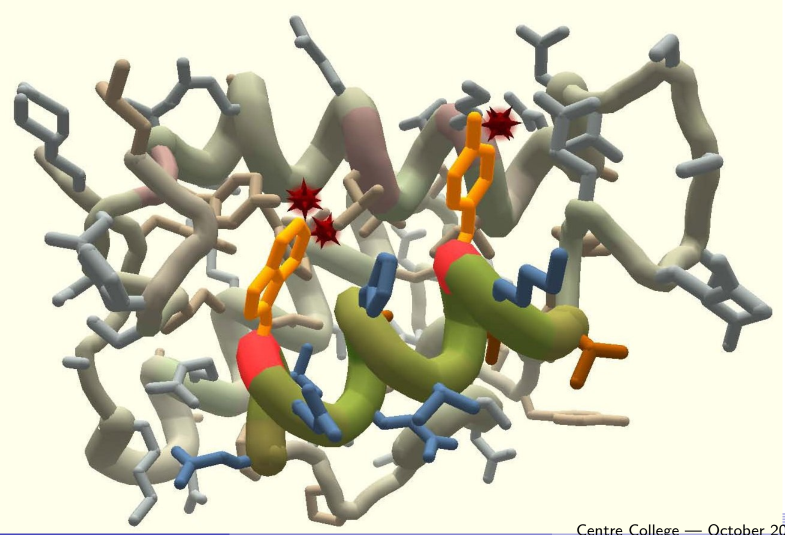
http://www.npr.org/2013/03/05/173435599/wanna-play-computer-gamers-help-push-frontier-of-brain-research

Center for Game Science:
http://www.centerforgamescience.org/site
"The Center for Game Science focuses on solving hard problems facing humanity today in a game based environment. Most of these problems are thus far unsolvable by either people alone or by computer-only approaches. We pursue solutions with a computational and creative symbiosis of humans and computers."

Foldit

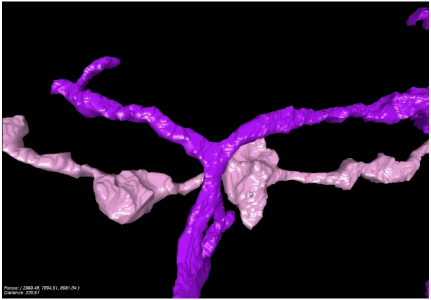
Foldit: `http://fold.it/portal`

Research level protein folding problems.



Eyewire

Eyewire: `http://eyewire.org`



"EyeWire is a game where you map the 3D structure of neurons. By playing EyeWire, you help map the retinal connectome and contribute to the neuroscience research conducted by Sebastian Seung's Computational Neuroscience Lab at MIT. The connectome is a map of all the connections between cells in the brain. Rather than mapping and entire brain, we're starting with a retina."

Galaxy Zoo

Galaxy Zoo: `http://www.galaxyzoo.org`

"To understand how galaxies formed we need your help to classify them according to their shapes. If you're quick, you may even be the first person to see the galaxies you're asked to classify."

What I Have No Time to Tell you About

- Solitaire games and puzzles; the Soma Cube and Conway's Cube
- Logic games; Wff 'n' Proof
- Mathematical magic tricks; the Kruskal Count
- Two-person zero-sum games and linear programming; Kuhn's Poker
- Dynamic programming; the game of Risk
- Multiperson cooperative and noncooperative games; Shapley value
- Etc.

Questions and Challenges

- How anomalous was my experience with mathematics outside the standard curriculum?
- What can (or should) we do to broaden students' experience of the sense and scope of mathematics before (or after) they enter college?
- Games and other areas of recreational mathematics offer portals to some serious mathematics. How can (or should) this be better exploited?

A Few References

- Ball and Coxeter, *Mathematical Recreations and Essays.*
- Berlekamp, Conway, and Guy, *Winning Ways for your Mathematical Plays.*
- Conway, *On Numbers and Games.*
- Cundy and Rollett, *Mathematical Models.*
- Martin Gardner's many books and articles, including *The Colossal Book of Mathematics: Classic Puzzles, Paradoxes, and Problems,* and *Martin Gardner's Mathematical Games* on searchable CD-ROM.
- Holden, *Shapes, Space, and Symmetry.*
- Knuth, *Surreal Numbers.*
- Steinhaus, *Mathematical Snapshots.*

Images

- Game of Set: http://sphotos-a.xx.fbcdn.net/hphotos-snc7/429302_10151045504031090_345331807_n.png
- Nim: http://www.gadial.net/wp-content/uploads/2012/05/nim.jpg
- Hex: http://mathworld.wolfram.com/images/eps-gif/HexGame_1000.gif
- Gale: http://kruzno.com/test2images/turn24b.jpg

Images

- Minecraft Computer: http://jdharper.com/blog/wp-content/uploads/HLIC/135f16156e9269aef638111dae649ede.png
- Foldit: http://msnbcmedia.msn.com/i/MSNBC/Components/Photo/_new/111107-coslog-foldit-1145a.jpg
- Eyewire: http://blog.eyewire.org/wp-content/gallery/image-gallery/synapse-discovered-by-eyewire-gamers.jpg
- Galaxy: http://zoo1.galaxyzoo.org/images/tutorial/example_face_on_spiral.jpg

GAME OVER!

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Thank you!