Counting Faces of Polytopes

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Convex Polytopes

A convex polytope $P$ is the convex hull of a finite set of points in $\mathbb{R}^d$.

Example: Cube
Face-Vectors

The face-vector of a $d$-dimensional polytope is $f = (f_0, f_1, \ldots, f_{d-1})$, where $f_j$ is the number of faces of dimension $j$. Define also $f_{-1} = f_d = 1$.

Example:
- Cube. $f = (8, 12, 6)$.
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- Cube. $f = (8, 12, 6)$.
- 4-Cube. $f = (16, 32, 24, 8)$. 
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- 4-Cube. $f = (16, 32, 24, 8)$.

Question: What are the possible face-vectors of polytopes?
Three-Dimensional Polytopes

Theorem (Euler’s Relation)

\[f_0 - f_1 + f_2 = 2\] for convex 3-polytopes.

Example: Cube. \(8 - 12 + 6 = 2\).
Three-Dimensional Polytopes

Sketch of proof: Sweep the polytope with a plane in general direction. (Think of immersing in water.) Count vertices, edges, and polygons only when fully swept (under water). Watch how \( \chi = f_0 - f_1 + f_2 \) changes when the plane hits each vertex.
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- Initially $\chi = 0$. 

Note: This proof technique generalizes to higher dimensions.
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- Intermediate vertex with $k$ incident lower edges. $\chi$ changes by $1 - k + (k - 1) = 0$.
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- Intermediate vertex with $k$ incident lower edges. $\chi$ changes by $1 - k + (k - 1) = 0$.
- Top vertex. If its degree is $k$, then $\chi$ changes by $1 - k + k = 1$.

Total change in $\chi$ is therefore 2.

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Total change in $\chi$ is therefore $2$.

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Three-Dimensional Polytopes

Other necessary conditions:

- $f_0$, $f_1$, $f_2$ are positive integers
- Theorem (Steinitz): A positive integer vector $(f_0, f_1, f_2)$ is the face-vector of a 3-polytope if and only if the following conditions hold:
  
  1. $f_0 - f_1 + f_2 = 2$
  2. $f_0 \leq 2f_2 - 4$
  3. $f_2 \leq 2f_0 - 4$
Three-Dimensional Polytopes

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- \( f_0 \leq 2f_2 - 4 \), and
- \( f_2 \leq 2f_0 - 4 \).
Three-Dimensional Polytopes

![Graph showing the relationship between the number of faces of various polytopes in three-dimensional space. The graph plots the number of faces for different polytopes, including Tetrahedron, Triangular Prism, Square Pyramid, Pentagon Pyramid, Octahedron, and Cube. Each polytope is represented by a point on the graph, and the graph shows the relationship between the number of vertices ($f_0$), edges ($f_1$), and faces ($f_2$).]
What is the characterization of face-vectors of 4-polytopes?
Four-Dimensional Polytopes

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We don’t know!
Four-Dimensional Polytopes

What is the characterization of face-vectors of 4-polytopes?

We don’t know!

But there are some partial results.
Theorem (Euler-Poincaré Formula)

For every $d$-polytope,

$$
\sum_{j=0}^{d-1} f_j = 1 - (-1)^d
$$
**d-Dimensional Polytopes**

**Theorem (Euler-Poincaré Formula)**

For every \( d \)-polytope,

\[
d - 1 \sum_{j=0}^{d-1} f_j = 1 - (-1)^d
\]

Early proofs (pre-Poincaré) relied upon the implicit or unproven assumption of “shellability,” not established until 1970 by Bruggesser and Mani.
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Early proofs (pre-Poincaré) relied upon the implicit or unproven assumption of “shellability,” not established until 1970 by Bruggesser and Mani.

Grünbaum developed a “sweeping-like” proof.
A facet of a $d$-polytope is a face of dimension $d - 1$. 

Theorem (Upper Bound Theorem, McMullen) 

The dual to the cyclic $d$-polytope with $n$ vertices has the largest number of faces of all dimensions of any $d$-polytope with $n$ facets.
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**Theorem (Upper Bound Theorem, McMullen)**

*The dual to the cyclic $d$-polytope with $n$ vertices has the largest number of faces of all dimensions of any $d$-polytope with $n$ facets.*
Barnette characterized the sets of various pairs of components of $(f_0, f_1, f_2, f_3)$. 
Simple Polytopes

A $d$-polytope is **simple** if every vertex is incident to precisely $d$ edges.
Simple Polytopes

A $d$-polytope is simple if every vertex is incident to precisely $d$ edges.

Example: The cube, (as well as the $d$-cube for all $d$).
Theorem (Lower Bound Theorem, Barnette)

The “truncation polytope” of dimension \( d \) with \( n \) facets has the smallest number of faces of all dimensions of any simple \( d \)-polytope with \( n \) facets.
**h-Vectors**

Sweep a simple $d$-polytope with a hyperplane in general direction.

Orient all edges in the direction of the sweep.

Let $h_i$ be the number of vertices of indegree $i$.

The **$h$-vector** is $(h_0, h_1, \ldots, h_d)$.

Note that it is a nonnegative vector of integers.
**$h$-Vectors**

Example: Cube
**h-Vectors**

Example: Cube

\[ h = (1, 3, 3, 1) \]
Theorem (McMullen)

\[ f_j = \sum_{i=j}^{d} \binom{i}{j} h_i, \quad j = 0, \ldots, d \]
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\[ f_j = \sum_{i=j}^{d} \binom{i}{j} h_i, \quad j = 0, \ldots, d \]

Idea: Every subset of \( j \) incoming edges to a vertex corresponds to an \( i \)-face.
These relations are invertible.

**Theorem (McMullen)**

\[ h_i = \sum_{j=i}^{d} (-1)^{i+j} \binom{j}{i} f_j, \quad i = 0, \ldots, d \]
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**Theorem (McMullen)**

\[ h_i = \sum_{j=i}^{d} (-1)^{i+j} \binom{j}{i} f_j, \quad i = 0, \ldots, d \]

This implies that \((h_0, \ldots, h_d)\) is independent of the choice of hyperplane!
“Stanley’s trick” to convert from the face-factor to the $h$-vector. Consider the face-vector $(36, 108, 141, 102, 43, 10)$. 
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```
1
1 10
1 9 43
1 8 34 102
1 7 26 68 141
1 6 19 42 73 108
1 5 13 23 31 35 36
1 4 8 10 8 4 1
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Dehn-Sommerville Relations

Reversing the sweep reverses the directions of all edges, and so swaps indegree with outdegree for each vertex.
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Theorem (Dehn-Sommerville Relations)
For every simple $d$-polytope, $h_i = h_{d-i}$ for all $i$. Besides being a nonnegative symmetric vector of integers, what other conditions must hold for the $h$-vector?
Dehn-Sommerville Relations

Reversing the sweep reverses the directions of all edges, and so swaps indegree with outdegree for each vertex. The invariance of the $h$-vector then implies

**Theorem (Dehn-Sommerville Relations)**

*For every simple $d$-polytope, $h_i = h_{d-i}$ for all $i$.***
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**Theorem (Dehn-Sommerville Relations)**

For every simple $d$-polytope, $h_i = h_{d-i}$ for all $i$.

Besides being a nonnegative symmetric vector of integers, what other conditions must hold for the $h$-vector?
Canonical Representations

For positive integers $a$ and $i$, $a$ can be written uniquely in the form

$$a = \binom{a_i}{i} + \binom{a_{i-1}}{i-1} + \cdots + \binom{a_j}{j}$$

where $a_i > a_{i-1} > \cdots > a_j \geq j \geq 1$. This is the $i$-canonical representation of $a$.

For example, the 4-canonical representation of 26 is

$$26 = \binom{6}{4} + \binom{5}{3} + \binom{2}{2}.$$
Canonical Representations

Now define $a^{<i>}$ by adding one to the top and bottom of every binomial coefficient in the $i$-canonical representation of $a$.

$$a^{<i>} = \binom{a_i + 1}{i + 1} + \binom{a_{i-1} + 1}{i} + \cdots + \binom{a_j + 1}{j + 1}$$

For example,

$$26^{<4>} = \binom{7}{5} + \binom{6}{4} + \binom{3}{3} = 37.$$

Define also $a^{<0>} = 0$. 

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Counting Faces of Polytopes  
James Madison University  
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For any symmetric vector \((h_0, h_1, \ldots, h_d)\) define \(g_0 = h_0\) and 
\(g_i = h_i - h_{i-1}, \quad i = 1, \ldots, \left\lfloor d/2 \right\rfloor.\) (That is to say, compute differences up to half way.)
**g-Theorem**

For any symmetric vector \((h_0, h_1, \ldots, h_d)\) define \(g_0 = h_0\) and \(g_i = h_i - h_{i-1}, i = 1, \ldots, \lfloor d/2 \rfloor\). (That is to say, compute differences up to half way.)

**Theorem (g-Theorem, Billera-L-Stanley, conjectured by McMullen)**

A vector \((h_0, h_1, \ldots, h_d)\) of positive integers is the h-vector of a simple d-polytope if and only if the following conditions hold.

- \(h_i = h_{d-1}, i = 0, \ldots, d,\)
- \(g_i \geq 0, i = 0, 1, \ldots, \lfloor d/2 \rfloor, \) and
- \(g_0 = 1 \text{ and } g_{i+1} \leq g_i^{<i>} \) for all \(i = 1, 2, \ldots \lfloor d/2 \rfloor - 1.\)
**g-Theorem**

For example, if we consider the potential face-vector $f = (36, 108, 141, 102, 43, 10)$ we compute $h = (1, 4, 8, 10, 8, 4, 1)$ and $g = (1, 3, 4, 2)$.

Now $h$ is symmetric, $g$ is nonnegative, $g_0 = 1$, $4 \leq 3^{1^{<1>}}$, and $2 \leq 4^{2^{<2>}}$, so this is a valid face-vector for a simple polytope.
The necessity of the conditions comes from considering a certain graded ring associated with the simple polytope (the Stanley-Reisner ring), and its relationship to the cohomology of a certain complex projective toric variety for which the hard Lefschetz Theorem holds. (McMullen later provided a more geometric proof using his “polytope algebra”.)

The sufficiency of the conditions comes from a direct construction.
Nonsimple Polytopes

It is fruitful to look beyond the face-vectors for nonsimple polytopes, and consider flag-vectors, which count chains of faces of various types and lengths.
Flag $f$-Vector and $cd$-Index

Fine; Bayer-Klapper

Example: Triangular bipyramid $P$. 
Flag $f$-Vector and $cd$-Index

Example: Triangular bipyramid $P$.
$f_S = \text{numbers of chains of faces of type } S$.

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Flag $f$-Vector and $cd$-Index

Example: Triangular bipyramid $P$.
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Example: Triangular bipyramid $P$.

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\[ h_S = \sum_{T \subseteq S} (-1)^{|S|-|T|} f_T \]

\[ c = a + b \text{ and } d = ab + ba \]

\[ \Phi(P) = c^3 + 4cd + 3dc. \]
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Ehrenborg has shown how to lift inequalities of cd-indices into new inequalities in higher dimensions.

We still cannot characterize the set of all cd-indices.
Four-Dimensional Polytopes

What we know about 4-polytopes besides the Euler-Poincaré relation:

- \( f_{02} - 3f_2 \geq 0 \)
- \( f_{02} - 3f_1 \geq 0 \)
- \( f_{02} - 3f_2 + f_1 - 4f_0 + 10 \geq 0 \)
- \( 6f_1 - 6f_0 - f_{02} \geq 0 \)
- \( f_0 - 5 \geq 0 \)
- \( f_2 - f_1 + f_0 \geq 0 \)
- \( 2(f_{02} - 3f_2) + f_1 \leq \binom{f_0}{2} \)
- \( 2(f_{02} - 3f_1) + f_2 \leq \binom{f_2-f_1+f_0}{2} \)
- \( f_{02} - 4f_2 + 3f_1 - 2f_0 \leq \binom{f_0}{2} \)
- \( f_{02} + f_2 - 2f_1 - 2f_0 \leq \binom{f_2-f_1+f_0}{2} \)
Four-Dimensional Polytopes

But there are still some huge gaps in what we know to be true about flag-vectors and the 4-polytopes we know how to construct.
What I am Skipping About Counting Faces

- Shellings
- Winding numbers in Gale transforms
- Dimensions of stress and motion spaces
- Formulas for volumes of polytopes
- The toric $h$-vector
- Hopf algebras
- Much more...
Thank you!