The g-Theorem

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February 2018

Simplicial Polytopes

How many faces of each dimension can a simplicial convex polytope have?

Landmark book: Grünbaum, *Convex Polytopes*, 1967. New edition with updates in 2003.

Simplicial Complexes

Collection of subsets of a finite set closed under inclusion.

$\emptyset$	1	12	123
	2	13	124
	3	23	134
	4	14	234
	5	24	125
	6	34	135
		15	235
		25	145
		35	245
		45	345
		16	
		26	
		36	

f-vector $f = (f_{-1}, f_0, f_1, f_2) = (1, 6, 13, 10)$.

Simplicial Complexes

To define $8^{(3)}$:

$$8 = \binom{4}{3}$$

Simplicial Complexes

To define $8^{(3)}$:

$$8 = \binom{4}{3} + \binom{3}{2}$$

Simplicial Complexes

To define $8^{(3)}$:

$$8 = \binom{4}{3} + \binom{3}{2} + \binom{1}{1}$$

Simplicial Complexes

To define $8^{(3)}$:

$$8 = \binom{4}{3} + \binom{3}{2} + \binom{1}{1}$$

$$8^{(3)} = \binom{4}{4} + \binom{3}{3} + \binom{1}{2} = 2$$

Also $0^{(0)} = 0$.

Simplicial Complexes

Theorem (Kruskal-Katona, 1963, 1968)

The vector $(f_{-1}, f_0, \ldots, f_{d-1})$ of positive integers is the f-vector of some simplicial $(d-1)$-dimensional complex Δ if and only if

1. $f_{-1} = 1$, *and*
2. $f_j \leq f_{j-1}^{(j)}$, $j = 1, 2, \ldots, d-1$.

Kruskal 1963.
Katona 1968: shorter proof.
Clements-Lindström 1969: generalized the shifting technique.

Simplicial Complexes

Sufficiency: For each j choose the first f_{j-1} j-subsets of $\mathbf{N}$ in co-lex order.

1	6	13	10
$\underline{\emptyset}$	1	12	123
	2	13	124
	3	23	134
	4	14	234
	5	24	125
	$\underline{6}$	34	135
		15	235
		25	145
		35	245
		45	$\underline{345}$
		16	126
		26	136
		$\underline{36}$	236
		46	146
		56	246
			346
			156
			256
			356
			456

Simplicial Complexes

Necessity: Given a simplicial complex. By application of a certain sequence of “shifting” or “compression” operations, transform it to a co-lex simplicial complex with the same f-vector. Then verify that the conditions must hold.

Dehn-Sommerville Equations

What about simplicial complexes that are the boundaries of simplicial convex polytopes?

$$f = (1, 10, 43, 102, 141, 108, 36)$$

$$f(t) = 1 + 10t + 43t^2 + 102t^3 + 141t^4 + 108t^5 + 36t^6$$

$$h(t) = (1-t)^6 f(\frac{t}{1-t}) = 1 + 4t + 8t^2 + 10t^3 + 8t^4 + 4t^5 + t^6$$

$$h = (1, 4, 8, 10, 8, 4, 1) = (h_0, \ldots, h_6)$$

This is the *h*-vector.

$$f(t) = (1+t)^d h(\frac{t}{1+t})$$

So knowing h is equivalent to knowing f.

Dehn-Sommerville Equations

Theorem (Dehn-Sommerville, 1905, 1927)

For a simplicial d-polytope, $h_i = h_{d-i}$, $i = 0, \ldots, \lfloor d/2 \rfloor$.

Dehn 1905: $d = 4$.
Sommerville 1927: general d.
Klee 1964: rediscovered but not formulated this way.
McMullen 1971: formulated them this way (with an index shift) and recognized the connection with shelling.
They hold also for simplicial homology spheres.

Dehn-Sommerville Equations

For a simplicial ball Δ, $h(\Delta)$ determines $h(\partial\Delta)$.
Let $\Sigma = \Delta \cup (v \cdot \partial\Delta)$. Note that $\partial\Delta$ and Σ are spheres.

$$
\begin{array}{rccccc}
h(\Delta) & 1 & 2 & 1 & 1 & 0 \\
+h(\partial\Delta) & & \cdot & \cdot & \cdot & \cdot \\
\hline
= h(\Sigma) & \cdot & \cdot & \cdot & \cdot & \cdot
\end{array}
$$

Dehn-Sommerville Equations

For a simplicial ball Δ, $h(\Delta)$ determines $h(\partial\Delta)$.
Let $\Sigma = \Delta \cup (v \cdot \partial\Delta)$. Note that $\partial\Delta$ and Σ are spheres.

$h(\Delta)$	1	2	1	1	0
$+h(\partial\Delta)$		·	·	·	·
$= h(\Sigma)$	·	·	·	·	·

$h(\Delta)$	1	2	1	1	0
$+h(\partial\Delta)$		1	2	2	1
$= h(\Sigma)$	1	3	3	3	1

McMullen-Walkup 1971.

Upper Bound Theorem

Theorem (Bruggesser-Mani, 1970)

The boundaries of convex polytopes are shellable.

Often implicitly assumed by early incomplete proofs of Euler's relation, pre-1900, pre-Poincaré.

Upper Bound Theorem

Shellings of simplicial polytopes. Facets (maximal faces) are ordered in such a way that among the new faces contributed by each new facet there is a unique minimal new face.

facet			type
1	2	5	0
2	3	5	1
3	4	5	1
1	4	5	2
1	2	6	1
2	3	6	2
3	4	6	2
1	4	6	3

h_i equals the number of facets of type i.
$h = (1, 3, 3, 1)$.
Reversible shellings imply the Dehn-Sommerville equations.

Upper Bound Theorem

A facet of type i contributes a Boolean algebra of faces, changing $f(t)$ by adding

$$(1+t)^{d-i}t^i = (1-t)^d(\frac{t}{1-t})^i$$

McMullen 1970.

Upper Bound Theorem

Cyclic polytope $C(n, d)$: the convex hull of any set of n distinct points on the moment curve $m(t) = (t, t^2, \ldots, t^d)$.

Theorem (Upper Bound Theorem, McMullen, 1970)

$f_j(P) \leq f_j(C(n, d))$, $j = 0, \ldots, d-1$, for all convex d-polytopes P with n vertices.

"Conjectured" by Motzkin in 1957.
McMullen used an h-vector reformulation, and shelling, observing that the Dehn-Sommerville equations are a consequence of the reversibility of shelling orders.
Proof discovered while McMullen and Shephard were writing the book *The Upper Bound Conjecture*. They did not change the title of the book.

Upper Bound Theorem

Gale's Evenness Condition. $v_i = m(t_i)$, $t_1 < \cdots < t_n$.

Facets of $C(8,5)$:

1	2	3	4	5	6	7	8
1	2	3	4	5			
1	2	3		5	6		
1		3	4	5	6		
1	2	3			6	7	
1		3	4		6	7	
1			4	5	6	7	
1	2	3				7	8
1		3	4			7	8
1			4	5		7	8
1				5	6	7	8
1	2	3	4				8
1	2		4	5			8
	2	3	4	5			8
1	2			5	6		8
	2	3		5	6		8
		3	4	5	6		8
1	2				6	7	8
	2	3			6	7	8
		3	4		6	7	8
			4	5	6	7	8

Upper Bound Theorem

Facet hyperplane for $\{m(t_{i_1}), \ldots, m(t_{i_d})\}$.

$$(t - t_{i_1}) \cdots (t - t_{i_d}) = a_0 + a_1 t + \cdots + a_d t^d$$

yields the hyperplane

$$a_1 x_1 + \cdots + a_d x_d = -a_0.$$

McMullen's Conjecture

Bold conjecture made in 1971.

To define $8^{<3>}$:

$$8 = \binom{4}{3} + \binom{3}{2} + \binom{1}{1}$$

McMullen's Conjecture

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To define $8^{<3>}$:

$$8 = \binom{4}{3} + \binom{3}{2} + \binom{1}{1}$$

$$8^{<3>} = \binom{5}{4} + \binom{4}{3} + \binom{2}{2} = 10$$

Also $0^{<0>} = 0$.

McMullen's Conjecture

The vector $(h_0, h_1, \ldots, h_d)$ of positive integers is the h-vector of some simplicial d-dimensional convex polytope if and only if

1. $h_0 = 1$,
2. $h_i = h_{d-i}$, $i = 0, \ldots \lfloor (d-1)/2 \rfloor$, and
3. $g_{i+1} \leq g_i^{<i>}$, $i = 1, 2, \ldots, \lfloor d/2 \rfloor - 1$,

where $g_0 = 1$ and $g_i = h_i - h_{i-1}$, $i = 0, \ldots \lfloor d/2 \rfloor$.

Example:

$$h = (1, 4, 8, 10, 8, 4, 1)$$

$$g = (1, 3, 4, 2)$$

McMullen proved it for $d \leq 5$ and also for $f_0 \leq d + 3$ (the latter using Gale diagrams).

M-Vectors

Order ideal of monomials: Collection of monomials over a finite set of variables, closed under divisor.

$$\begin{array}{llll} 1 & x_1 & x_1^2 & x_2x_3^2 \\ & x_2 & x_2^2 & x_3^3 \\ & x_3 & x_3^2 & \\ & & x_2x_3 & \end{array}$$

M-vector $= (1, 3, 4, 2)$, counting number of monomials of each degree.

M-Vectors

Theorem (Stanley, 1975)

The vector of nonnegative integers, $(h_0, h_1, \ldots, h_d)$, is an M-vector if and only if

1. $h_0 = 1$, *and*
2. $h_{i+1} \leq h_i^{<i>}$, $i = 1, 2, \ldots, d$.

Stanley 1975, using a result of Macauley 1927.

M-Vectors

Sufficiency: For each i choose the first h_i monomials of degree i in co-lex order.

1	3	4	2
$\underline{1}$	x_1	x_1^2	x_1^3
	x_2	$x_1 x_2$	$\underline{x_1^2 x_2}$
	$\underline{x_3}$	x_2^2	$x_1 x_2^2$
		$\underline{x_1 x_3}$	x_2^3
		$x_2 x_3$	$x_1^2 x_3$
		x_3^2	$x_1 x_2 x_3$
			$x_2^2 x_3$
			$x_1 x_3^2$
			$x_2 x_3^2$
			x_3^3

M-Vectors

Necessity: Given an order ideal of monomials. By application of a certain sequence of "shifting" or "compression" operations, transform it to a co-lex order of monomials with the same M-vector. Then verify that the conditions must hold.

Clements-Lindström 1969: generalized the shifting technique.

Shellable Simplicial Complexes

Theorem (Stanley 1975)

The vector of nonnegative integers, $h = (h_0, h_1, \ldots, h_d)$, is the h-vector of some shellable simplicial $(d-1)$-complex if and only if it is an M-vector.

Shellable Simplicial Complexes

Sufficiency: Not published by Stanley in 1975 that I could find, so below is the proof I came up with.

List all monomials in h_1 variables of degree at most d in co-lex order. Next to these, list all cardinality d subsets of $\{1, \ldots, h_1 + d\}$, also in co-lex order.

Select the co-lex order ideal of monomials associated with h. Select the associated subsets. These will be the facets of the desired simplicial complex, the order is a shelling order, and the type of each facet is the degree of the associated monomial.

Shellable Simplicial Complexes

Example: $h = (1, 3, 4, 2)$.

monomial	degree	1	2	3	4	5	6
1	$*0$	1	2	3			
x_1	$*1$	1	2		4		
x_1^2	$*2$	1		3	4		
x_1^3	$*3$		2	3	4		
x_2	$*1$	1	2			5	
x_1x_2	$*2$	1		3		5	
$x_1^2x_2$	$*3$		2	3		5	
x_2^2	$*2$	1			4	5	
$x_1x_2^2$	3		2		4	5	
x_2^3	3			3	4	5	
x_3	$*1$	1	2				6
x_1x_3	$*2$	1		3			6
$x_1^2x_3$	3		2	3			6
x_2x_3	2	1			4		6
$x_1x_2x_3$	3		2		4		6
$x_2^2x_3$	3			3	4		6
x_3^2	2	1				5	6
$x_1x_3^2$	3		2			5	6
$x_2x_3^2$	3			3		5	6
x_3^3	3				4	5	6

Shellable Simplicial Complexes

Necessity.

Let Δ be a simplicial $(d-1)$-complex with n vertices $1, \ldots, n$.
Consider the polynomial ring $R = \mathbf{R}[x_1, \ldots, x_n]$.
For a monomial $m = x_1^{a_1} \cdots x_n^{a_n}$ in R the support of m is $\text{supp}(m) = \{i : a_i > 0\}$. Let I be the ideal of R generated by square-free monomials m such that $\text{supp}(m) \notin \Delta$.
The Stanley-Reisner ring or face ring of Δ is $A_\Delta := R/I$. There is a natural grading of $A = A_0 \oplus A_1 \oplus A_2 \oplus \cdots$ by degree.
Informally, we do calculations as in R but set any monomial to zero whose support does not correspond to a face.
HIlbert series of A_Δ:

$$\sum_{i=0}^{\infty} \dim A_i t^i = f\left(\frac{t}{1-t}\right)$$

Shellable Simplicial Complexes

Stanley proved that if Δ is shellable, then there exist d elements $\theta_1, \ldots, \theta_d \in A_1$ (a homogeneous system of parameters) such that θ_i is not a zero-divisor in $A_\Delta/(\theta_1, \ldots, \theta_{i-1})$, $i = 1, \ldots, d$. (I.e., multiplication by θ_i in $A_\Delta/(\theta_1, \ldots, \theta_{i-1})$ is an injection.) Equivalently, A is Cohen-Macaulay.

Let $B = A_\Delta/(\theta_1, \ldots, \theta_d) = B_0 \oplus B_1 \oplus \cdots \oplus B_d$. Then

$$\sum \dim B_i t^i = (1-t)^d f(\frac{t}{1-t}) = h(t).$$

So $\dim B_i = h_i$, $i = 0, \ldots, d$.

Macaulay proved that there exists a basis for B as an **R**-vector space that is an order ideal of monomials. Therefore h is an M-vector.

Shellable Simplicial Complexes

Kind-Kleinschmidt 1979: Another proof that shellable complexes are Cohen-Macaulay. (In German—my language exam.)

Stanley 1975: Proved simplicial spheres, shellable or not, are Cohen-Macaulay, using a homological characterization of Cohen-Macaulay complexes (see also Reisner 1976), and extended the Upper Bound Theorem to them.

Polytope Pairs

What is the maximum value of f_j for convex d-polytopes with n vertices, one of which has degree exactly k?

Klee 1975: Derived some bounds including a construction placing a new point outside of $C(n-1, d)$ and taking the convex hull.

Billera 1978: Suggested using approaching this problem in light of Stanley's work.

L. 1978–79: Solution, plus an idea. . .

The g-Theorem

Theorem (Billera-L 1981, Stanley 1980)

McMullen's conjecture is true.

Comments on the letter "g".
Sufficiency: Billera-L.
Necessity: Stanley.

The g-Theorem

Sufficiency.
Given $h = (h_0, \ldots, h_d)$ satisfying McMullen's conditions:

1. $h_0 = 1$,
2. $h_i = h_{d-i}$, $i = 0, \ldots \lfloor (d-1)/2 \rfloor$, and
3. $g_{i+1} \leq g_i^{<i>}$, $i = 1, 2, \ldots, \lfloor d/2 \rfloor - 1$,

where $g_0 = 1$ and $g_i = h_i - h_{i-1}$, $i = 0, \ldots \lfloor d/2 \rfloor$.

Example:

$$h = (1, 4, 6, 4, 1)$$

$$g = (1, 3, 2)$$

The g-Theorem

List all monomials in g_1 variables of degree at most $\lfloor d/2 \rfloor$ in co-lex order. Next to these, list all facets of $C(f_0, d+1)$ containing v_1 and having even right-end set, also in co-lex order. ($f_0 = h_1 + d$).

$$h = (1,4,6,4,1),\ g = (1,3,2)$$

monomial	degree	1	2	3	4	5	6	7	8
1	0	1	2	3	4	5			
x_1	1	1	2	3		5	6		
x_1^2	2	1		3	4	5	6		
x_2	1	1	2	3			6	7	
x_1x_2	2	1		3	4		6	7	
x_2^2	2	1			4	5	6		
x_3	1	1	2	3				7	8
x_1x_3	2	1		3	4			7	8
x_2x_3	2	1			4	5		7	8
x_3^2	2	1				5	6	7	8

The g-Theorem

Select the co-lex order ideal of monomials associated with g. Select the associated subsets. These will be the facets of a shellable simplicial complex, the order is a shelling order, and the type of each facet is the degree of the associated monomial. Example: $g = (1, 3, 2)$.

monomial	degree	1	2	3	4	5	6	7	8
1	$*0$	1	2	3	4	5			
x_1	$*1$	1	2	3		5	6		
x_1^2	$*2$	1		3	4	5	6		
x_2	$*1$	1	2	3			6	7	
x_1x_2	$*2$	1		3	4		6	7	
x_2^2	2	1			4	5	6		
x_3	$*1$	1	2	3				7	8
x_1x_3	2	1		3	4			7	8
x_2x_3	2	1			4	5		7	8
x_3^2	2	1				5	6	7	8

The g-Theorem

The resulting simplicial complex, Δ, a "patch" on the boundary of $C(f_0, d+1)$, is a simplicial d-ball with h-vector equal to g padded with a final string of 0's.

Use the "boundary calculation" to determine the h-vector of its boundary, $\partial\Delta$.

$$\begin{array}{cccccccc} & h(\Delta) & 1 & 3 & 2 & 0 & 0 & 0 \\ + & h(\partial\Delta) & & \cdot & \cdot & \cdot & \cdot & \cdot \\ \hline = & h(\Sigma) & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \end{array}$$

The g-Theorem

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$$\begin{array}{cc|cccccc} & h(\Delta) & 1 & 3 & 2 & 0 & 0 & 0 \\ + & h(\partial\Delta) & & 1 & 4 & 6 & 4 & 1 \\ \hline = & h(\Sigma) & 1 & 4 & 6 & 6 & 4 & 1 \end{array}$$

The g-Theorem

Using indeterminate t_i for the points on the moment curve and the cyclic polytope facet equations, carefully select a new point z outside of $C(f_0, d+1)$ and determine inequalities that must hold for Δ to be precisely visible from z. Then show that one can choose specific values of t_i. This part of the proof explicitly references the order ideal of monomials and facet selection. (This is the hardest part of the proof.)

Take the convex hull Q of $C(f_0, d+1)$ and z, and let P be a vertex-figure of z—the intersection of Q and a hyperplane separating z from the other vertices. Then $h(P) = h(\partial\Delta)$.

The g-Theorem

Necessity.
Recall the ring $B = B_0 \oplus B_1 \oplus \cdots \oplus B_d$ with Hilbert series

$$h_0 + h_1 t + \cdots + h_d t^d.$$

The Hard Lefschetz Theorem implies there is an element $\omega \in B_1$ such that multiplication by ω^{d-2i} is a bijection from B_i to B_{d-i}, $i = 0, \ldots, \lfloor d/2 \rfloor$, and so ω is not a zero divisor in $B_0 \oplus B_1 \oplus \cdots \oplus B_{\lfloor d/2 \rfloor - 1}$.
Thus the Hilbert series for $B/(\omega) = C_0 \oplus C_1 \oplus \cdots \oplus C_{\lfloor d/2 \rfloor}$ is

$$g_0 + g_1 t + \cdots g_{\lfloor d/2 \rfloor} t^{\lfloor d/2 \rfloor}$$

(Multiply first half of $h_0 + h_1 t + \cdots + h_d t^d$ by $(1 - t)$.)
By Macaulay there is a basis for C that is an order ideal of monomials. Therefore g is an M-vector.

The g-Theorem

McMullen 1993 and 1996: New proof of necessity using weights and his polytope algebra.

Some Reflections