Problems

MA310

Spring 2001

1 Some Games

1. This is a two-player game. There is a deck of 9 cards consisting of the ace through the 9 of clubs displayed on the table. Players alternately select a card from the currently unchosen cards. The first player to acquire, among his cards, a set of three cards that sum to 15, wins. (An ace counts as 1.) If no player is able to make a set after all 9 cards have been selected, the game is a tie. What is a good strategy? Who can win?

2. This is another two-player game. The first player selects a whole number between 1 and 6, inclusive. To this number the second player adds a number from 1 through 6. The players continue to alternately increase the current sum, each by adding a number from 1 through 6. The first player to achieve the exact sum 100 wins. What is a good strategy? Who will win?

3. In this cousin of the previous game, there are initially 100 pennies on the table. Two players alternately choose a number from 1 to 6 and remove that many pennies from the pile. The first person to take the last penny wins.

4. This is another two-player game. The initial position consists of three piles of pennies. The first pile has 3 pennies, the second pile has 5 pennies, and the third has 7 pennies. Players alternately take turns, which consists of removing any positive number of pennies from any one pile. The winner is the player who removes the very last penny.

5. Yet another money game! Initially there is no money on the table. Two players alternately add one penny, one nickel, one dime, or one quarter to the money on the table. It is not permissible to allow the total to exceed one dollar, and the first player to reach exactly one dollar wins.

2 Some More Games

The following problems are from Chapter 7 of *Mathematical Circles* by Fomin, Genkin and Itenberg.

1. Two children take turns breaking up a rectangular chocolate bar 6 squares wide by 8 squares long. They may break the bar only along the divisions between the squares. If the bar breaks into several pieces, they keep breaking the pieces up until only the individual squares remain. The player who cannot make a break loses the game. Who will win?

2. There are three piles of stones: one with 10 stones, one with 15 stones, and one with 20 stones. At each turn, a player can choose one of the piles and divide it into two smaller piles. The loser is the player who cannot do this. Who will win, and how?

3. The numbers 1 through 20 are written in a row. Two players take turns putting plus signs and minus signs between the numbers. When all such signs have been placed, the resulting expression is evaluated (i.e., the additions and subtractions are performed). The first player wins if the sum is even, and the second wins if the sum is odd. Who will win and how?

4. Two players take turns placing rooks (castles) on a chessboard so that they cannot capture each other. The loser is the player who cannot place a castle. Who will win?

5. Ten 1's and ten 2's are written on a blackboard. In one turn, a player may erase any two figures. If the two figures erased are identical, they are replaced with a 2. If they are different, they are replaced with a 1. The first player wins if a 1 is left at the end, and the second player wins if a 2 is left.

6. The numbers 25 and 36 are written on a blackboard. At each turn, a player writes on the blackboard the (positive) difference between two numbers already on the blackboard—if this number does not already appear on the blackboard. The loser is the player who cannot write a number.

7. Given a checkerboard with dimensions (a) 9×10; (b) 10×12; (c) 9×11. In one turn, a player is allowed to cross out one row or one column if at the beginning of the turn there is at least one square of the row or column remaining. The player who cannot make a move loses.

8. Two players take turns putting pennies on a round table, without piling one penny on top of another. The player who cannot place a penny loses.

9. Two players take turns placing bishops on the squares of a chessboard, so that they cannot capture each other (the bishops may be placed on squares of any color). The player who cannot move loses.

10. There are two piles of 7 stones each. At each turn, a player may take as many stones as he chooses, but only from one of the piles. The loser is the player who cannot move.

11. Two players take turns placing knights on the squares of a chessboard, so that no knight can take another. The player who is unable to do this loses.

12. Two players take turns placing kings on the squares of a 9×9 chessboard, so that no king can capture another. The player who is unable to do this loses.

13. (a) Two players turn turns placing bishops on the squares of a chessboard. At each turn, the bishop must threaten at least one square not threatened by another bishop. (We regard the square a bishop is placed on as one of the squares it "threatens.") The player who cannot move is the loser. (b) The same game, but with rooks (castles).

14. Given a 10×10 chessboard, two players take turns covering pairs of squares with dominoes. Each domino consists of a rectangle 1 square in width and 2 squares in length (which can be held either way). The dominoes cannot overlap. The player who cannot place a domino loses.

15. A checker is placed on each square of an 11×11 checkerboard. Players take turns removing any number of checkers which lie next to each other along a row or column. the winner is the player who removes the last checker.

16. There are two piles of stones. One has 30 stones, and the other has 20 stones. Players take turns removing as many stones as they please, but from one pile only. The player removing the last stone wins.

17. Twenty points are placed around a circle. Players take turns joining two of the points with a line segment which does not cross a segment already drawn in. The player who cannot do so loses.

18. A daisy has (a) 12 petals; (b) 11 petals. Players take turns tearing off either a single petal, or two petals right next to each other. The player who cannot do so loses.

19. Given a rectangular parallelepiped of dimensions (a) $4 \times 4 \times 4$; (b) $4 \times 4 \times 3$; (c) $4 \times 3 \times 3$, consisting of unit cubes. Players take turns skewering a row of cubes (parallel to the edges of the figure), so long as there is at least one cube which is not yet skewered in the row. The player who cannot do so loses.

20. Two players take turns breaking a piece of chocolate consisting of 5×10 small squares. At each turn, they may break along the division lines of the squares. The player who first obtains a single square of chocolate wins.

21. Two players take turns placing **x**'s and **o**'s on a 9×9 checkerboard. The first player places **x**'s, and the second player places **o**'s. At the end of the play, the first player gets a point for each row or column which contains more **x**'s than **o**'s. The second player

gets a point for each row or column which contains more **o**'s than **x**'s. The player with the most points wins.

22. On a chessboard, a rook stands on square $a1$. Players take turns moving the rook as many squares as they want, either horizontally to the right or vertically upward. The player who can place the rook on square $h8$ wins.

23. A king is placed on square $a1$ of a chessboard. Players take turns moving the king either upwards, to the right, or along a diagonal going upwards and to the right. The player who places the king on square $h8$ is the winner.

24. There are two piles of candy. One contains 20 pieces, and other 21. Players take turns eating all the candy in one pile, and separating the remaining candy into two (not necessarily equal) non-empty piles. The player who cannot move loses.

25. A checker is placed at each end of a strip of squares measuring 1×20. Players take turns moving either checker in the direction of the other, each by one or by two squares. A checker cannot jump over another checker. The player who cannot move loses.

26. A box contains 300 matches. Players take turns removing no more than half the matches in the box. The payer who cannot move loses.

27. There are three piles of stones. The first contains 50 stones, the second 60 stones, and the third 70. A turn consists in dividing each of the piles containing more than one stone into two smaller piles. The player who leaves piles of individual stones is the winner.

28. The number 60 is written on a blackboard. Players take turns subtracting from the number on the blackboard any of its divisors, and replacing the original number with the result of this subtraction. The player who writes the number 0 loses.

29. There are two piles of matches: (a) a pile of 101 matches and a pile of 201 matches; (b) a pile of 100 matches and a pile of 201 matches. Players take turns removing a number of matches from one pile which is equal to one of the divisors of the number of matches in the other pile. The player removing the last match wins.

30. A queen stands on square $c1$ of a chessboard. Players take turns moving the queen any number of squares to the right, upwards, or along a diagonal to the right and upwards. The player who can place the queen in square $h8$ wins.

31. Of two piles of stones, one contains 7 stones, and the other 5. Players alternate taking any number of stones from one of the piles, or an equal number from each pile. The player who cannot move loses.

32. A knight is placed on square $a1$ of a chessboard. Players alternate moving the knight either two squares to the right and one square up or down, or two squares up and one square right or left (and usual knight moves but in restricted directions). The player who cannot move loses.

33. (a) There are two piles of 7 stones each. In each turn, a player may take a single stone from one of the piles, or a stone from each pile. The player who cannot move loses.
(b) In addition to the moves described above, players are allowed to take a stone from the first pile and place it on the second pile. Other rules remain the same.

34. There are two piles of 11 matches each. In one turn, a player must take two matches from one pile and one match from the other. The player who cannot move loses.

35. This game begins with the number 0. In one turn, a player can add to the current number any natural number from 1 through 9. The player who reaches the number 100 wins.

36. This game begins with the number 1. In one turn, a player can multiply the current number by any natural number from 2 through 9. The player who first names a number greater than 1000 wins.

37. This game begins with the number 2. In one turn, a player can add to the current number any natural number smaller than it. The player who reaches the number 1000 wins.

38. This game begins with the number 1000. In one turn, a player can subtract from the current number any natural number less than it which is a power of 2 (note that $1 = 2^0$). The player who reaches the number 0 wins.

3 The Lady or the Tiger

The following problems are from *The Lady or the Tiger* by Raymond Smullyan.

1. There are two rooms. Each room contains either a lady or a tiger, but it *could* be that there are tigers in both rooms, or ladies in both rooms. The doors to the rooms bear signs:

    I. In this room there is a lady, and in the other room there is a tiger.

    II. In one of these rooms there is a lady, and in one of these rooms there is a tiger.

    One of the signs is true and the other is false. Which room should you choose?

2. There are two rooms. Each room contains either a lady or a tiger, but it *could* be that there are tigers in both rooms, or ladies in both rooms. The doors to the rooms bear signs:

    I. At least one of these rooms contains a lady.

    II. A tiger is in the other room.

    The signs are either both true or both false. Which room should you choose?

3. There are two rooms. Each room contains either a lady or a tiger, but it *could* be that there are tigers in both rooms, or ladies in both rooms. The doors to the rooms bear signs:

    I. Either a tiger is in this room or a lady is in the other room.

    II. A lady is in the other room.

    The signs are either both true or both false. Which room should you choose?

4. There are two rooms. Each room contains either a lady or a tiger, but it *could* be that there are tigers in both rooms, or ladies in both rooms. The doors to the rooms bear signs:

    I. Both rooms contain ladies.

    II. Both rooms contain ladies.

    If the lady is in Room I, then the sign on the door is true, but if a tiger is in it, the sign is false. In Room II, the situation is the opposite: A lady in the room means the sign on the door is false, and a tiger in the room means the sign is true. Which room should you choose?

5. There are two rooms. Each room contains either a lady or a tiger, but it *could* be that there are tigers in both rooms, or ladies in both rooms. The doors to the rooms bear signs:

   I. At least one room contains a lady.

   II. The other room contains a lady.

   If the lady is in Room I, then the sign on the door is true, but if a tiger is in it, the sign is false. In Room II, the situation is the opposite: A lady in the room means the sign on the door is false, and a tiger in the room means the sign is true. Which room should you choose?

6. There are two rooms. Each room contains either a lady or a tiger, but it *could* be that there are tigers in both rooms, or ladies in both rooms. The doors to the rooms bear signs:

   I. It makes no difference which room you pick.

   II. There is a lady in the other room.

   If the lady is in Room I, then the sign on the door is true, but if a tiger is in it, the sign is false. In Room II, the situation is the opposite: A lady in the room means the sign on the door is false, and a tiger in the room means the sign is true. Which room should you choose?

7. There are two rooms. Each room contains either a lady or a tiger, but it *could* be that there are tigers in both rooms, or ladies in both rooms. The doors to the rooms bear signs:

   I. It does make a difference which room you pick.

   II. You are better off choosing the other room.

   If the lady is in Room I, then the sign on the door is true, but if a tiger is in it, the sign is false. In Room II, the situation is the opposite: A lady in the room means the sign on the door is false, and a tiger in the room means the sign is true. Which room should you choose?

8. There are two rooms. Each room contains either a lady or a tiger, but it *could* be that there are tigers in both rooms, or ladies in both rooms. The signs have fallen off the doors:

- This room contains a tiger.
- Both rooms contain tigers.

When properly attached, a lady in Room I means the sign which should be on that door is true and a tiger in it means the sign should be false, and the reverse is true for Room II. Which room should you choose?

9. There are three rooms, with a lady in one room and a tiger in each of the other two rooms.

   I. A tiger is in this room.
   II. A lady is in this room.
   III. A tiger is in Room II.

   At most one of the three signs is true. Which rooms contains the lady?

10. Again there are three rooms, with a lady in one room and a tiger in each of the other two rooms. The sign on the door of the room containing the lady is true, and at least one of the other two signs is false.

    I. A tiger is in Room II.
    II. A tiger is in this room.
    III. A tiger is in Room I.

    Which room contains the lady?

11. One of the three rooms contains a lady, another a tiger, and the third is empty. The sign on the door of the room containing the lady is true, the sign on the door of the room with the tiger is false, and the sign on the door of the empty room is either true or false.

    I. Room III is empty.
    II. The tiger is in Room I.
    III. This room is empty.

    Which rooms contain what?

12. There are nine rooms. One contains a lady. Each of the others either contains a tiger or is empty. The sign on the door of the room containing the lady is true. The signs on the doors of all rooms containing tigers are false. The signs on the doors of the empty rooms may be either true or false.

    I. The lady is in an odd-numbered room.
    II. This room is empty.
    III. Either Sign V is right or Sign VII is wrong.
    IV. Sign I is wrong.
    V. Either Sign II or Sign IV is right.
    VI. Sign III is wrong.
    VII. The lady is not in Room I.
    VIII. This room contains a tiger and Room IX is empty.
    IX. This room contains a tiger and VI is wrong.

    Show that you cannot determine where the lady is. But suppose you learn that someone solves this puzzle after being told whether Room Eight is empty or not. Where is the lady?

4 More Thinking about Thinking

1. Three students — Alfred, Beth, and Carla — are blindfolded and told that either a red or a green hat will be placed on each of them. After this is done, the blindfolds are removed; the students are asked to raise a hand if they see a red hat, and to leave the room as soon as they are sure of the color of their own hat. All three hats happen to be red, so all three students raise a hand. Several minutes go by until Carla, who is more astute than the others, leaves the room. How did she deduce the color of her hat?

2. Two students, A and B, are chosen from a math class of highly logical individuals. They are each given one positive integer. Each knows his/her own number, and is trying to determine the other's number. They are informed that their numbers are consecutive. In each of the following scenarios, what can you deduce about the two numbers?

   (a) First Scenario

   A: I know your number.

   B: I know your number.

   (b) Second Scenario

   A: I don't know your number.

   B: I know your number.

   A: I know your number.

   (c) Third Scenario

   A: I don't know your number.

   B: I don't know your number.

   A: I know your number.

   B: I know your number.

   (d) Fourth Scenario

   A: I don't know your number.

   B: I don't know your number.

   A: I don't know your number.

   B: I don't know your number.

   A: I know your number.

   B: I know your number.

3. "I hear some youngsters playing in the back yard," said Jones, a graduate student in mathematics. "Are they all yours?"

   "Heavens, no," exclaimed Professor Smith, the eminent number theorist. "My children are playing with friends from three other families in the neighborhood, although our family happens to be largest. The Browns have a smaller number of children, the Greens have a still smaller number, and the Blacks the smallest of all."

   "How many children are there altogether?" asked Jones.

   "Let me put it this way," said Smith. "There are fewer than 18 children, and the product of the numbers in the four families happens to be my house number which you saw when you arrived."

   Jones took a notebook and pencil from his pocket and started scribbling. A moment later he looked up and said, "I need more information. Is there more than one child in the Black family?"

   As soon as Smith replied, Jones smiled and correctly stated the number of children in each family.

   Knowing the house number and whether the Blacks had more than one child, Jones found the problem trivial. it is a remarkable fact, however, that the number of children in each family can be determined solely on the basis of the information given above!

4. A logician vacationing in the South Seas finds himself on an island inhabited by the two proverbial tribes of liars and truth-tellers. Members of one tribe always tell the truth, members of the other always lie. He comes to a fork in a road and has to ask a native bystander which branch he should take to reach a village. He has no way of telling whether the native is a truth-teller or a liar. The logician thinks a moment, then asks one question only. From the reply he knows which road to take. What question does he ask?

5. Imagine that you have three boxes, one containing two black marbles, one containing two white marbles, and the third, one black marble and one white marble. The boxes were labeled for their contents—BB, WW and BW—but someone has switched the labels so that every box is now incorrectly labeled. You are allowed to take one marble at a time out of any box, without looking inside, and by this process of sampling you are to determine the contents of all three boxes. What is the smallest number of drawings needed to do this?

5 Search for Extra-Terrestrial Intelligence

The following is from *The Sixth Book of Mathematical Games from Scientific American* by Martin Gardner.

Suppose you want to send a message to extra-terrestrials that stands a reasonable chance of being understood, even though it is quite unlikely that any recipient is familiar with any human language. One way is to exploit the assumption of the universality of arithmetic. (How reasonable is this assumption? I suggest reading: R. W. Hamming, Mathematics on a distant planet, *American Mathematical Monthly* **105** (1998) 640–650.) The following proposed message to extraterrestrials was devised by Ivan Bell in 1960.

Letters from A through Z (omitting O and X) provide the 24 symbols. (Each symbol is presumably radioed by a combination of beeps, but we need not be concerned with those details.) The punctuation marks are not part of the message but indications of time lapses. Adjacent letters are sent with short pauses between them. A space between letters means a longer pause. Commas, semicolons, and periods represent progressively longer pauses. The longest time lapses come between paragraphs, which are numbered for the reader's convenience; the numbers are not part of the message. To minds in any solar system the message should be crystal clear except for the last paragraph, which is somewhat ambiguous; even if properly deciphered, it could be understood fully only by inhabitants of one of our solar system's planets.

1. A. B. C. D. E. F. G. H. I. J. K. L. M. N. P. Q. R. S. T. U. V. W. Y. Z.
2. A A, B; A A A, C; A A A A, D; A A A A A, E; A A A A A A, F; A A A A A A A, G; A A A A A A A A, H; A A A A A A A A A, I; A A A A A A A A A A, J.
3. A K A L B; A K A K A L C; A K A K A K A L D. A K A L B; B K A L C; C K A L D; D K A L E. B K E L G; G L E K B. F K D L J; J L F K D.
4. C M A L B; D M A L C; I M G L B.
5. C K N L C; H K N L H. D M D L N; E M E L N.
6. J L AN; J K A L AA; J K B L AB; AA K A L AB. J K J L BN; J K J K J L CN. FN K G L FG.
7. B P C L F; E P B L J; F P J L FN.
8. F Q B L C; J Q B L E; FN Q F L J.
9. C R B L I; B R E L CB.

10. J P J L J R B L S L ANN; J P J P J L J R C L T L ANNN. J P S L T; J P T L J R D.
11. A Q J L U; U Q J L A Q S L V.
12. U L WA; U P B L WB; AWD M A L WD L D P U. V L WNA; V P C L WNC. V Q J L WNNA; V Q S L WNNNA. J P EWFGH L EFWGH; S P EWFGH L EFGWH.
13. GIWIH Y HN; T K C Y T. Z Y CWADAF.
14. D P Z P WNNIB R C Q C.

6 Some Counting Problems

1. What is the next term in the sequence? In each case, think of at least three different "plausible" answers.
   (a) $1, 2, 3, \ldots$.
   (b) $1, 2, 4, \ldots$.
   (c) $1, 2, 5, \ldots$.

2. With an unlimited supply of toothpicks of n different colors, how many different triangles can be formed on a flat surface, using three toothpicks for the three sides of each triangle? (Reflections are considered different, but not rotations.)

3. The same as the previous question, but this time with squares.

4. How many different tetrahedra can be produced by coloring each face a solid color and using n different colors? (Two tetrahedra are the same if they can be turned and placed side by side so that corresponding sides match in color.)

5. The same as the previous question, but this time with cubes.

6. What is the maximum number of regions space can be divided into using 10 planes?

7. What is the maximum number of pieces into which a pancake can be cut by n straight cuts, each of which crosses each of the others?

8. What is the maximum number of pieces that can be produced by n simultaneous straight cuts of a flat figure shaped like a crescent moon?

9. Into how many parts can the plane be divided by intersecting circles of the same size? Of different sizes? By intersecting ellipses of different sizes?

10. How many pieces of cheesecake can be produced by n simultaneous plane cuts of a cylindrical cake?

11. Into how many regions can space be divided by intersecting spheres?

12. What is the maximum number of pieces that can be produced by n simultaneous plane cuts through a doughnut?

13. Guess and prove a formula for the sum of the first n positive integers.

14. Guess and prove a formula for the sum of the first n positive odd integers.

15. Guess and prove a formula for the sum of the squares of the first n positive integers.

16. Guess and prove a formula for the sum of the cubes of the first n positive integers.

17. Guess and prove a formula for the sum of the fourth powers of the first n positive integers.

18. The Fibonacci numbers are defined by $F_1 = F_2 = 1$ and thereafter $F_n = F_{n-2} + F_{n-1}$, $n = 3, 4, 5, \ldots$. So the first 8 Fibonacci numbers are: 1, 1, 2, 3, 5, 8, 13, 21. Prove the formula:
$$F_n = \frac{1}{\sqrt{5}}\left[\left(\frac{1+\sqrt{5}}{2}\right)^n - \left(\frac{1-\sqrt{5}}{2}\right)^n\right].$$

19. Find and prove a formula for the number of subsets of the set $\{1, 2, 3, \ldots, n\}$.

20. For nonnegative integers n and k, define $\binom{n}{k}$ to be the number of subsets of size k contained in a set of size n. Prove that
$$\binom{n}{k} + \binom{n}{k+1} = \binom{n+1}{k+1}.$$
Then prove that
$$\binom{n}{k} = \frac{n!}{k!(n-k)!},$$
remembering that we define $0! = 1$.

21. Prove the binomial formula:
$$(a+b)^n = \sum_{i=0}^{n} \binom{n}{i} a^{n-i} b^i.$$

22. Find and prove a formula for the number of ways n different men can be completely paired up with n different women.

23. A group of $2n$ individuals are to be paired off into n pairs to play checkers. Find and prove a formula for the number of ways to do this.

24. The Towers of Hanoi puzzle consists of three posts and n disks of different sizes that are initially stacked on one of the posts, from largest to smallest with the largest disk on the bottom. The goal is to move this stack of disks to a different post by a sequence of moves. Each move consists of moving the disk from top of the pile on any post to the top of the pile on any other post, as long as a larger disk is never placed directly on a smaller one. Try to solve this puzzle with $n = 7$. Can you find a general procedure for n disks? What is the minimum number of moves required?

25. A convex polygon with n sides can be dissected (triangulated) into triangles by drawing a certain number of non-crossing diagonals.

    (a) How many diagonals are needed?

    (b) How many triangles result?

    (c) A triangle in the triangulation consisting of two consecutive edges of the boundary together with one internal diagonal is called an ear of the triangulation. Prove that if $n \geq 4$ then every triangulation has at least two ears.

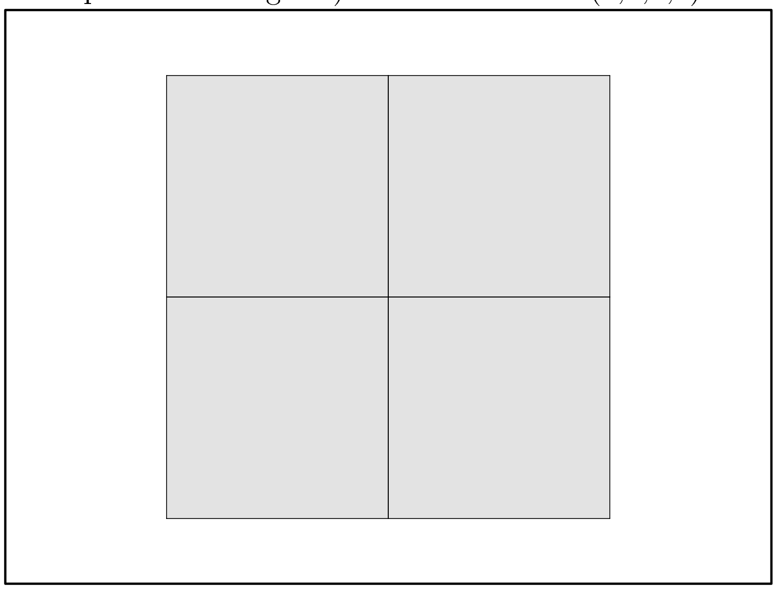
26. Let t_n be the number of ways that a convex polygon with n sides can be triangulated using non-crossing diagonals. Verify that $t_3 = 1$, $t_4 = 2$ and $t_5 = 5$. Calculate t_6 and t_7.

    (a) Find a way to calculate t_n if you already know $t_3, t_4, \ldots, t_{n-1}$.

    (b) Can you find a formula for t_n?

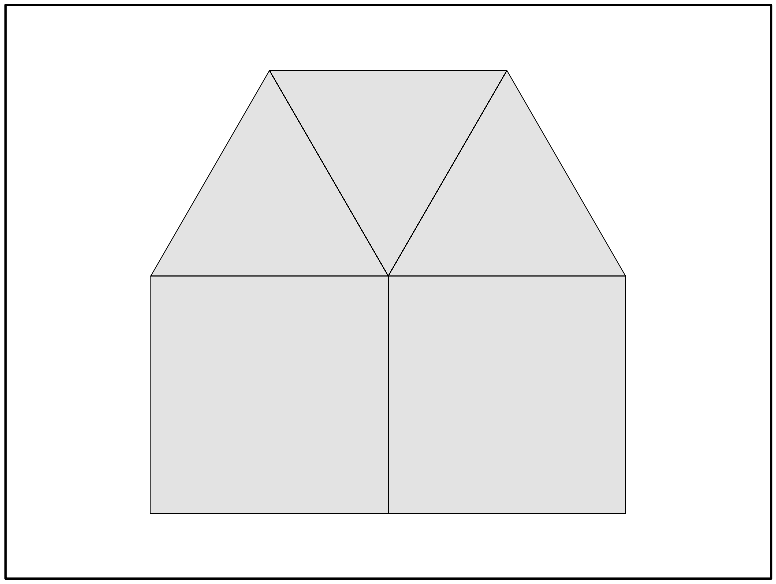
7 Plane Clusters

Four squares can be fit together perfectly in the plane surrounding a common corner (since each interior angle of a square is 90 degrees). Let's call this a (4,4,4,4) cluster.

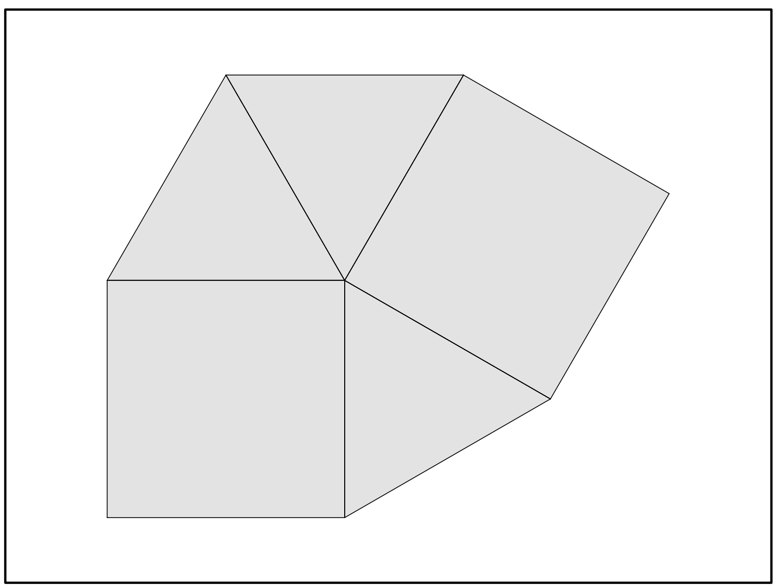


(4,4,4,4) Cluster

Similarly, two squares and three equilateral triangles can fit together perfectly surrounding a common corner. There are essentially two different ways to do this: (4,4,3,3,3) (where the squares are adjacent) and (4,3,4,3,3) (where the squares are not adjacent).



(4,4,3,3,3) Cluster



(4,3,4,3,3) Cluster

Note that we could have called this last cluster $(3,3,4,3,4)$ as well—it still refers to the same cluster. However, $(4,4,3,3,3)$ and $(4,3,4,3,3)$ are *not* the same.

1. You have just seen three clusters. Try to determine all possible clusters that can be formed by placing combinations of regular polygons in the plane surrounding a common corner. Be *systematic* in some fashion, so that you can be certain you have found all of them.

2. Some of the clusters can be extended to cover (tile) the plane so that at every corner point of the tiling, exactly the same cluster appears—the same sequence of polygons, in either clockwise or counterclockwise order. For example, if you tile the plane with squares, you have a $(4, 4, 4, 4)$ cluster at every single corner. Of the clusters you have found, determine which ones can be extended. Make a good drawing of each one you have found.