# SMSG Geometry Summary

February 26, 2014

# Contents


**1 Lines** **1**

1.1 Measurement of Distance . . . 1

1.2 A Choice of a Unit of Distance . . . 1

1.3 An Infinite Ruler . . . 1

1.4 The Ruler Placement Postulate — Betweenness — Segments And Rays . . . 2

**2 Lines, Planes and Separation** **4**

2.1 Lines and Planes in Space . . . 4

2.2 Convex Sets . . . 5

**3 Angles and Triangles** **6**

3.1 The Basic Definitions . . . 6

3.2 Measurement of Angles . . . 6

3.3 Perpendicularity, Right Angles, and Congruence of Angles . . . 7

**4 Congruences** **9**

4.1 Congruences Between Triangles . . . 9

4.2 The Basic Congruence Postulate . . . 9

4.3 The Isosceles Triangle Theorem and the Angle Bisector Theorem . . . 9

4.4 The Angle Side Angle Theorem . . . . . . . . . . . . . . . . . . . . . . . . . 10

4.5 The Side Side Side Theorem . . . . . . . . . . . . . . . . . . . . . . . . . . . 10

**5 Perpendicular Lines** **11**

5.1 Theorems About Perpendiculars . . . . . . . . . . . . . . . . . . . . . . . . 11

5.2 Betweenness and Separation . . . . . . . . . . . . . . . . . . . . . . . . . . 11

**6 Geometric Inequalities** **12**

6.1 The Basic Inequality Theorems . . . . . . . . . . . . . . . . . . . . . . . . 12

6.2 Altitudes . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . 13

**7 Perpendicular Lines and Planes in Space** **14**

7.1 The Basic Definition . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . 14

7.2 The Basic Theorem . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . 14

7.3 Existence and Uniqueness Theorems . . . . . . . . . . . . . . . . . . . . . 14

**8 Parallel Lines in a Plane** **16**

8.1 Conditions Which Guarantee Parallelism . . . . . . . . . . . . . . . . . . . 16

8.2 Corresponding Angles . . . . . . . . . . . . . . . . . . . . . . . . . . . . . 16

8.3 The Parallel Postulate . . . . . . . . . . . . . . . . . . . . . . . . . . . . . 17

8.4 Triangles . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . 17

8.5 Quadrilaterals in the Plane . . . . . . . . . . 18

8.6 Rhombus, Rectangle and Square . . . . . . . . . . 19

8.7 Transversals to Many Parallel Lines . . . . . . . . . . 19

**9 Parallels in Space** **21**

9.1 Parallel Planes . . . . . . . . . . 21

9.2 Dihedral Angles, Perpendicular Planes . . . . . . . . . . 21

9.3 Projections . . . . . . . . . . 22

**10 Areas of Polygonal Regions** **23**

10.1 Areas of Triangles and Quadrilaterals . . . . . . . . . . 23

10.2 The Pythagorean Theorem . . . . . . . . . . 24

**11 Similarity** **25**

11.1 The Idea of Similarity . . . . . . . . . . 25

11.2 Similarities Between Triangles . . . . . . . . . . 25

11.3 The Basic Similarity Theorems . . . . . . . . . . 25

11.4 Similarities in Right Triangles . . . . . . . . . . 26

11.5 Areas of Similar Triangles . . . . . . . . . . 26

**12 Circles and Spheres** **27**

12.1 Basic Definitions . . . . . . . . . . 27

12.2 Tangent Lines and the Fundamental Theorem for Circles . . . . . . . . . . 27

12.3 Tangent Planes and the Fundamental Theorem for Spheres . . . . . . . . . . 28

12.4 Arcs of Circles . . . . . . . . . . 29

12.5 Lengths of Tangent and Secant Segments . . . . . . . . . . 31

**13 Characterization of Sets and Constructions** **32**

13.1 Basic Characterizations and Concurrence Theorems . . . . . . . . . . 32

13.2 Constructions with Straight-edge and Compass . . . . . . . . . . 32

13.3 Elementary Constructions . . . . . . . . . . 33

13.4 Inscribed and Circumscribed Circles . . . . . . . . . . 33

13.5 The Impossible Construction Problems of Antiquity . . . . . . . . . . 33

**14 Areas of Circles and Sectors** **34**

14.1 Polygons . . . . . . . . . . 34

14.2 Regular Polygons . . . . . . . . . . 34

14.3 The Circumference of a Circle and the Number $\pi$ . . . . . . . . . . 35

14.4 Area of a Circle . . . . . . . . . . 35

14.5 Lengths of Arcs and Areas of Sectors . . . . . . . . . . 35

**15 Volumes of Solids** **37**

15.1 Prisms . . . . . . . . . . 37

15.2 Pyramids . . . . . . . . . . 38

15.3 Volumes of Prisms and Pyramids, Cavalieri's Principle . . . . . . . . . . 38

15.4 Cylinders and Cones . . . . . . . . . . 39

15.5 Spheres; Volume and Area . . . . . . . . . . 39

**16 Plane Coordinate Geometry** **40**

16.1 Coordinate Systems in a Plane . . . . . . . . . . 40

16.2 The Slope of a Non-Vertical Line . . . . . . . . . . 40

16.3 Parallel and Perpendicular Lines . . . . . . . . . . 41

16.4 The Distance Formula . . . . . . . . . . 41

16.5 The Mid-Point Formula . . . . . . . . . . 41

16.6 Proofs of Geometric Theorems . . . . . . . . . . 41

16.7 The Graph of a Condition . . . . . . . . . . 41

16.8 How to Describe a Line by an Equation . . . . . . . . . . 42

16.9 Various Forms of the Equation of a Line . . . . . . . . . . 42

16.10 The General Form of the Equation of a Line . . . . . . . . . . 42

16.11 Intersections of Lines . . . . . . . . . . 42

16.12 Circles . . . . . . . . . . 43

**17 Rigid Motion** **44**

17.1 Rigid Motion of Segments . . . . . 44

17.2 Rigid Motion of Rays, Angles and Triangles . . . . . 44

17.3 Rigid Motion of Circles and Arcs . . . . . 44

17.4 Reflections . . . . . 45

**18 Trigonometry** **46**

18.1 Trigonometric Ratios . . . . . 46

18.2 Relations Among the Trigonometric Ratios . . . . . 46

# 1 Lines

## 1.1 Measurement of Distance

1. Postulate 1. Given any two different points, there is exactly one line which contains both of them.

2. Notation. To designate the line determined by two points $P$ and $Q$ we use the notation $\overleftrightarrow{PQ}$.

## 1.2 A Choice of a Unit of Distance

1. Postulate 2. (The Distance Postulate.) To every pair of different points there corresponds a unique positive number.

2. Definition. The *distance* between two points is the positive number given by the Distance Postulate. If the points are $P$ and $Q$, then the distance is denoted by $PQ$.

## 1.3 An Infinite Ruler

1. Postulate 3. (The Ruler Postulate.) The points of a line can be placed in correspondence with the real numbers in such a way that

   (a) To every point of the line there corresponds exactly one real number,

   (b) To every real number there corresponds exactly one point of the line, and

   (c) The distance between two points is the absolute value of the difference of the corresponding numbers.

2. Definition. A correspondence of the sort described in Postulate 3 is called a *coordinate system* for the line. The number corresponding to a given point is called the *coordinate* of the point.

## 1.4 The Ruler Placement Postulate — Betweenness — Segments And Rays

1. Postulate 4. (The Ruler Placement Postulate.) Given two points $P$ and $Q$ of a line, the coordinate system can be chosen in such a way that the coordinate of $P$ is zero and the coordinate of $Q$ is positive.

2. Definition. $B$ is *between* $A$ and $C$ if (1) $A$, $B$ and $C$ are distinct points on the same line and (2) $AB + BC = AC$.

3. Theorem 2-1. Let $A$, $B$, $C$ be three points of a line, with coordinates $x$, $y$, $z$. If $x < y < z$, then $B$ is between $A$ and $C$.

4. Theorem 2-2. Of any three different points on the same line, one is between the other two.

5. Theorem 2-3. Of three different points on the same line, only one is between the other two.

6. Definitions. For any two points $A$ and $B$ the *segment* $\overline{AB}$ is the set whose points are $A$ and $B$, together with all points that are between $A$ and $B$. The points $A$ and $B$ are called the *end-points* of $\overline{AB}$.

7. Definition. The distance $AB$ is called the *length* of the segment $\overline{AB}$.

8. Definition. Let $A$ and $B$ be points of a line $L$. The *ray* $\overrightarrow{AB}$ is the set which is the union of (1) the segment $\overline{AB}$ and (2) the set of all points $C$ for which it is true that $B$ is between $A$ and $C$. The point $A$ is called the *end-point* of $\overrightarrow{AB}$.

9. Definition. If $A$ is between $B$ and $C$, then $\overrightarrow{AB}$ and $\overrightarrow{AC}$ are called *opposite rays.*

10. Theorem 2-4. (The Point Plotting Theorem.) Let $\overrightarrow{AB}$ be a ray, and let $x$ be a positive number. Then there is exactly one point $P$ of $\overrightarrow{AB}$ such that $AP = x$.

11. Definition. A point $B$ is called a *mid-point* of a segment $\overline{AC}$ if $B$ is between $A$ and $C$, and $AB = BC$.

12. Theorem 2-5. Every segment has exactly one midpoint.

13. Definition. The mid-point of a segment is said to *bisect* the segment. More generally, any figure whose intersection with a segment is the mid-point of the segment is said to *bisect* the segment.

# 2 Lines, Planes and Separation

## 2.1 Lines and Planes in Space

1. Definition. The set of all points is called *space.*
2. Definition. A set of points is *collinear* if there is a line which contains all the points of the set.
3. Definition. A set of points is *coplanar* if there is a plane which contains all the points of the set.
4. Postulate 5.
   (a) Every plane contains at least three non-collinear points.
   (b) Space contains at least four non-coplanar points.
5. Theorem 3-1. Two different lines intersect in at most one point.
6. Postulate 6. If two points lie in a plane, then the line containing these points lies in the same plane.
7. Theorem 3-2. If a line intersects a plane not containing it, then the intersection is a single point.
8. Postulate 7. Any three points lie in at least one plane, and any three non-collinear points lie in exactly one plane. More briefly, any three points are coplanar, and any three non-collinear points determine a plane.
9. Theorem 3-3. Given a line and a point not on the line, there is exactly one plane containing both of them.
10. Theorem 3-4. Given two intersecting lines, there is exactly one plane containing them.
11. Postulate 8. If two different planes intersect, then their intersection is a line.

## 2.2 Convex Sets

1. Definition. A set $A$ is called *convex* if for every two points $P$ and $Q$ on $A$, the entire segment $\overline{PQ}$ lies in $A$.

2. Postulate 9. (The Plane Separation Postulate.) Given a line and a plane containing it. The points of the plane that do not lie on the line form two sets such that (1) each of the sets is convex and (2) if $P$ is in one set and $Q$ is in the other then the segment $\overline{PQ}$ intersects the line.

3. Definitions. Given a line $L$ and a plane $E$ containing it, the two sets determined by Postulate 9 are called *half-planes,* and $L$ is called an *edge* of each of them. We say that $L$ *separates* $E$ into the two half-planes. If two points $P$ and $Q$ or $E$ lie in the same half-plane, we say that they lie *on the same side* of $L$; if $P$ lies in one of the half-planes and $Q$ in the other they lie *on opposite sides* of $L$.

4. Postulate 10. (The Space Separation Postulate.) The points of space that do not lie in a given plane form two sets such that (1) each of the sets is convex and (2) if $P$ is one set and $Q$ is in the other, then the segment $\overline{PQ}$ intersects the plane.

5. Definitions. The two sets determined by Postulate 10 are called *half-spaces,* and the given plane is called the *face* of each of them.

# 3 Angles and Triangles

## 3.1 The Basic Definitions

1. Definitions. An *angle* is the union of two rays which have the same end-point but do not lie in the same line. The two rays are called the *sides* of the angle, and their common end-point is called the *vertex.*

2. Notation. The angle which is the union of $\overrightarrow{AB}$ and $\overrightarrow{AC}$ is denoted by $\angle BAC$, or by $\angle CAB$, or simply by $\angle A$ if it is clear which rays are meant.

3. Definitions. If $A$, $B$, and $C$ are any three non-collinear points, then the union of the segments $\overline{AB}$, $\overline{BC}$ and $\overline{AC}$ is called a *triangle,* and is denoted by $\Delta ABC$; the points $A$, $B$ and $C$ are called its *vertices,* and the segments $\overline{AB}$, $\overline{BC}$ and $\overline{AC}$ are called its *sides.* Every triangle determines three angles; $\Delta ABC$ determines the angles $\angle BAC$, $\angle ABC$ and $\angle ACB$, which are called the *angles of* $\Delta ABC$*.* For short, we will often write them simply as $\angle A$, $\angle B$, and $\angle C$.

4. Definitions. Let $\angle BAC$ be an angle lying in plane $E$. A point $P$ of $E$ lines in the *interior* of $\angle BAC$ if (1) $P$ and $B$ are on the same side of the line $\overleftrightarrow{AC}$ and also (2) $P$ and $C$ are on the same side of the line $\overleftrightarrow{AB}$. The *exterior* of $\angle BAC$ is the set of all points of $E$ that do not line in the interior and do not lie on the angle itself.

5. Definitions. A point lies in the *interior* of a triangle if it lie in the interior of each of the angles of the triangle. A point lies in the *exterior* of a triangle if it lies in the plane of the triangle but is not a point of the triangle or of its interior.

## 3.2 Measurement of Angles

1. Postulate 11. (The Angle Measurement Postulate.) To every angle $\angle BAC$ there corresponds a real number between 0 and 180.

2. Definition. The number specified by Postulate 11 is called the *measure of the angle,* and written as $\text{m}\angle BAC$.

3. Postulate 12. (The Angle Construction Postulate.) Let $\overrightarrow{AB}$ be a ray on the edge of the half-plane $H$. For every number $r$ between 0 and 180 there is exactly one ray $\overrightarrow{AP}$, with $P$ in $H$, such that m$\angle PAB = r$.

4. Postulate 13. (The Angle Addition Postulate.) If $D$ is a point in the interior of $\angle BAC$, then m$\angle BAC = m\angle BAD + m\angle DAC$.

5. Definition. If $\overrightarrow{AB}$ and $\overrightarrow{AC}$ are opposite rays, and $\overrightarrow{AD}$ is another ray, then $\angle BAD$ and $\angle DAC$ form a *linear pair.*

6. Definition. If the sum of the measures of two angles is 180, then the angles are called *supplementary,* and each is called a *supplement* of the other.

7. Postulate 14. (The Supplement Postulate.) If two angles form a linear pair, then they are supplementary.

## 3.3 Perpendicularity, Right Angles, and Congruence of Angles

1. Definitions. If the two angles of a linear pair have the same measure, then each of the angles is a *right angle.*

2. Definition. Two intersecting sets, each of which is either a line, a ray or a segment, are *perpendicular* if the two lines which contain them determine a right angle.

3. Definition. If the sum of the measures of two angles is 90, then the angles are called *complementary,* and each of them is called a *complement* of the other.

4. Definition. An angle with measure less than 90 is called *acute,* and an angle with measure greater than 90 is called *obtuse.*

5. Definition. Angles with the same measure are called *congruent angles.*

6. Theorem 4-1. If two angles are complementary, then both of them are acute.

7. Theorem 4-2. Every angle is congruent to itself.

8. Theorem 4-3. Any two right angles are congruent.

9. Theorem 4-4. If two angles are both congruent and supplementary, then each of them is a right angle.

10. Theorem 4-5. Supplements of congruent angles are congruent.
11. Theorem 4-6. Complements of congruent angles are congruent.
12. Definition. Two angles are *vertical angles* if their sides form two pairs of opposite rays.
13. Theorem 4.7. Vertical angles are congruent.
14. Theorem 4-8. If two intersecting lines form one right angle, then they form four right angles.

# 4 Congruences

## 4.1 Congruences Between Triangles

1. Definitions. Angles are *congruent* if they have the same measure. Segments are *congruent* if they have the same length.

2. Theorem 5-1. Every segment is congruent to itself. (This is called an *identify congruence.*

3. Notation: $\angle A \cong \angle B$ and $\overline{AB} \cong \overline{CD}$.

4. We do not write = between two geometric figures unless we mean the figures are exactly the same. Example: Two descriptions of exactly the same angle or exactly the same line using different points.

5. Definition. Given a correspondence $ABC \longleftrightarrow DEF$ between the vertices of two triangles. If every pair of corresponding sides are congruent, and every pair of corresponding angles are congruent, then the correspondence $ABC \longleftrightarrow DEF$ is a *congruence between the two triangles.*

6. Notation. $\Delta ABC \cong \Delta DEF$.

## 4.2 The Basic Congruence Postulate

1. Postulate 15. (The S.A.S. Postulate.) Given a correspondence between two triangles (or between a triangle and itself). If two sides and the included angle of the first triangle are congruent to the corresponding parts of the second triangle, then the correspondence is a congruence.

## 4.3 The Isosceles Triangle Theorem and the Angle Bisector Theorem

1. Theorem 5-2. If two sides of a triangle are congruent, then the angles opposite these sides are congruent.

2. Definitions. A triangle with two congruent sides is called *isosceles.* The remaining side is the *base.* The two angles that include the base are *base angles.*

3. Definitions. A triangle whose three sides are congruent is called *equilateral.* A triangle no two of whose sides are congruent is called *scalene.*

4. Definition. A triangle is *equiangular* if all three of its angle are congruent.

5. Corollary 5-2-1. Every equilateral triangle is equiangular.

6. Definition. A ray $\overrightarrow{AD}$ *bisects,* or is a *bisector* of, an angle $\angle BAC$ if $D$ is in the interior of $\angle BAC$, and $\angle BAD \cong \angle DAC$.

7. Theorem 5-3. Every angle has exactly one bisector.

8. Definition. A *median of a triangle* is a segment whose end-points are one vertex of the triangle and the mid-point of the opposite side.

9. Definition. An *angle bisector of a triangle* is a segment whose end-points are one vertex of the triangle and a point of the opposite side which lies in the ray bisecting the angle at the given vertex.

## 4.4 The Angle Side Angle Theorem

1. Theorem 5-4. (The A.S.A. Theorem.) Given a correspondence between two triangles, (or between a triangle and itself). If two angles and the included side of the first triangle are congruent to the corresponding parts of the second triangle, then the correspondence is a congruence.

2. Theorem 5-5. If two angles of a triangle are congruent, the sides opposite these angles are congruent.

3. Corollary 5-5-1. An equiangular triangle is equilateral.

## 4.5 The Side Side Side Theorem

1. Theorem 5-6. (The S.S.S. Theorem.) Given a correspondence between two triangles (or between a triangle and itself). If all three pairs of corresponding sides are congruent, then the correspondence is a congruence.

# 5 Perpendicular Lines

## 5.1 Theorems About Perpendiculars

1. Theorem 6-1. In a given plane, through a given point of a given line of the plane, there passes one and only one line perpendicular to the given line.
2. Definition. The *perpendicular bisector* of a segment, in a plane, is the line in the plane which is perpendicular to the segment and contains the mid-point.
3. Theorem 6-2. The perpendicular bisector of a segment, in a plane, is the set of all points of the plane that are equidistant from the end-points of the segment.
4. Theorem 6-3. Through a given external point there is at most one line perpendicular to a given line.
5. Corollary 6-3-1. At most one angle of a triangle can be a right angle.
6. Definitions. A *right triangle* is a triangle one of whose angles is a right angle. The side opposite the right angle is the *hypotenuse;* the sides adjacent to the right angle are the *legs.*
7. Theorem 6-4. Through a given external point there is at least one line perpendicular to a given line.

## 5.2 Betweenness and Separation

1. Theorem 6-5. If $M$ is between $A$ and $C$ on a line $L$, then $M$ and $A$ are on the same side of any other line that contains $C$.
2. Theorem 6-6. If $M$ is between $A$ and $C$, and $B$ is any point not on the line $\overleftrightarrow{AC}$, then $M$ is in the interior of $\angle ABC$.

# 6 Geometric Inequalities

## 6.1 The Basic Inequality Theorems

1. Definition. In triangle $\Delta ABC$, if $C$ is between $A$ and $D$, then $\angle BCD$ is an *exterior angle* of $\Delta ABC$.

2. Definition. $\angle A$ and $\angle B$ of the above triangle are called the *remote interior angles* of the exterior angle $\angle BCD$.

3. Theorem 7-1. (The Exterior Angle Theorem.) An exterior angle of a triangle is larger than either remote interior angle.

4. Corollary 7-1-1. If a triangle has a right angle, then the other two angles are acute.

5. Theorem 7-2. (The S.A.A. Theorem.) Given a correspondence between two triangles. If two angles and a side opposite one of them in one triangle are congruent to the corresponding parts of the second triangle, then the correspondence is a congruence.

6. Theorem 7-3. (The Hypotenuse-Leg Theorem.) Given a correspondence between two right triangles. If the hypotenuse and one leg of one triangle are congruent to the corresponding parts of the second triangle, then the correspondence is a congruence.

7. Theorem 7-4. If two sides of a triangle are not congruent, then the angles opposite these two sides are not congruent, and the larger angle is opposite the longer side.

8. Theorem 7-5. If two angles of a triangle are not congruent, then the sides opposite them are not congruent, and the longer side is opposite the larger angle.

9. Theorem 7-6. The shortest segment joining a point to a line is the perpendicular segment.

10. Definition. The *distance between a line and a point* not on it is the length of the perpendicular segment from the point to the line. The distance between a line and a point on the line is defined to be zero.

11. Theorem 7-7. (The Triangle Inequality.) The sum of the lengths of any two sides of a triangle is greater than the length of the third side.

12. Theorem 7-8. If two sides of one triangle are congruent respectively to two sides of a second triangle, and the included angle of the first triangle is larger than the included angle of the second, then the opposite side of the first triangle is longer than the opposite side of the second.

13. Theorem 7-9. If two sides of one triangle are congruent respectively to two sides of a second triangle, and the third side of the first triangle is longer than the first side of the second, then the included angle of the first triangle is larger than the included angle of the second.

## 6.2 Altitudes

1. Definition. An *altitude* of a triangle is the perpendicular segment joining a vertex of the triangle to the line that contains the opposite side.

# 7 Perpendicular Lines and Planes in Space

## 7.1 The Basic Definition

1. Definition. A line and a plane are *perpendicular* if they intersect and if every line lying in the plane and passing through the point of intersection is perpendicular to the given line.

## 7.2 The Basic Theorem

1. Theorem 8-1. If each of two points of a line is equidistant from two given points, then every point of the line is equidistant from the given points.
2. Theorem 8-2. If each of three non-collinear points of a plane is equidistant from two points, then every point of the plane is equidistant from these two points.
3. Theorem 8-3. If a line is perpendicular to each of two intersecting lines at their point of intersection, then it is perpendicular to the plane of these two lines.
4. Theorem 8-4. Through a given point on a given line there passes a plane perpendicular to the line.
5. Theorem 8-5. If a line and a plane are perpendicular, then the plane contains every line perpendicular to the given line at its point of intersection with the given plane.
6. Theorem 8-6. Through a given point on a given line there is at most one plane perpendicular to the line.
7. Theorem 8-7. The perpendicular bisecting plane of a segment is the set of all points equidistant from the end-points of the segment.
8. Theorem 8-8. Two lines perpendicular to the same plane are coplanar.

## 7.3 Existence and Uniqueness Theorems

1. Theorem 8-9. Through a given point there passes one and only one plane perpendicular to a given line.

2. Theorem 8-10. Through a given point there passes one and only one line perpendicular to a given plane.

3. Definition. The *distance* to a plane from an external point is the length of the perpendicular segment from the point to the plane.

4. Theorem 8-11. The shortest segment to a plane from an external point is the perpendicular segment.

# 8 Parallel Lines in a Plane

## 8.1 Conditions Which Guarantee Parallelism

1. Definition. Two lines which are not coplanar are said to be *skew.*
2. Definition. Two lines are *parallel* if they are coplanar and do not intersect.
3. Theorem 9-1. Two parallel lines lie in exactly one plane.
4. Notation. $L_1 \parallel L_2$.
5. Theorem 9-2. Two lines in a plane are parallel if they are both perpendicular to the same line.
6. Theorem 9-3. Let $L$ be a line, and let $P$ be a point not on $L$. Then there is at least one line through $P$, parallel to $L$.
7. Definition. A *transversal* of two coplanar lines is a line which intersects them in two different points.
8. Definition. Let $L$ be a transversal of $L_1$ and $L_2$, intersecting them in $P$ and $Q$. Let $A$ be a point of $L_1$ and $B$ a point of $L_2$ such that $A$ and $B$ are on opposite sides of $L$. Then $\angle PQB$ and $\angle QPA$ are *alternate interior angles* formed by the transversal to the two lines.
9. Theorem 9-4. If two lines are cut by a transversal, and if one pair of alternate interior angles are congruent, then the other pair of alternate interior angles are also congruent.
10. Theorem 9-5. If two lines are cut by a transversal, and if a pair of alternate interior angles are congruent, then the lines are parallel.

## 8.2 Corresponding Angles

1. Definition. If two lines are cut by a transversal, and if $\angle x$ and $\angle y$ are alternate interior angles, and if $\angle y$ and $\angle z$ are vertical angles, then $\angle x$ and $\angle z$ are *corresponding angles.*

2. Theorem 9-6. If two lines are cut by a transversal, and if one pair of corresponding angles are congruent, then the other three pairs of corresponding angles have the same property.

3. Theorem 9-7. If two lines are cut by a transversal, and if a pair of corresponding angles are congruent, then the lines are parallel.

## 8.3 The Parallel Postulate

1. Postulate 16. (The Parallel Postulate.) Through a given external point there is at most one line parallel to a given line.

2. Theorem 9-8. If two parallel lines are cut by a transversal, then alternate interior angles are congruent.

3. Theorem 9-9. If two parallel lines are cut by a transversal, each pair of corresponding angles are congruent.

4. Theorem 9-10. If two parallel lines are cut by a transversal, interior angles on the same side of the transversal are supplementary.

5. Theorem 9-11. In a plane, two lines parallel to the same line are parallel to each other.

6. Theorem 9-12. In a plane, if a line is perpendicular to one of two parallel lines it is perpendicular to the other.

## 8.4 Triangles

1. Theorem 9-13. The sum of the measures of the angles of a triangle is 180.

2. Corollary 9-13-1. Given a correspondence between two triangles. If two pairs of corresponding angles are congruent, then the third pair of corresponding angles are also congruent.

3. Corollary 9-13-2. The acute angles of a right triangle are complementary.

4. Corollary 9-13-3. For any triangle, the measure of an exterior angle is the sum of the measures of the two remote interior angles.

## 8.5 Quadrilaterals in the Plane

1. Definition. Let $A$, $B$, $C$ and $D$ be four points lying in the same plane, such that no three of them are collinear, and such that the segments $\overline{AB}$, $\overline{BC}$, $\overline{CD}$ and $\overline{DA}$ intersect only in their end-points. Then the union of these four segments is as *quadrilateral.*

2. Notation: $ABCD$.

3. Definition. *Opposite sides* of a quadrilateral are two sides that do not intersect. Two of its *angles* are *opposite* if they do not contain a common side. Two sides are called *consecutive* if they have a common vertex. Similarly, two angles are called *consecutive* if they contain a common side. A *diagonal* is a segment joining two non-consecutive vertices.

4. Definition. A *trapezoid* is a quadrilateral in which two, and only two, opposite sides are parallel.

5. Definition. A *parallelogram* is a quadrilateral in which both pairs of opposite sides are parallel.

6. Theorem 9-14. Either diagonal separates a parallelogram into two congruent triangles. That is, if $ABCD$ is a parallelogram, then $\Delta ABC \cong \Delta CDA$.

7. Theorem 9-15. In a parallelogram, any two opposite sides are congruent.

8. Corollary 9-15-1. If $L_1 \parallel L_2$ and if $P$ and $Q$ are any two points on $L_1$, then the distances of $P$ and $Q$ from $L_2$ are equal.

9. Definition. The *distance between two parallel lines* is the distance from any point of one line to the other line.

10. Theorem 9-16. In a parallelogram, any two opposite angles are congruent.

11. Theorem 9-17. In a parallelogram, any two consecutive angles are supplementary.

12. Theorem 9-18. The diagonals of a parallelogram bisect each other.

13. Theorem 9-19. Given a quadrilateral in which both pairs of opposite sides are congruent. Then the quadrilateral is a parallelogram.

14. Theorem 9-20. If two sides of a quadrilateral are parallel and congruent, then the quadrilateral is a parallelogram.

15. Theorem 9-21. If the diagonals of a quadrilateral bisect each other, then the quadrilateral is a parallelogram.

16. Theorem 9-22. The segment between the mid-points of two sides of a triangle is parallel to the third side and half as long as the third side.

## 8.6 Rhombus, Rectangle and Square

1. Definitions. A *rhombus* is a parallelogram all of whose sides are congruent. A *rectangle* is a parallelogram all of whose angles are right angles. Finally, a *square* is a rectangle all of whose sides are congruent.

2. Theorem 9-23. If a parallelogram has one right angle, then it has four right angles, and the parallelogram is a rectangle.

3. Theorem 9-24. In a rhombus, the diagonals are perpendicular to one another.

4. Theorem 9-25. If the diagonals of a quadrilateral bisect each other and are perpendicular, then the quadrilateral is a rhombus.

## 8.7 Transversals to Many Parallel Lines

1. Definitions. If a transversal intersects two lines $L_1$, $L_2$ in points $A$ and $B$, then we say that $L_1$ and $L_2$ *intercept* the segment $\overline{AB}$ on the transversal. Suppose that we have given three lines $L_1$, $L_2$, $L_3$ and a transversal intersecting them in points $A$, $B$ and $C$. If $AB = BC$, then we say that the three lines *intercept congruent segments* on the transversal.

2. Theorem 9-26. If three parallel lines intercept congruent segments on one transversal, then they intercept congruent segments on any other transversal.

3. Corollary 9-26-1. If three or more parallel lines intercept congruent segments on one transversal, then they intercept congruent segments on any other transversal.

4. Definition. Two or more sets are *concurrent* if there is a point which belongs to all of the sets.

5. Theorem 9-27. The medians of a triangle are concurrent in a point two-thirds the way from any vertex to the mid-point of the opposite side.

6. Definition. The *centroid* of a triangle is the point of concurrency of the medians.

# 9 Parallels in Space

## 9.1 Parallel Planes

1. Definition. Two planes, or a plane and a line, are *parallel* if they do not intersect.
2. Notation. $E_1 \parallel E_2$.
3. Theorem 10-1. If a plane intersects two parallel planes, then it intersects them in two parallel lines.
4. Theorem 10-2. If a line is perpendicular to one of two parallel planes, it is perpendicular to the other.
5. Theorem 10-3. Two planes perpendicular to the same line are parallel.
6. Corollary 10-3-1. If two planes are each parallel to a third plane, they are parallel to each other.
7. Theorem 10-4. Two lines perpendicular to the same plane are parallel.
8. Corollary 10-4-1. A plane perpendicular to one of two parallel lines is perpendicular to the other.
9. Corollary 10-4-2. If two lines are each parallel to a third they are parallel to each other.
10. Theorem 10-5. Two parallel planes are everywhere equidistant. That is, all segments perpendicular to the two planes and having their end-points in the planes have the same length.

## 9.2 Dihedral Angles, Perpendicular Planes

1. Definitions. A *dihedral angle* is the union of a line and two non-coplanar half-planes having this line as their common edge. The line is called the *edge* of the dihedral angle. The union of the edge and either half-plane is called a *face,* or *side,* of the dihedral angle.
2. Notation. If $\overleftrightarrow{PQ}$ is the edge, and $A$ and $B$ points on different sides, we denote the dihedral angle by $\angle A\text{-}PQ\text{-}B$.

3. Definition. Through any point on the edge of the dihedral angle pass a plane perpendicular to the edge, intersecting each of the sides in a ray. The angle formed by these rays is called a *plane angle* of the dihedral angle.

4. Theorem 10-6. Any two plane angles of a given dihedral angle are congruent.

5. Definitions. The *measure* of a dihedral angle is the real number which is the measure of any of its plane angles. A dihedral angle is a *right dihedral angle* if its plane angles are right angles. Two planes are *perpendicular* if they determine right dihedral angles.

6. Corollary 10-6-1. If a line is perpendicular to a plane, then any plane containing this line is perpendicular to the given plane.

7. Corollary 10-6-2. If two planes are perpendicular, then any line in one of them perpendicular to their line of intersection, is perpendicular to the other plane.

## 9.3 Projections

1. Definition. The *projection of a point* onto a plane is the foot of the perpendicular from the point to the plane.

2. Definition. The *projection of a line* into a plane is the set of points which are projections into the plane of the points of the line.

3. Theorem 10-7. The projection of a line into a plane is a line, unless the line and the plane are perpendicular.

# 10 Areas of Polygonal Regions

1. Definitions. A *triangular* region is the union of a triangle and its interior. A *polygonal region* is the union of a finite number of coplanar triangular regions, such that if any two of these intersect, the intersection is either a segment or a point.

2. Postulate 17. To every polygonal region there corresponds a unique positive number.

3. Definition. The *area* of a polygonal region is the number assigned to it by Postulate 17.

4. Notation. area $R$.

5. Postulate 18. If two triangles are congruent, then the triangular regions have the same area.

6. Postulate 19. Suppose that the region $R$ is the union of two regions $R_1$ and $R_2$. Suppose that $R_1$ and $R_2$ intersect at most in a finite number of segments and points. Then the area of $R$ is the sum of the areas of $R_1$ and $R_2$.

7. Postulate 20. The area of a rectangle is the product of the length of its base and the length of its altitude.

## 10.1 Areas of Triangles and Quadrilaterals

1. Theorem 11-1. The area of a right triangle is half the product of its legs.

2. Theorem 11-2. The area of a triangle is half the product of any base and the altitude to that base.

3. Theorem 11-3. The area of a parallelogram is the product of any base and the corresponding altitude.

4. Theorem 11-4. The area of a trapezoid is half the product of its altitude and the sum of its bases.

5. Theorem 11-5. If two triangles have equal altitudes, then the ration of their areas is equal to the ratio of their bases.

6. Theorem 11-6. If two triangles have equal altitudes and equal bases, then they have equal areas.

## 10.2 The Pythagorean Theorem

1. Theorem 11-7. (The Pythagorean Theorem.) In a right triangle, the square of the hypotenuse is equal to the sum of the squares of the legs.

2. Theorem 11-8. If the square of one side of a triangle is equal to the sum of the squares of the other two sides, then the triangle is a right triangle, with a right angle opposite the first side.

3. Theorem 11-9. (The 30-60 Triangle Theorem.) The hypotenuse or a right triangle is twice as long as a leg if and only if the measures of the acute angles are 30 and 60.

4. Theorem 11-10. (The Isosceles Right Triangle Theorem.) A right triangle is isosceles if and only if the hypotenuse is $\sqrt{2}$ times as long as leg.

# 11 Similarity

## 11.1 The Idea of Similarity

1. Two sequences of numbers, $a, b, c, \ldots$ and $p, q, r, \ldots$, none of which is zero, are *proportional* if $\frac{a}{p} = \frac{b}{q} = \frac{c}{r} = \cdots$ or $\frac{p}{a} = \frac{q}{b} = \frac{r}{c} = \cdots$.

2. Algebraic Properties of a Simple Proportion. If $\frac{a}{b} = \frac{c}{d}$, with $a$, $b$, $c$, $d$ all different from zero, then

   (a) $ad = bc$,

   (b) $\frac{a}{c} = \frac{b}{d}$,

   (c) $\frac{a+b}{b} = \frac{c+d}{d}$,

   (d) $\frac{a-b}{b} = \frac{c-d}{d}$.

3. Definition. If $a$, $b$, $c$ are positive numbers and $\frac{a}{b} = \frac{b}{c}$, then $b$ is the *geometric mean* between $a$ and $c$.

## 11.2 Similarities Between Triangles

1. Definition. Given a correspondence between the vertices of two triangles. If corresponding angles are congruent and the corresponding sides are proportional, then the correspondence is a *similarity,* and the triangles are said to be *similar.*

2. Notation. $\Delta ABC \sim \Delta A'B'C'$.

## 11.3 The Basic Similarity Theorems

1. Theorem 12-1. (The Basic Proportionality Theorem.) If a line parallel to one side of a triangle intersects the other two sides in distinct points, then it cuts off segments which are proportional to these sides.

2. Theorem 12-2. If a line intersects two sides of a triangle, and cuts off segments proportional to these two sides, then it is parallel to the third side.

3. Theorem. (The A.A.A. Similarity Theorem.) Given a correspondence between two triangles. If corresponding angles are congruent, then the correspondence is a similarity.

4. Corollary 12-3-1. (The A.A. Corollary.) Given a correspondence between two triangles. If two pairs of corresponding angles are congruent, then the correspondence is a similarity.

5. Corollary 12-3-2. If a line parallel to one side of a triangle intersects the other two sides in distinct points, then it cuts off a triangle similar to the given triangle.

6. Theorem 12-4. (The S.A.S. Similarity Theorem.) Given a correspondence between two triangles. If two pairs of corresponding sides are proportional, and the included angles are congruent, then the correspondence is a similarity.

7. Theorem 12-5. (The S.S.S. Similarity Theorem.) Given a correspondence between two triangles. If corresponding sides are proportional, then the correspondence is a similarity.

## 11.4 Similarities in Right Triangles

1. Theorem 12-6. In any right triangle, the altitude to the hypotenuse separates the triangle into two triangles which are similar to each other and to the original triangle.

2. Corollary 12-6-1. Given a right triangle and the altitude from the right angle to the hypotenuse:

   (a) The altitude is the geometric mean of the segments into which it separates the hypotenuse.

   (b) Either leg is the geometric mean of the hypotenuse and the segment of the hypotenuse adjacent to the leg.

## 11.5 Areas of Similar Triangles

1. Theorem 12-7. The ratio of the areas of two similar triangles is the square of the ratio of any two corresponding sides.

# 12 Circles and Spheres

## 12.1 Basic Definitions

1. Definitions. A *sphere* is the set of points each of which is at a given distance from a given point. A *circle* is the set of points in a given plane each of which is at a given distance from a given point in the plane. In each case the given point is called the *center* and the given distance the *radius* of the sphere or circle. Two or more spheres or circles with the same center are said to be *concentric.*

2. Theorem 13-1. The intersection of a sphere with a plane through its center is a circle with the same center and radius.

3. Definition. The circle of intersection of a sphere with a plane through the center is called a *great circle* of the sphere.

4. Definition. A *chord* of a circle or a sphere is a segment whose end-points are points of the circle or the sphere. The line containing a chord is a *secant.* A *diameter* is a chord containing the center. A *radius* is a segment one of whose end-points is the center and the other one a point of the circle or sphere. The latter end-point is called the *outer end* of the radius.

## 12.2 Tangent Lines and the Fundamental Theorem for Circles

1. Definitions. The *interior* of a circle is the union of its center and the set of all points in the plane of the circle whose distances from the center are less than the radius. The *exterior* of the circle is the set of all points in the plane of the circle whose distances from the center are greater than the radius.

2. Definitions. A *tangent* to a circle is a line in the plane of the circle which intersects the circle in only one point. This point is called the *point of tangency,* or *point of contact,* and we say that the line and the circle *are tangent* at this point.

3. Theorem 13-2. Given a line and a circle in the same plane. Let $P$ be the center of the circle, and let $F$ be the foot of the perpendicular from $P$ to the line. Then either

   (a) Every point of the line is outside the circle, or

(b) $F$ is on the circle, and the line is tangent to the circle at $F$, or

(c) $F$ is inside the circle, and the line intersects the circle in exactly two points, which are equidistant from $F$.

4. Corollary 13-2-1. Every line tangent to a circle $C$ is perpendicular to the radius drawn to the point of contact.

5. Corollary 13-2-2. Any line in a plane $E$ containing a given circle, perpendicular to a radius at its outer end, is tangent to the circle.

6. Corollary 13-2-3. Any perpendicular from the center of a circle $C$ to a chord bisects the chord.

7. Corollary 13-2-4. The segment joining the center of a circle $C$ to the mid-point of a chord is perpendicular to the chord.

8. Corollary 13-2-5. In the plane of a circle, the perpendicular bisector of a chord passes through the center of the circle.

9. Corollary 13-2-6. If a line in the plane of a circle intersects the interior of the circle, then it intersects the circle in exactly two points.

10. Definition: Circles of congruent radii are called *congruent.*

11. Theorem 13-3. In the same circle or in congruent circles, chords equidistant from the center are congruent.

12. Theorem 13-4. In the same circle or in congruent circles, any two congruent chords are equidistant from the center.

13. Definitions. Two circles are tangent if they are each tangent to the same line at the same point. If tangent circles are coplanar they are internally or externally tangent according as their centers lie on the same side or on opposite sides of the common tangent line.

## 12.3 Tangent Planes and the Fundamental Theorem for Spheres

1. Definitions. The *interior* of a sphere is the union of its center and the set of all points whose distances from the center are less than the radius. The *exterior* of the sphere is the set of all points whose distances from the center are greater than the radius.

2. Definitions. A plane that intersects a sphere in exactly one point is called a *tangent plane* to the sphere. If the tangent plane intersects the sphere in the point $Q$ then we say that the plane is *tangent to the sphere at* $Q$. $Q$ is called the *point of tangency,* or the *point of contact.*

3. Theorem 13-5. Given a plane $E$ and a sphere $S$ with center $P$. Let $F$ be the foot of the perpendicular segment from $P$ to $E$. Then either

   (a) Every point of $E$ is outside $S$, or

   (b) $F$ is on $S$, and $E$ is tangent to $S$ at $F$, or

   (c) $F$ is inside $S$, and $E$ intersects $S$ in a circle with center $F$.

4. Corollary 13-5-1. A plane tangent to a sphere $S$ is perpendicular to the radius drawn to the point of contact.

5. Corollary 13-5-2. A plane perpendicular to a radius of a sphere $S$ at its outer end is tangent to $S$.

6. Corollary 13-5-3. A perpendicular from the center $P$ of a sphere $S$ to a chord of $S$ bisects the chord.

7. Corollary 13-5-4. The segment joining the center of a sphere $S$ to the mid-point of a chord is perpendicular to the chord.

## 12.4 Arcs of Circles

1. Definition. A *central angle* of a given circle is an angle whose vertex is the center of the circle.

2. Definitions. If $A$ and $B$ are two points of a circle with center $P$, not the end-points of a diameter, the union of $A$, $B$, and all the points of the circle in the interior of $\angle APB$ is a *minor arc* of the circle. The union of $A$, $B$, and all points of the circle in the exterior of $\angle APB$ is a *major arc* of the circle. If $\overline{AB}$ is a diameter the union of $A$, $B$, and all points of the circle in one of the two half-planes lying in the plane of the circle with edge $\overleftrightarrow{AB}$ is a *semi-circle.* An *arc* is either a minor arc, a major arc or a semi-circle. $A$ and $B$ are the *end-points* of the arc.

3. Notation. $\widehat{AB}$ or $A\widehat{X}B$.

4. Definition: The *degree measure* m $\overset{\frown}{AXB}$ of an arc $\overset{\frown}{AXB}$ is defined in the following way:

   (a) If $\overset{\frown}{AXE}$ is a minor arc, then m $\overset{\frown}{AXB}$ is the measure of the corresponding central angle.

   (b) If $\overset{\frown}{AXB}$ is a semi-circle, then m $\overset{\frown}{AXB}$= 180.

   (c) If $\overset{\frown}{AXB}$ is a major arc, and $\overset{\frown}{AYB}$ is the corresponding minor arc, then m $\overset{\frown}{AXB}$= $360 -$ m $\overset{\frown}{AYB}$.

5. Theorem 13-6. If $\overset{\frown}{AB}$ and $\overset{\frown}{BC}$ are arcs of the same circle having only the point $B$ in common, and if their union is an arc $\overset{\frown}{AC}$, then m $\overset{\frown}{AB}$ +m $\overset{\frown}{BC}$= m $\overset{\frown}{AC}$.

6. Definition. An angle is *inscribed* in an arc if (1) the two end-points of the arc lie on the two sides of the angle and (2) the vertex of the angle is a point, but not an end-point, of the arc. More concisely, $\angle ABC$ is inscribed in $\overset{\frown}{ABC}$.

7. Definition. An angle *intercepts* an arc if (1) the end-points of the arc lie on the angle, (2) each side of the angle contains at least one end-point of the arc and (3) except for its end-points, the arc lies in the interior of the angle.

8. Theorem 13-7. The measure of an inscribed angle is half the measure of its intercepted arc.

9. Corollary 13-7-1. An angle inscribed in a semi-circle is a right angle.

10. Corollary 13-7-2. Angles inscribed in the same arc are congruent.

11. Definition. In the same circle, or in congruent circles, two arcs are called *congruent* if they have the same measure.

12. Theorem 13-8. In the same circle or in congruent circles, if two chords are congruent, then so also are the corresponding minor arcs.

13. Theorem 13-9. In the same circle or in congruent circles, if two arcs are congruent, then so are the corresponding chords.

14. Theorem 13-10. Given an angle with vertex on the circle formed by a secant ray and a tangent ray. The measure of the angle is half the measure of the intercepted arc.

## 12.5 Lengths of Tangent and Secant Segments

1. Definition. If the line $\overleftrightarrow{QR}$ is tangent to a circle at $R$, then the segment $\overline{QR}$ is a *tangent segment* from $Q$ to the circle.

2. Theorem 13-11. The two tangent segments to a circle from an external point are congruent, and form congruent angles with the line Joining the external point to the center of the circle.

3. Theorem 13-12. Given a circle $C$ and an external point $Q$, let $L_1$ be a secant line through $Q$, intersecting $C$ in points $R$ and $S$; and let $L_2$ be another secant line through $Q$, intersecting $C$ in points $T$ and $U$. Then $QR \cdot QS = QU \cdot QT$.

4. Theorem 13-13. Given a tangent segment $\overline{QT}$ to a circle, and a secant line through $Q$, intersecting the circle in points $R$ and $S$. Then $QR \cdot QS = QT^2$.

5. Theorem 13-14. If two chords intersect within a circle, the product of the lengths of the segments of one equals the product of the lengths of the segments of the other.

# 13 Characterization of Sets and Constructions

## 13.1 Basic Characterizations and Concurrence Theorems

1. Restatement of some earlier results:
   (a) A sphere is the set of points at a given distance from a given point.
   (b) A circle is the set of points in a given plane at a given distance from a given point of the plane.
   (c) The perpendicular bisecting plane of a given segment is the set of points equidistant from the end-points of the segment.
   (d) The perpendicular bisector, in a given plane, of a given segment in the plane, is the set of points in the plane equidistant from the end-points of the segment.
2. Theorem 14-1. The bisector of an angle, minus its end-point, is the set of points in the interior of the angle equidistant from the sides of the angle.
3. Theorem 14-2. The perpendicular bisectors of the sides of a triangle are concurrent in a point equidistant from the three vertices of the triangle.
4. Corollary 14-2-1. There is one and only one circle through three non-collinear points.
5. Corollary 14-2-2. Two distinct circles can intersect in at most two points.
6. Theorem 14-3. The three altitudes of a triangle are concurrent.
7. Theorem 14-4. The angle bisectors of a triangle are concurrent in a point equidistant from the three sides.

## 13.2 Constructions with Straight-edge and Compass

1. Theorem 14-5. (The Two Circle Theorem.) If two circles have radii $a$ and $b$, and if $c$ is the distance between their centers, then the circles intersect in two points, one on each side of the line of centers, provided each one of $a$, $b$, $c$ is less than the sum of the other two.

## 13.3 Elementary Constructions

1. Construction 14-6. To copy a given triangle.
2. Construction 14-7. To copy a given angle.
3. Construction 14-8. To construct the perpendicular bisector of a given segment.
4. Corollary 14-8-1. To bisect a given segment.
5. Construction 14-9. To construct a perpendicular to a given line through a given point.
6. Construction 14-10. To construct a parallel to a given line, through a given external point.
7. Construction 14-11. To divide a segment into a given number of congruent segments.

## 13.4 Inscribed and Circumscribed Circles

1. Definitions. A circle is *inscribed in a triangle,* or the triangle is *circumscribed about the circle,* if each side of the ' . triangle is tangent to the circle. A circle is *circumscribed about a triangle,* or the triangle is *inscribed in the circle* if each vertex of the triangle lies on the circle.
2. Construction 14-12. To circumscribe a circle about a given triangle.
3. Construction 14-13. To bisect a given angle.
4. Construction 14-14. To inscribe a circle in a given triangle.

## 13.5 The Impossible Construction Problems of Antiquity

1. The angle-trisection problem.
2. The duplication of the cube.
3. Squaring the circle.

# 14 Areas of Circles and Sectors

## 14.1 Polygons

1. Definitions. Let $P_1, P_2, P_3, \ldots, P_{n-1}, P_n$ be $n$ distinct points in a plane ($n \geq 3$). Let the $n$ segments $\overline{P_1P_2}, \overline{P_2P_3}, \ldots, \overline{P_{n-1}P_n}, \overline{P_nP_1}$ have the properties:

    1. No two segments intersect except at their end-points, as specified;
    2. No two segments with a common end-point are collinear.

    Then the union of the $n$ segments is a *polygon.*

    The $n$ given points are *vertices* of the polygon, the $n$ segments are *sides* of the polygon. By (2.), any two segments with a common vertex determine an angle, which is called an *angle of the polygon.*

    Notice that triangles are polygons of 3 vertices and 3 sides, and quadrilaterals are polygons of 4 vertices and 4 sides. Polygons of $n$ vertices and $n$ sides are sometimes called *$n$-gons.* Thus a triangle is a 3-gon and a quadrilateral is a $4s$-gon (although the terms $3s$-gon and 4-gon are almost never used.) 5-gons are called *pentagons,* 6-gons are *hexagons.* 8-gons are *octagons,* and 10-gons are *decagons.* The other $n$-gons, for reasonably small numbers $n$, also have special names taken from the Greek, but the rest of these special names are not very commonly used.

    Each side of a polygon lies in a line, which separates the plane into two half-planes. If, for each side, the rest of the polygon lies entirely in one of the half-planes having that side on its edge, then the polygon is called a *convex polygon.*

## 14.2 Regular Polygons

1. Definitions. A convex polygon is *regular* if all its sides are congruent and all its angles are congruent. A polygon is *inscribed* in a circle if all of its vertices lie on the circle.

2. Definition. The number $a$ (the distance from the center of a regular polygon to the midpoint of one of its sides) is called the *apothem* of the polygon. The sum of the lengths of the sides is called the *perimeter.*

3. The area of a regular polygon is $\frac{1}{2}ap$, where $a$ is the apothem and $p$ is the perimeter.

## 14.3 The Circumference of a Circle and the Number $\pi$

1. Definition. The *circumference* of a circle is the limit of the perimeters of the inscribed regular polygons.

2. Theorem 15-1. The ratio $\frac{C}{2r}$ of the circumference to the diameter, is the same for all circles.

3. The number $\frac{C}{2r}$, which is the same for all circles, is designated by $\pi$. We can therefore express the conclusion of Theorem 15-1 in the well-known form, $C = 2\pi r$.

## 14.4 Area of a Circle

1. Definition. A *circular region* is the union of a circle and its interior.

2. Definition. The *area of a circle* is the limit of the areas of the inscribed regular polygons.

3. Theorem 15-2. The area of a circle of radius $r$ is $\pi r^2$.

## 14.5 Lengths of Arcs and Areas of Sectors

1. Definition. The *length of arc* $\widehat{AB}$ is the limit of $AP_1 + P_1P_2 + \cdots + P_{n-1}B$ as we take $n$ larger and larger. ($P_1, \ldots, P_{n-1}$ are sequential points on the arc $\widehat{AB}$.

2. Theorem 15-3. If two arcs have equal radii, then their lengths are proportional to their measures.

3. Theorem 15-4. An arc of measure $q$ and radius $r$ has length $\frac{\pi}{180}qr$.

4. Definitions. If $\widehat{AB}$ is an arc of a circle with center $Q$ and radius $r$, then the union of all segments $\overline{QP}$, where $P$ is any point of $\widehat{AB}$, is a *sector*. $\widehat{AB}$ is the *arc of the sector* and $r$ is the *radius of the sector.*

5. Theorem 15-5. The area of a sector is half the product of its radius by the length of its arc.

6. Theorem 15-6. The area of a sector of radius $r$ and arc measure $qs$ is $\frac{\pi}{360}qr^2$.

# 15 Volumes of Solids

## 15.1 Prisms

1. Definitions. Let $E_1$ and $E_2$ be two parallel planes, $L$ a transversal, and $K$ a polygonal region in $E_1$ which does not intersect $L$. For each point $P$ of $K$ let $\overline{PP'}$ be a segment parallel to $L$ with $P'$ in $E_2$. The union of all such segments is called a *prism.*

2. Definitions. The polygonal region $K$ is called the *lower base,* or just *the base,* of the prism. The set of all the points $P'$, that is, the part of the prism that lies in $E_2$, is called the *upper base.* The distance $h$ between $E_1$ and $E_2$ is the *altitude* of the prism. If $L$ is perpendicular to $E_1$ and $E_2$ the prism is called a *right prism.*

   Prisms are classified according to their bases: a *triangular prism* is one whose base is a triangular region, a *rectangular prism* is one whose base is a rectangular region, and so on.

3. Definition. A *cross-section* of a prism is its intersection with a plane parallel to its base, provided this intersection is not empty.

4. Theorem 16-1. All cross-sections of a triangular prism are congruent to the base.

5. Corollary 16-1-1. The upper and lower bases of a triangular prism are congruent.

6. Theorem 16-2. (Prism Cross-Section Theorem.) All cross-sections of a prism have the same area.

7. Corollary 16-2-1. The two bases of a prism have equal areas.

8. Definitions. A *lateral edge* of a prism is a segment $\overline{AA'}$, where $A$ is a vertex of the base of the prism. A *lateral face* is the union of all segments $PP'$ for which $P$ is a point in a given side of the base. The *lateral surface* of a prism is the union of its lateral faces. The *total surface* of a prism is the union of its lateral surface and its bases.

9. Theorem 16-3. The lateral faces of a prism are parallelogram regions, and the lateral faces of a right prism are rectangular regions.

10. Definitions: A *parallelepiped* is a prism whose base is a parallelogram region. A *rectangular parallelepiped* is a right rectangular prism.

## 15.2 Pyramids

1. Definitions. Let $K$ be a polygonal region in a plane $E$, and $V$ a point not in $E$. For each point $P$ in $K$ there is a segment $\overline{PV}$. The union of all such segments is called a *pyramid* with *base* $K$ and vertex $V$. The distance $h$ from $V$ to $E$ is the *altitude* of the pyramid.

2. Theorem 16-4. A cross-section of a triangular pyramid, by a plane between the vertex and the base, is a triangular region similar to the base. If the distance from the vertex to the cross-section plane is $k$ and the altitude is $h$, then the ratio of the area of the cross-section to the area of the base is $\left(\frac{k}{h}\right)^2$.

3. Theorem 16-5. In any pyramid, the ratio of the area of a cross-section and the area of the base is $\left(\frac{k}{h}\right)^2$, where $h$ is the altitude of the pyramid and $k$ is the distance from the vertex to the plane of the cross-section.

4. Theorem 16-6. (The Pyramid Cross-Section Theorem.) Given two pyramids with the same altitude. If the bases have the same area, then cross-sections equidistant from the bases also have the same area;

## 15.3 Volumes of Prisms and Pyramids, Cavalieri's Principle

1. Postulate 21. The volume of a rectangular parallelepiped is the product of the altitude and the area of the base.

2. Postulate 22. (Cavalieri's Principle.) Given two solids and a plane. If for every plane which intersects the solids and is parallel to the given plane the two intersections have equal areas, then the two solids have the same volume.

3. Theorem 16-7. The volume of any prism is the product of the altitude and the area of the base.

4. Theorem 16-8. If two pyramids have the same altitude and the same base area, then they have the same volume.

5. Theorem 16-9. The volume of a triangular pyramid is one-third the product of its altitude and its base area.

6. Theorem 16-10. The volume of a pyramid is one-third the product of its altitude and its base area.

## 15.4 Cylinders and Cones

1. Definitions. *Circular cylinder* and *circular cone.*
2. Definition. If the center of the base circle of a circular cone is the foot of the perpendicular from $V$ to $E$, the cone is called a *right circular cone.*
3. Theorem 16-11. A cross-section of a circular cylinder is a circular region congruent to the base.
4. Theorem 16-12. The area of a cross-section of a circular cylinder is equal to the area of the base.
5. Theorem 16-13. A cross-section of a cone of altitude $h$, made by a plane at a distance $k$ from the vertex, is a circular region whose area has a ratio to the area of the base of $\left(\frac{k}{h}\right)^2$.
6. Theorem 16-14. The volume of a circular cylinder is the product of the altitude and the area of the base.
7. Theorem 16-15. The volume of a circular cone is one-third the product of the altitude and the area of the base.

## 15.5 Spheres; Volume and Area

1. By the *volume of a sphere* we mean the volume of the solid which is the union of the sphere and its interior.
2. Theorem 16-16. The volume of a sphere of radius $r$ is $\frac{4}{3}\pi r^3$.
3. Theorem 16-17. The surface area of a sphere of radius $r$ is $S = 4\pi r^2$.

# 16 Plane Coordinate Geometry

## 16.1 Coordinate Systems in a Plane

1. First we take a line $X$ in the plane, and set up a coordinate system on $X$. This line will be called the *$x$-axis.* In a figure we usually use an arrow-head to emphasize the positive direction on the $x$-axis.

   Next we let $Y$ be the perpendicular to the $x$-axis through the point $O$ whose coordinate is zero, and we set up a coordinate system on $Y$. By the Ruler Placement Postulate this can be done so that point $O$ also has coordinate zero on $Y$. $Y$ will be called the *$y$-axis.* As before, we indicate the positive direction by an arrow-head. The intersection $O$ of the two axes is called the *origin.*

   We can now describe any point in the plane by a pair of numbers. The scheme is this. Given a point $P$, we drop a perpendicular to the $x$-axis, ending at a point $M$, with coordinate $x$. We drop a perpendicular to the $y$-axis, ending at a point $N$, with coordinate $y$.

2. Definitions. The numbers $x$ and $y$ are called the *coordinates* of the point $P$; $x$ is the *$x$-coordinate* and $y$ is the *$y$-coordinate.*

3. Just as a single line separates the plane into two parts (called half-planes) so the two axes separate the plane into four parts, called *quadrants.*

4. We have a one-to-one correspondence between points in the plane and ordered pairs of numbers . Such a correspondence is called a *coordinate system* in the plane.

## 16.2 The Slope of a Non-Vertical Line

1. The $x$-axis, and all lines parallel to it, are called *horizontal.* The $y$-axis, and all lines parallel to it, are called *vertical.*

2. Definition: The *slope* of $\overline{P_1P_2}$ is the number $m = \frac{y_2 - y_1}{x_2 - x_1}$.

3. Theorem 17-1. On a non-vertical line, all segments have the same slope.

## 16.3 Parallel and Perpendicular Lines

1. Theorem 17-2. Two non-vertical lines are parallel if and only if they have the same slope.

2. Theorem 17-3. Two non-vertical lines are perpendicular if and only if their slopes are the negative reciprocals of each other.

## 16.4 The Distance Formula

1. Theorem 17-4. (The Distance Formula.) The distance between the points $(x_1, y_1)$ and $(x_2, y_2)$ is equal to $\sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$.

## 16.5 The Mid-Point Formula

1. Theorem 17-5. (The Mid-Point Formula.) Let $P_1 = (x_1, y_1)$ and let $P_2 = (x_2, y_2)$. Then the midpoint of $\overline{P_1P_2}$ is the point $P = \left(\frac{x_1+x_2}{2}, \frac{y_1+y_2}{2}\right)$.

## 16.6 Proofs of Geometric Theorems

1. Theorem A. The segment between the mid-points of two sides of a triangle is parallel to the third side and half as long.

2. Theorem B. If the diagonals of a parallelogram are congruent, the parallelogram is a rectangle.

## 16.7 The Graph of a Condition

1. By a *graph* we mean simply a figure in the plane, that is, a set of points.

## 16.8 How to Describe a Line by an Equation

1. Theorem 17-6. Let $L$ be a non-vertical line with slope $m$, and let $P$ be a point of $L$, with coordinates $(x_1, y_1)$. For every point $Q = (x, y)$ of $L$, the equation $y - y_1 = m(x - x_1)$ is satisfied.

2. Theorem 17-7. The graph of the equation $y - y_1 = m(x - x_1)$ is the line that passes through the point $(x_1, y_1)$ and has slope $m$.

3. The equation given in Theorem 17-7 is called the *point-slope form* of the equation of the line.

## 16.9 Various Forms of the Equation of a Line

1. Definition. The point where the line crosses the $y$-axis is called the *$y$-intercept*. If this is the point $(O, b)$, then the point-slope equation takes the form $y - b = m(x - O)$, $y = mx + b$. This is called the *slope-intercept* form. The number $b$ is also called the *$y$-intercept* of the line.

2. Theorem 17-8. The graph of the equation $y = mx + b$ is the line with slope $m$ and $y$-intercept $b$.

## 16.10 The General Form of the Equation of a Line

1. Definition. By a *linear equation in $x$ and $y$* we mean an equation of the form $Ax + By + C = 0$, where $A$ and $B$ are not both zero.

2. Theorem 17-9. Every line in the plane is the graph of a linear equation in $x$ and $y$.

3. Theorem 17-10. The graph of a linear equation in $x$ and $y$ is always a line.

## 16.11 Intersections of Lines

1. The geometric problem of finding the point $P$ that is the intersection of two lines is equivalent to the algebraic problem of solving a system of two linear equations in two unknowns.

## 16.12 Circles

1. Theorem 17-11. The graph of the equation $(x-a)^2+(y-b)^2=r^2$ is the circle with center at $(a,b)$ and radius $r$.

2. Theorem 17-12. Every circle is the graph of an equation of the form $x^2+y^2+Ax+By+C=0$.

3. Theorem 17-13. Given the equation $x^2+y^2+Ax+By+C=O$. The graph of this equation is (1) a circle, (2) a point or (3) the empty set.

# 17 Rigid Motion

1. Definition. Given two figures $F$ and $F'$, a rigid motion between $F$ and $F$ is a one-to-one correspondence $P \longleftrightarrow P'$ between the points of $F$ and the points of $F'$, preserving distances.

2. Notation: $F \approx F'$, read $F$ is *isometric* to $F'$.

## 17.1 Rigid Motion of Segments

1. Theorem VIII-I. If $AB = CD$, then there is a rigid motion $\overline{AB} \approx \overline{CD}$.

2. This rigid motion between the two segments is completely described if we explain how the end-points are to be matched up. We therefore will call it the rigid motion *induced by the correspondence* $A \longleftrightarrow C$, $B \longleftrightarrow D$.

3. Theorem VIII-2. If there is a rigid motion $\overline{AB} \approx \overline{CD}$ between two segments, then $AB = CD$.

## 17.2 Rigid Motion of Rays, Angles and Triangles

1. Theorem VIII-3. Given any two rays $\overrightarrow{AB}$ and $\overrightarrow{CD}$, there is a rigid motion $\overrightarrow{AB} \approx \overrightarrow{CD}$.

2. Theorem VIII-4. If $\angle ABC \cong \angle DEF$, then there is a rigid motion $\angle ABC \approx \angle DEF$ between these two angles.

3. Theorem VIII-5. If $\Delta ABC \cong \Delta A'B'C'$, then there is a rigid motion $\Delta ABC \approx \Delta A'B'C'$, under which $A$, $B$ and $C$ correspond to $A'$, $B'$ and $C'$.

## 17.3 Rigid Motion of Circles and Arcs

1. Theorem VIII-6. Let $C$ and $C'$ be circles of the same radius $r$. Then there is a rigid motion $C \approx C'$ between $C$ and $C'$.

2. Theorem VIII-7. Let $C$ and $C'$ be circles with the same radius, as in Theorem VIII-6. Let $\angle XPB$ and $\angle X'P'B'$ be congruent central angles of $C$ and $C'$, respectively. Then a rigid motion $C \approx C'$ can be chosen in such a way that $B \longleftrightarrow B'$, $X \longleftrightarrow X'$, and $\widehat{BX} \approx \widehat{B'X'}$.

3. Theorem VIII-8. Given any two congruent arcs, there is a rigid motion between them.

## 17.4 Reflections

1. Definitions. A one-to-one correspondence between two figures is a *reflection* if there is a line $L$, such that for any pair of corresponding points $P$ and $P's$, either (1) $P = P'$ and lies on $L$ or (2) $L$ is the perpendicular bisector of $\overline{PP'}$. $L$ is called the *axis of reflection,* and each figure is said to be the *reflection,* or the *image,* of the other figure in $L$.

2. Theorem VIII-9. A reflection is a rigid motion.

3. Corollary VIII-9-1. A chain of reflections carrying $F$ into $F'$ determines a rigid motion between $F$ and $F'$.

4. Theorem VIII-10. Let $A$, $B$, $C$, $A'$, $B'$, $C'$ be six points such that $AB = A'B'$, $AC = A'C'$, $BC = B'C'$. Then there is a chain of at most three reflections that carries $A$, $B$, $C$ into $A'$, $B'$, $C'$.

5. Theorem VIII-II. Any rigid motion is the result of a chain of at most three reflections.

# 18 Trigonometry

## 18.1 Trigonometric Ratios

1. Theorem X-I. If an acute angle of one right triangle is congruent to an acute angle of another right triangle, then the two triangles are similar.
2. Definitions of *sine, cosine,* and *tangent* of an acute angle. These three quantities are called *trigonometric ratios.*

## 18.2 Relations Among the Trigonometric Ratios

1. Theorem X-2. For any acute $\angle A$, $\sin A < 1$, $\cos A < 1$.
2. Theorem X-3. For any acute angle $A$, $\frac{\sin A}{\cos A} = \tan A$, and $(\sin A)^2 + (\cos A)^2 = 1$.
3. Theorem X-4. If $\angle A$ and $\angle B$ are complementary acute angles, then $\sin A = \cos B$, $\cos A = \sin B$, and $\tan A = \frac{1}{\tan B}$.