Midterm Exam

Instructions: You may use your course notes and other outside materials, but you must properly cite your sources. You may not consult each other or any other individuals, though you may ask me questions if some point needs clarification. Your responses are due by 11 pm, Sunday, March 19, either uploaded in a single file to moodle, emailed to mathhist@ms.uky.edu, or sent by physical mail postmarked by the deadline. Justify all of your answers.

1. A polynomial equation is an equation of the form $p(x) = 0$, where $p(x)$ is a polynomial. Consider the sets $\mathbf{W}$ (whole numbers, $0, 1, 2, \ldots$), $\mathbf{Z}$ (integers), $\mathbf{Q}$ (rational numbers), $\mathbf{R}$ (real numbers), and $\mathbf{C}$ (complex numbers). Each set contains the previous one.

   (a) Write a polynomial equation with whole number coefficients that has no whole number solution, but has an integer solution.

   (b) Write a polynomial equation with integer coefficients that has no integer solution, but has a rational solution.

   (c) Write a polynomial equation with rational coefficients that has no rational solution, but has real solution.

   (d) Write a polynomial equation with real coefficients that has no real solution, but has a complex solution.

   (e) Each of the above cases provides motivation for enlarging the sets. For each of the above four cases, give one specific example from the history of mathematics that illustrates how mathematicians were motivated to consider and accept these enlargements. (Though technically these were not really considered as “sets” yet.)

   (f) What result in the history of mathematics shows that we cannot write a polynomial equation with complex coefficients that has no complex solution? Who is responsible for this result?

   (g) Find all complex solutions to the equation $x^3 = i$.

2. Several times we have seen mathematics relating and connecting algebraic and geometric ideas.

   (a) In each of the following cases, provide one specific example from the history of mathematics as an illustration.

      i. The formulas for the areas of various types of polygons can be obtained by geometric dissection.

ii. Algebraic formulas can be derived from geometric diagrams.

iii. Good numerical approximations to the number π can be found by geometric reasoning.

iv. The solutions to certain algebraic equations can be found by geometrical arguments.

v. Certain areas can be obtained by calculus that had not been found by pre-calculus methods.

(b) Here is an example of deriving a volume formula by geometric dissection. Consider a regular tetrahedron T with side length $2a$. At each corner slice off a regular tetrahedron of side length a. What is left is an octahedron P with side length a. Use this dissection to determine the volume of P in terms of a. (Extra Credit: Can you find a reference to this construction in the history of mathematics? I don't know where to look.)

3. For each of the seven "Great Theorems" we have seen so far in the book by Dunham, make your best estimate as to where they are likely first to appear in the K–16 curriculum as a *statement without proof*, and where they are likely to be first *proved* (if at all).