Counting Faces of Polytopes

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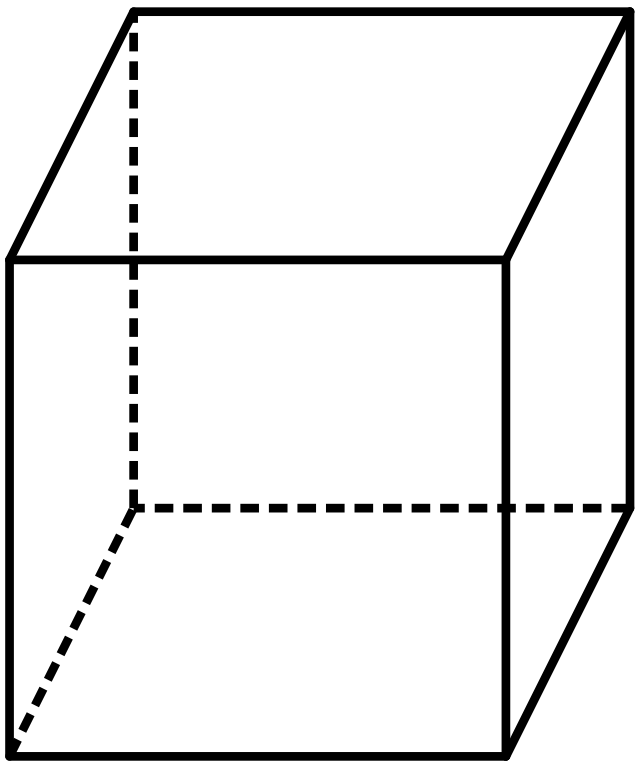
James Madison University—March 2015

Convex Polytopes

A convex polytope P is the convex hull of a finite set of points in $\mathbf{R}^d$.

Example: Cube



Face-Vectors

The face-vector of a d-dimensional polytope is $f = (f_0, f_1, \ldots, f_{d-1})$, where f_j is the number of faces of dimension j. Define also $f_{-1} = f_d = 1$.

Example:

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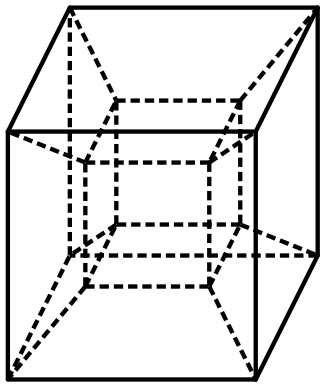

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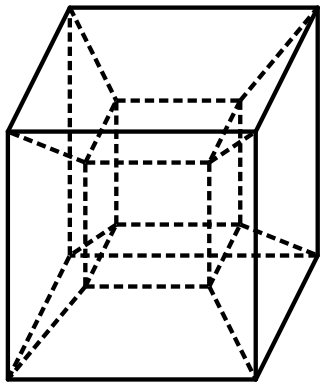

Question: What are the possible face-vectors of polytopes?

Three-Dimensional Polytopes

Theorem (Euler's Relation)

$f_0 - f_1 + f_2 = 2$ for convex 3-polytopes.

Example: Cube. $8 - 12 + 6 = 2$.

Three-Dimensional Polytopes

Sketch of proof: Sweep the polytope with a plane in general direction. (Think of immersing in water.) Count vertices, edges, and polygons only when fully swept (under water). Watch how $\chi = f_0 - f_1 + f_2$ changes when the plane hits each vertex.

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Note: This proof technique generalizes to higher dimensions.

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What else?

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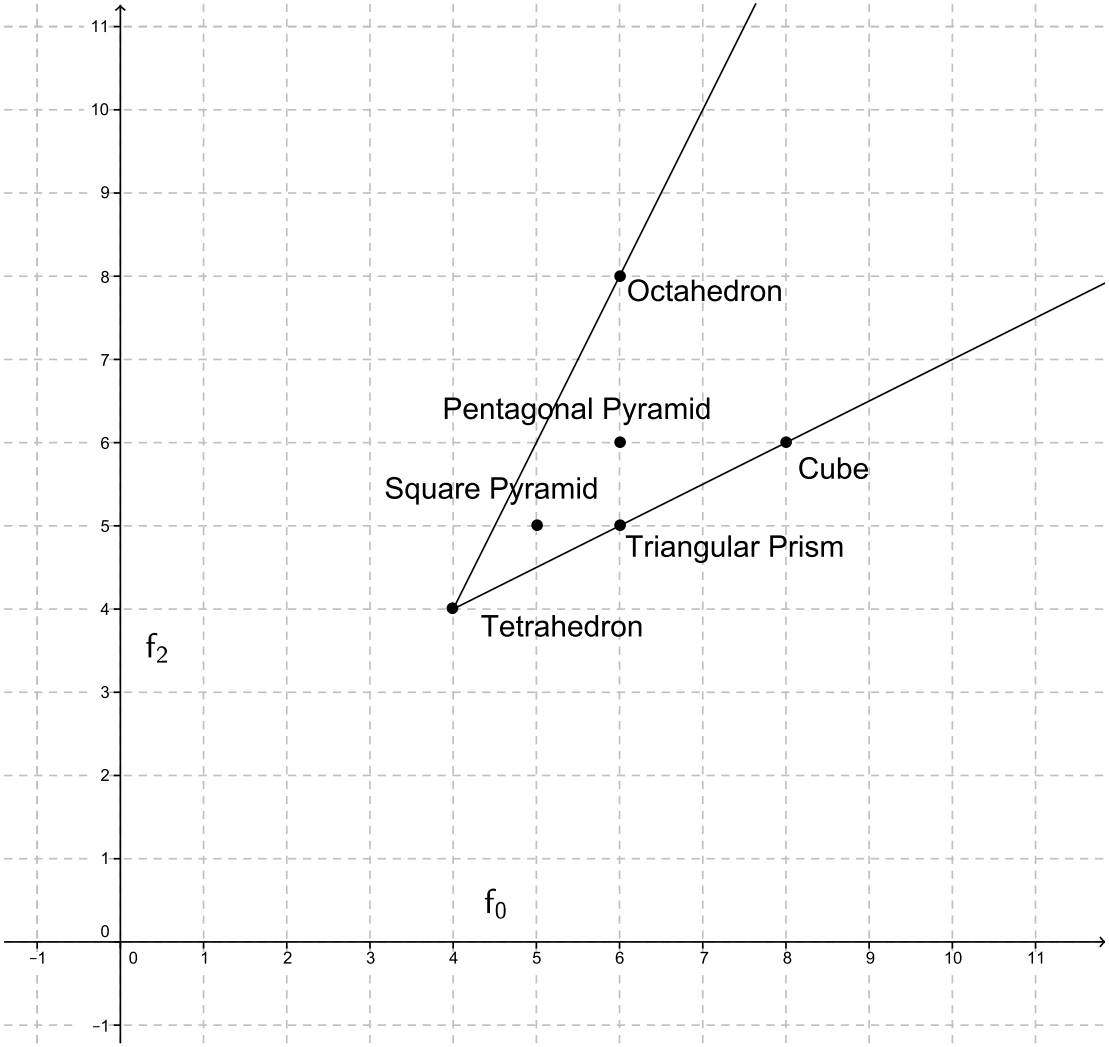
Theorem (Steinitz)

A positive integer vector (f_0, f_1, f_2) is the face-vector of a 3-polytope if and only if the following conditions hold.

- $f_0 - f_1 + f_2 = 2$,
- $f_0 \leq 2f_2 - 4$, *and*
- $f_2 \leq 2f_0 - 4$.

Three-Dimensional Polytopes



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But there are some partial results.

d-Dimensional Polytopes

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Grünbaum developed a "sweeping-like" proof.

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A facet of a d-polytope is a face of dimension $d-1$.

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Theorem (Upper Bound Theorem, McMullen)

The dual to the cyclic d-polytope with n vertices has the largest number of faces of all dimensions of any d-polytope with n facets.

Four-Dimensional Polytopes

Barnette characterized the sets of various pairs of components of (f_0, f_1, f_2, f_3).

Simple Polytopes

A d-polytope is simple if every vertex is incident to precisely d edges.

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Example: The cube, (as well as the d-cube for all d).

Simple Polytopes

Theorem (Lower Bound Theorem, Barnette)

The "truncation polytope" of dimension d with n facets has the smallest number of faces of all dimensions of any simple d-polytope with n facets.

h-Vectors

Sweep a simple d-polytope with a hyperplane in general direction.

Orient all edges in the direction of the sweep.

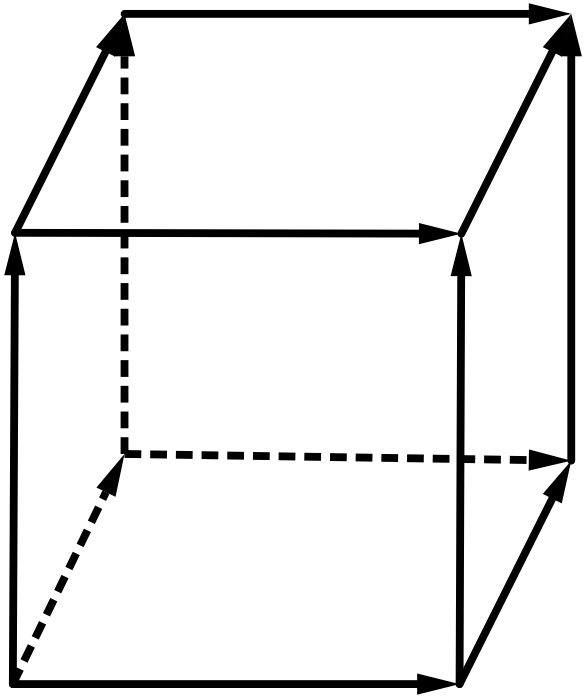
Let h_i be the number of vertices of indegree i.

The h-vector is $(h_0, h_1, \ldots, h_d)$.

Note that it is a nonnegative vector of integers.

h-Vectors

Example: Cube



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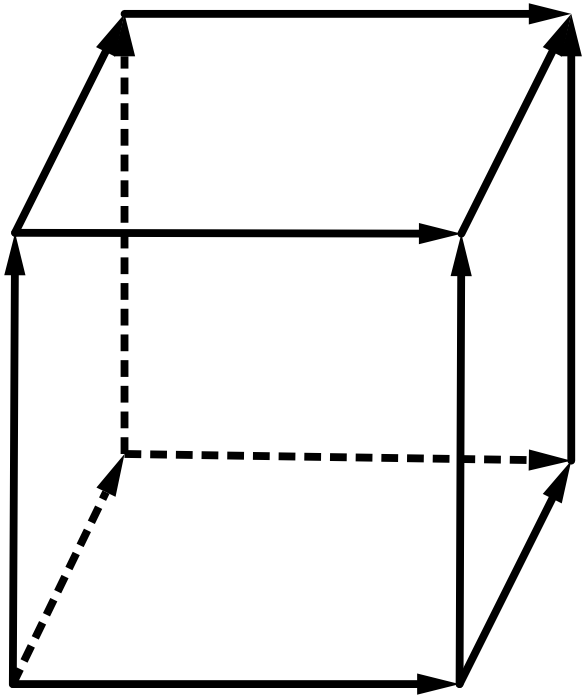

$h = (1, 3, 3, 1)$

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Idea: Every subset of j incoming edges to a vertex corresponds to an i-face.

h-Vectors

These relations are invertible.

Theorem (McMullen)

$$h_i = \sum_{j=i}^{d} (-1)^{i+j} \binom{j}{i} f_j, \;\; i = 0, \ldots, d$$

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Theorem (McMullen)

$$h_i = \sum_{j=i}^{d} (-1)^{i+j} \binom{j}{i} f_j, \;\; i = 0, \ldots, d$$

This implies that $(h_0, \ldots, h_d)$ is independent of the choice of hyperplane!

h-Vectors

"Stanley's trick" to convert from the face-factor to the h-vector.
Consider the face-vector $(36, 108, 141, 102, 43, 10)$.

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						1						
					1		10					
				1		9		43				
			1		8		34		102			
		1		7		26		68		141		
	1		6		19		42		73		108	
1		5		13		23		31		35		36
1	4		8		10		8		4		1	

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Reversing the sweep reverses the directions of all edges, and so swaps indegree with outdegree for each vertex.

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Theorem (Dehn-Sommerville Relations)

For every simple d-polytope, $h_i = h_{d-i}$ for all i.

Besides being a nonnegative symmetric vector of integers, what other conditions must hold for the h-vector?

Canonical Representations

For positive integers a and i, a can be written uniquely in the form

$$a = \binom{a_i}{i} + \binom{a_{i-1}}{i-1} + \cdots + \binom{a_j}{j}$$

where $a_i > a_{i-1} > \cdots > a_j \geq j \geq 1$. This is the *i*-canonical representation of *a*.

For example, the 4-canonical representation of 26 is

$$26 = \binom{6}{4} + \binom{5}{3} + \binom{2}{2}.$$

Canonical Representations

Now define $a^{<i>}$ by adding one to the top and bottom of every binomial coefficient in the i-canonical representation of a.

$$a^{<i>} = \binom{a_i + 1}{i + 1} + \binom{a_{i-1} + 1}{i} + \cdots + \binom{a_j + 1}{j + 1}$$

For example,

$$26^{<4>} = \binom{7}{5} + \binom{6}{4} + \binom{3}{3} = 37.$$

Define also $a^{<0>} = 0$.

g-Theorem

For any symmetric vector $(h_0, h_1, \ldots, h_d)$ define $g_0 = h_0$ and $g_i = h_i - h_{i-1}$, $i = 1, \ldots, \lfloor d/2 \rfloor$. (That is to say, compute differences up to half way.)

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Theorem (g-Theorem, Billera-L-Stanley, conjectured by McMullen)

A vector $(h_0, h_1, \ldots, h_d)$ of positive integers is the h-vector of a simple d-polytope if and only if the following conditions hold.

- $h_i = h_{d-1}$, $i = 0, \ldots, d$,
- $g_i \geq 0$, $i = 0, 1, \ldots, \lfloor d/2 \rfloor$, *and*
- $g_0 = 1$ *and* $g_{i+1} \leq g_i^{<i>}$ *for all* $i = 1, 2, \ldots \lfloor d/2 \rfloor - 1$.

g-Theorem

For example, if we consider the potential face-vector $f = (36, 108, 141, 102, 43, 10)$ we compute $h = (1, 4, 8, 10, 8, 4, 1)$ and $g = (1, 3, 4, 2)$.

Now h is symmetric, g is nonnegative, $g_0 = 1$, $4 \leq 3^{<1>}$, and $2 \leq 4^{<2>}$, so this is a valid face-vector for a simple polytope.

g-Theorem

The necessity of the conditions comes from considering a certain graded ring associated with the simple polytope (the Stanley-Reisner ring), and its relationship to the cohomology of a certain complex projective toric variety for which the hard Lefschetz Theorem holds. (McMullen later provided a more geometric proof using his "polytope algebra".)

The sufficiency of the conditions comes from a direct construction.

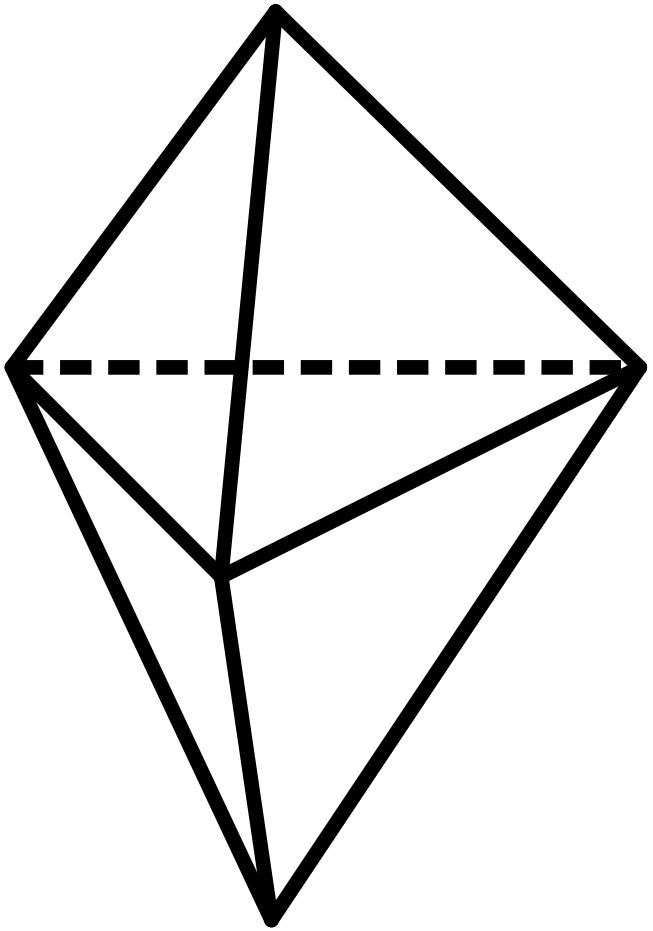
Nonsimple Polytopes

It is fruitful to look beyond the face-vectors for nonsimple polytopes, and consider flag-vectors, which count chains of faces of various types and lengths.

Flag f-Vector and **cd**-Index

Fine; Bayer-Klapper

Example: Triangular bipyramid P.



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Example: Triangular bipyramid P.
f_S = numbers of chains of faces of type S.

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$\emptyset$						
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1						
2						
01						
02						
12						
012						

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$$\mathbf{c} = \mathbf{a} + \mathbf{b} \text{ and } \mathbf{d} = \mathbf{ab} + \mathbf{ba}$$

$$\Phi(P) = \mathbf{c}^3 + 4\mathbf{cd} + 3\mathbf{dc}.$$

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We still cannot characterize the set of all **cd**-indices.

Four-Dimensional Polytopes

What we know about 4-polytopes besides the Euler-Poincaré relation:

- $f_{02} - 3f_2 \geq 0$
- $f_{02} - 3f_1 \geq 0$
- $f_{02} - 3f_2 + f_1 - 4f_0 + 10 \geq 0$
- $6f_1 - 6f_0 - f_{02} \geq 0$
- $f_0 - 5 \geq 0$
- $f_2 - f_1 + f_0 \geq 0$
- $2(f_{02} - 3f_2) + f_1 \leq \binom{f_0}{2}$
- $2(f_{02} - 3f_1) + f_2 \leq \binom{f_2 - f_1 + f_0}{2}$
- $f_{02} - 4f_2 + 3f_1 - 2f_0 \leq \binom{f_0}{2}$
- $f_{02} + f_2 - 2f_1 - 2f_0 \leq \binom{f_2 - f_1 + f_0}{2}$

Four-Dimensional Polytopes

But there are still some huge gaps in what we know to be true about flag-vectors and the 4-polytopes we know how to construct.

What I am Skipping About Counting Faces

- Shellings
- Winding numbers in Gale transforms
- Dimensions of stress and motion spaces
- Formulas for volumes of polytopes
- The toric h-vector
- Hopf algebras
- Much more...

Thank you!