**Covering and Surrounding**
**Extensions 1**

1. The Snowflake Curve.

   Begin with an equilateral triangle. Let's assume that each side of the triangle has length one. Remove the middle third of each line segment and replace it with two sides of an "outward-pointing" equilateral triangle of side length 1/3. Now you have a six-pointed star formed from 12 line segments of length 1/3. Replace the middle third of each of these line segments with two sides of outward equilateral triangle of side length 1/9. Now you have a star-shaped figure with 48 sides. Continue to repeat this process, and the figure will converge to the "Snowflake Curve." Shown below are the first three stages in the construction of the Snowflake Curve.

   (a) In the limit, what is the length of the Snowflake Curve?

   (b) In the limit, what is the area enclosed by the Snowflake Curve?

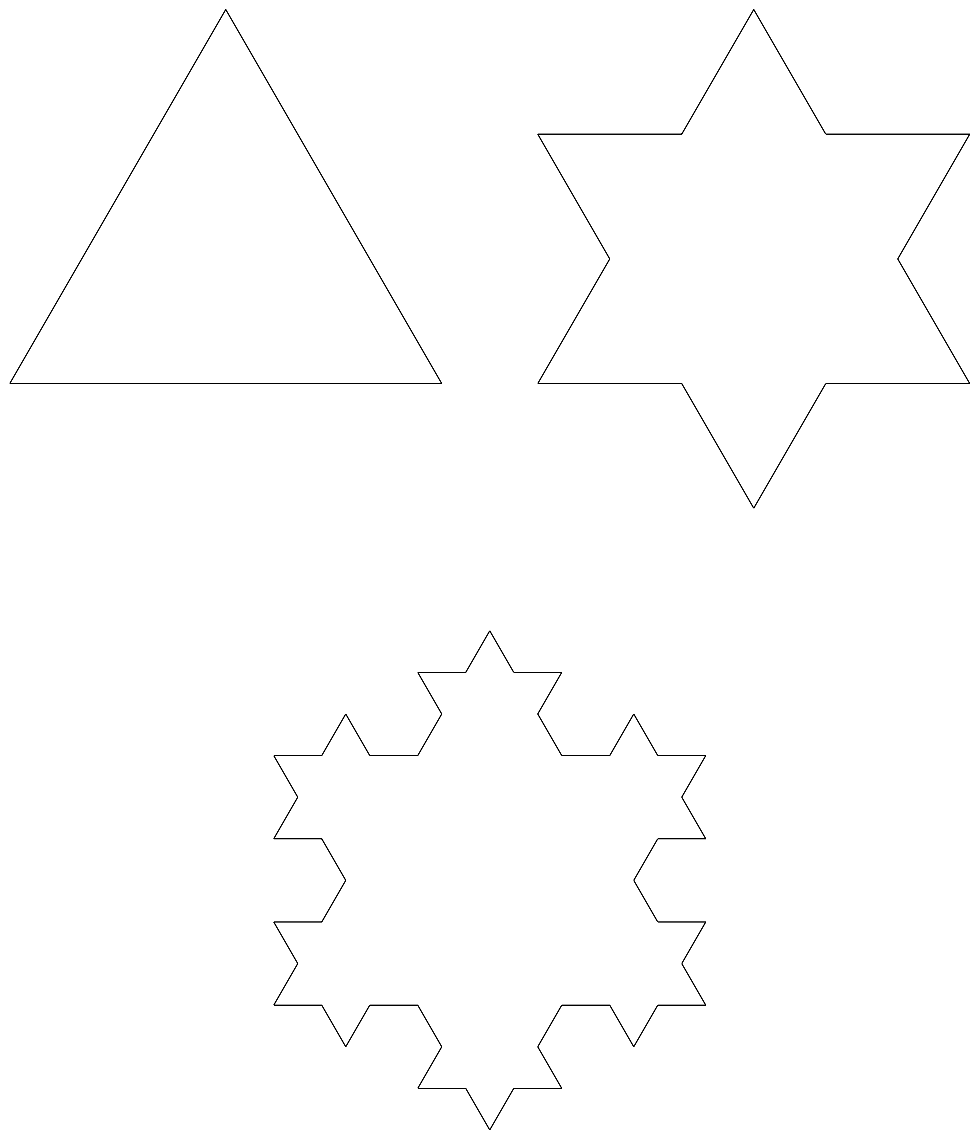

Figure 1: Constructing the Snowflake Curve

**Covering and Surrounding**
**Extensions 2**

1. Draw a region on a blank sheet of paper roughly in the shape of an oval (it needn't be any precise shape). Use grids to get upper and and lower estimates on its area. For the lower estimate, count only the squares that lie completely within the region. For the upper estimate, also count the squares that intersect the region but are not completely contained within the region. Use grids of different sizes. What do you expect will happen to these two estimates as the grids involve smaller and smaller squares?

2. 
   (a) What happens if you apply this method to determine the area of a line segment?

   (b) Find a set in the plane for which the above method will not work to obtain a measurement for its area.

3. Look up the method of determining the area of a region using Riemann sums. How does this relate to the above problems?

4. Discuss how you might estimate the area of a continent on the surface of the earth. What problems do you encounter when you try to use the method of problem (1)? How might you overcome them?

5. How can we measure the lengths of curves in "real life?" There are devices consisting of wheels with some sort of dial that you can roll over a map to estimate distances, and larger versions that you can roll in front of you on, e.g., paths, to measure distance (what are these things called?). You can also estimate the distance that you walk by wearing a pedometer.

   Here is another way to estimate the length of a curve on a map, using a simple device called a *longimeter*. On a transparent sheet of plastic create a square grid, each square having side length of, say 1 mm. Superimpose this grid your curve in three different orientations, differing one from the other by a rotation of 30°. In each of the three cases, count how many squares the curve passes through. Let the sum of these three numbers be $S$. Then an estimate of the length of the curve is $S/3.82$ mm.

   In the example below, I rotated the figure rather than the grid. Each square has side length 0.25 in. The sum $S$ is $16 + 16 + 15 = 47$, so the estimate of the length of the curve is $47/3.82 \approx 12.30$ units of length 0.25 in, or 3.07 in.

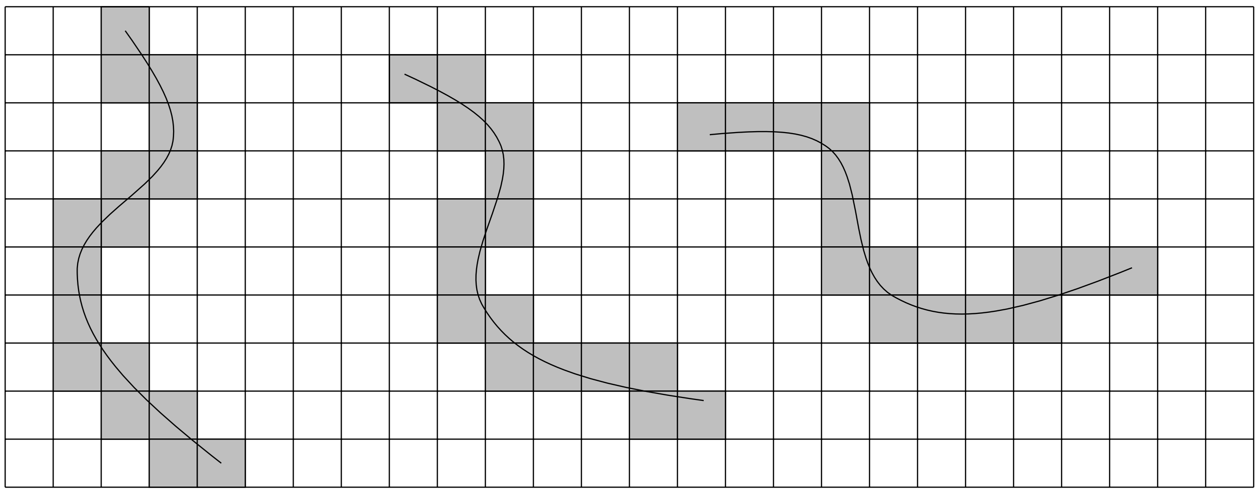

Figure 2: Using a Longimeter

Research question: Read the reference below and write up an explanation of why this method works. In particular, where does the number 3.82 come from? (This is not explicitly explained in the book.)

Reference: H. Steinhaus, *Mathematical Snapshots*, Oxford University Press, New York, 1989, pp. 105–107.

6. Fractals

The notion of "length" of certain naturally occurring objects can, however, be tricky, and can lead one into the notion of fractals. The following quote comes from a book by Mandelbrot. Read this quote and then explain how you think measurements of lengths of coastlines are calculated in practice.

> *To introduce a first category of fractals, namely curves whose fractal dimension is greater than* 1*, consider a stretch of coastline. It is evident that its length is at least equal to the distance measured along a straight line between its beginning and its end. However, the typical coastline is irregular and winding, and there is no question it is much longer than the straight line between its end points.*
>
> *There are various ways of evaluating its length more accurately. . . The result is most peculiar: coastline length turns out to be an elusive notion that slips between the fingers of one who wants to grasp it. All measurement methods*

*ultimately lead to the conclusion that the typical coastline's length is very large and so ill determined that it is best considered infinite. . . .*

*Set dividers to a prescribed opening $\epsilon$, to be called the yardstick length, and walk these dividers along the coastline, each new step starting where the previous step leaves off. The number of steps multipled by $\epsilon$ is an approximate length $L(\epsilon)$. As the dividers' opening becomes smaller and smaller, and as we repeat the operation, we have been taught to expect $L(\epsilon)$ to settle rapidly to a well-defined value called the* true length. *But in fact what we expect does not happen. In the typical case, the observed $L(\epsilon)$ tends to increase without limit.*

*The reason for this behavior is obvious: When a bay or peninsula noticed on a maa scaled to* $1/100,000$ *is reexamined on a map at* $1/10,000$*, subbays and subpeninsulas become visible. On a* $1/1,000$ *scale map, sub-subbays and sub-subpeninsulas appear, and so forth. Each adds to the measured length.*

—B.B. Mandelbrot, "How Long is the Coast of Britain," *The Fractal Geometry of Nature*, W.H. Freeman and Company, New York, 1983, Chapter 5, p. 25.

**Covering and Surrounding**
**Extensions 3**

1. Consider regions of the plane that are formed by gluing $n$ unit squares together edge-to-edge. Investigate the problem of finding the region with the minimum perimeter and with the maximum perimeter for low values of $n$; e.g., $n = 1, \ldots, 25$.

2. Make a set of pentominoes. Find at least two ways of fitting them together to make rectangles.

3. Find all the possible hexominoes.

4. Look up a description of the Soma Cube puzzle. Make a set for yourself.

5. Consider the problem of gluing together $n$ unit cubes face-to-face to create a solid object. Investigate the problem of finding the object with the minimum surface area and with the maximum surface area for low values of $n$; e.g., $n = 1, \ldots, 27$.

6. Solve the previous problem if it is required that the cubes be arranged in a rectangular solid ("brick").

7. A farmer wishes to enclose a rectangular garden of 100 square meters with fencing. What is the minimum length of fence needed to do this?

8. Repeat question (7) if the area is $A$ square meters.

9. Repeat question (7) if the area is $A$ square meters and one side of the garden is along a river, so that only three sides of fencing is required.

10. Repeat question (7) if the area is $A$ square meters and the garden is in the shape of a rectangle with additional fencing down the middle dividing it into two congruent rectangles.

11. Repeat question (7) if the area is $A$ square meters and the garden is in the shape of a rectangle with two rows of additional fencing down the middle dividing it into three congruent rectangles.

12. Repeat question (7) if the area is $A$ square meters and the garden is in the shape of a rectangle with $k$ rows of additional fencing down the middle dividing it into $k + 1$ congruent rectangles.

**Covering and Surrounding**
**Extensions 4**

1. A farmer wishes to enclose a rectangular garden with 40 meters of fencing. What is the maximum area that he can enclose?

2. Repeat question (1) if the length of fencing is $L$ meters.

3. Repeat question (1) if the the length of fencing is $L$ meters and one side of the garden is along a river, so that only three sides of fencing is required.

4. Repeat question (1) if the length of fencing is $L$ meters and the garden is in the shape of a rectangle with additional fencing down the middle dividing it into two congruent rectangles.

5. Repeat question (1) if the length of fencing $L$ meters and the garden is in the shape of a rectangle with two rows of additional fencing down the middle dividing it into three congruent rectangles.

6. Repeat question (1) if the length of fencing is $L$ meters and the garden is in the shape of a rectangle with $k$ rows of additional fencing down the middle dividing it into $k+1$ congruent rectangles.

7. You desire to construct a cylinder that has a given volume $V$. (For example, you might be trying to design a soda can.) Find the radius and the height of the cylinder with least surface area that accomplishes this.

8. Look up the statement of the Isoperimetric Problem and write a short description of this problem and its solution in your own words. What does this have to do with Investigations 3 and 4?

## Covering and Surrounding
## Extensions 5

1. Assume that the area of a rectangle of length $\ell$ and width $w$ is $\ell w$. Use dissection arguments to derive the formulas for the area of

   (a) A triangle of base $b$ and height $h$.

   (b) A parallelogram of base $b$ and height $h$. What is the definition of the base and height of a parallelogram?

   (c) A trapezoid with bases $a$ and $b$ and height $h$.

   (d) A regular polygon with perimeter $p$ and radius $r$, where the radius is measured from the center of the polygon to the midpoint of a side.

2. Construct many polygonal regions whose vertices lie on the vertices of a grid. For each region determine its area $A$, and also determine the number $I$ of grid points that lie completely inside each region and the number $B$ of grid points that lie on the boundary of each region. Sketchpad is very helpful in doing this. Guess a formula for the area $A$ in terms of $I$ and $B$ and test your conjecture on more examples.

**Covering and Surrounding**
**Extensions 6**

1. It's time to put some algebra and trigonometry to use. In this problem we will use the triangle in Figure 3. In this triangle all angles have measure less than 90°; however, the results hold true for general triangles.

   The lengths of $BC$, $AC$ and $AB$ are $a$, $b$ and $c$, respectively. Segment $AD$ has length $c'$ and $DB$ length $c''$. Segment $CD$ is the altitude of the triangle from $C$, and has length $h$.

   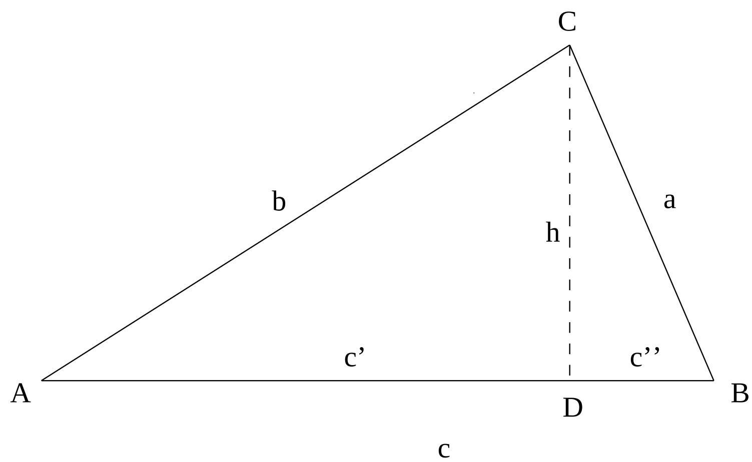


   Figure 3: Triangle $ABC$

   The usual formula for the area of a triangle is $\frac{1}{2}$(base)(height).

   (a) Using the given labeling, Area($ABC$) =______________________________

   (b) Since triangle ADC is a right triangle, $\sin A =$____________ so $h =$____________

   (c) Thus, Area($ABC$) $= \frac{1}{2}ch =$ ______________________________

   (d) What is a formula for Area($ABC$) using $\sin B$? Using $\sin C$?? (Note: you will have to use the altitude from $A$ or $B$).

   The conclusion we have made is that the area of a triangle is one-half the product of the lengths of any two sides and the sine of the included angle.

2. Using the same triangle, the *Law of Sines* is:

$$\frac{\sin A}{a} = \frac{\sin B}{b} = \frac{\sin C}{c}$$

The result holds for arbitrary triangles, but we shall prove it for our triangle $ABC$.

(a) We showed that the area of this triangle was given by three different formulas. What are they?

(b) From these three formulas, prove the Law of Sines.

3. Using the above triangle, the *Law of Cosines* is:

$$a^2 = b^2 + c^2 - 2bc\cos A$$

(a) Show that $c' = b\cos A$.

(b) Verify that $c'' = c - c'$.

(c) Verify that $h^2 = b^2 - (c')^2$.

(d) Apply the Pythagorean Theorem to triangle $CDB$, then use the facts above to make the appropriate substitutions to prove the Law of Cosines.

(e) What happens when you Apply the Law of Cosines in the case that $\angle A$ is a right angle?

4. Assume that you have triangle $ABC$ such that the coordinates of the three (distinct) points $A$, $B$, and $C$ are $(0, 0)$, $(x_1, y_1)$, and $(x_2, y_2)$, respectively. The Law of Cosines can be used to prove that

$$\cos A = \frac{x_1x_2 + y_1y_2}{\sqrt{x_1^2 + y_1^2}\sqrt{x_2^2 + y_2^2}}.$$

Use the Law of Cosines to prove this formula. Recall that the length of a line segment joining points $(x_1, y_1)$ and $(x_2, y_2)$ is $\sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$.

5. Referring again to the triangle $ABC$, we can prove another area formula.

$$\text{area}(ABC) = \frac{1}{2}|x_1y_2 - x_2y_1|.$$

Use $\text{area}(ABC) = \frac{1}{2}bc\sin A$, the cosine formula from the previous problem, and $\sin^2 A + \cos^2 A = 1$ to prove this formula.

6. What is the area of a lune (two-sided polygon formed by two half-great circles joining two antipodal points) on a sphere?

7. What is the formula for the area of a "triangle" on a sphere?

8. What is the area of a spherical polygonal region?

9. Prove that two spherical triangles having congruent corresponding angles are in fact congruent.

## Covering and Surrounding
## Extensions 7

1. It is a theorem that the ratio of the circumference to the diameter is the same for any circle in the plane. Explain why this leads to a *definition* of the number we refer to as $\pi$.

2. Cut up a circle into very thin pie-shaped wedges with their points meeting at the center of the circle. Explain how to use this to guess the formula for the area of a circle in terms of its radius.